Thai Journal of Mathematics
Volume 19 Number 1 (2021)
Pages 67–76

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

σ-Intertwinings, σ-Cocycles and Automatic Continuity

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Abstract Let $\mathcal{A}$ be an algebra, $\mathcal{X}$ an $\mathcal{A}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{A}$ a continuous homomorphism. In this paper, we show a continuous linear one to one correspondence between $Z^1_\sigma(\mathcal{A}, \mathcal{F})$, the set of all module valued σ-derivations and $LI_\sigma(\mathcal{A}, \mathcal{X})$, the set of all left σ-intertwining mappings, where $\mathcal{F} = B(\mathcal{A}_+, \mathcal{X})$ and that $B(\mathcal{A}_+, \mathcal{X})$ is a $\sigma(\mathcal{A})$-bimodule. A similar fact is proved between $Z^n_\sigma(\mathcal{A}, \mathcal{F})$, the set of all n-σ-cocycles, and $LI^n_\sigma(\mathcal{A}, \mathcal{X})$, the set of all σ-intertwining mappings in the last variables. Also there exists a linear homeomorphism between $\mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{F})$, the set of all continuous module valued σ-derivations, and $B(\mathcal{A}, \mathcal{X})$. Moreove, it is proved that the same relation satisfies between $\mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{F})$ and $B^n(\mathcal{A}, \mathcal{X})$.

MSC: 47B47; 46H40
Keywords: derivation; σ-derivation; (σ, τ)-derivation; intertwining; cocycle

Submission date: 29.04.2016 / Acceptance date: 29.01.2018

1. Introduction

Let $\mathcal{A}$ be an algebra. A linear operator d on $\mathcal{A}$ is said to be a *deivation* if it satisfies the Libnitz Rule $d(ab) = d(a)b + ad(b)$ for each $a, b \in \mathcal{A}$. Furthermore if σ is a homomorphism on $\mathcal{A}$, then $d\sigma$ has the property that $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for each $a, b \in \mathcal{A}$; a linear mapping with such a property, is called a *σ-derivation.*

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ an $\mathcal{A}$-bimodule. We say that a function $S : \mathcal{A} \to \mathcal{X}$ is *intertwining* if $\Delta^1 : L^1(\mathcal{A}, \mathcal{X}) \to L^2(\mathcal{A}, \mathcal{X})$ defined by

$$\Big(\Delta^1(S)\Big)(a, b) = aS(b) - S(ab) + S(a)b, \quad \forall a, b \in \mathcal{A} \tag{1.1}$$

is continuous bilinear mapping. The function S is *left intertwining* if for each $a \in \mathcal{A}$, the function $\varphi_a : \mathcal{A} \to \mathcal{X}$ defined by $\varphi_a(b) = aS(b) - S(ab)$ is continuous. In the same manner, S is called *right intertwining* if for each $a \in \mathcal{A}$, the function $\phi_a : \mathcal{A} \to \mathcal{X}$ defined by $\varphi_a(b) = S(ba) - S(b)a$ is continuous; the set of all left-intertwining mappings (or right-intertwining mappings) of $\mathcal{A}$ to $\mathcal{X}$, denoted by $LI(\mathcal{A}, \mathcal{X})$ (or $RI(\mathcal{A}, \mathcal{X})$). At the same

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Published by The Mathematical Association of Thailand.

time, the set of all intertwining mappings of $\mathcal{A}$ to $\mathcal{X}$ is denoted by $I(\mathcal{A},\mathcal{X})$. In view of the uniform bounded theorem, $S:\mathcal{A}\to\mathcal{X}$ is intertwining if and only if, it is both left intertwining and right intertwining.

Similar to the definition Δ^1, one can consider for each natural number n, the function $\Delta^n : L^n(\mathcal{A},\mathcal{X}) \to L^{n+1}(\mathcal{A},\mathcal{X})$ defined by

$$\begin{aligned}((\Delta^n)S)(a_1,a_2,...a_n,a_{n+1}) &= (-1)^0 a_1 S(a_2,a_3,...a_{n+1})\\ &+ (-1)^1 S(a_1a_2,a_3,...a_{n+1})\\ &+ (-1)^2 S(a_1,a_2a_3,a_4,...a_{n+1})\\ &+ ... + (-1)^{n+1} S(a_1,a_2,...a_n)a_{n+1}\end{aligned}$$

in where $S\in L^n(\mathcal{A},\mathcal{X})$ and $(a_1,a_2,...a_n,a_{n+1})\in\mathcal{A}^{n+1}$. For each nonnegative integer n, we denote $Kerl(\Delta^n)$ by $Z^n(\mathcal{A},\mathcal{X})$ and call each of its elements a *n-cocycle*; for more about this fact, we refer the reader to [1, 2]. In [1], it was also proved that if $S:\mathcal{A}\to\mathcal{X}$ is left intertwining, then there exists a module valued derivation $D:\mathcal{A}\to B(\mathcal{A}_+,\mathcal{X})$ and a continuous left $\mathcal{A}$-module homomorphism $U : B(\mathcal{A}_+,\mathcal{X})\to\mathcal{X}$ such that $U\circ D = S$.

In this paper, prior to anything, the facts and the notations such as σ-intertwining, σ-cocycle, $Z^1_\sigma(\mathcal{A},\mathcal{F})$, $LI_\sigma(\mathcal{A},\mathcal{X})$, $Z^n_\sigma(\mathcal{A},\mathcal{F})$ will be defined and after that we extend some of theorems and results stated on the concepts intertwining and cocycle in [1, 2].

2. σ-Intertwinings and σ-Cocycles

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule and $\sigma:\mathcal{A}\to\mathcal{A}$ a continuouse homomorphism linear mapping. We say that a function $S:\mathcal{A}\to\mathcal{X}$ is a *σ-intertwining mapping* if $\Delta^1 : L^1(\mathcal{A},\mathcal{X})\to L^2(\mathcal{A},\mathcal{X})$ defined by

$$\Big(\Delta^1(S)\Big)(a,b) = \sigma(a)S(b) - S(ab) + S(a)\sigma(b), \quad \forall a,b\in\mathcal{A} \tag{2.1}$$

is continuous bilinear mapping.

Definition 2.2. Let $\mathcal{A}$, $\mathcal{X}$, σ be as in Definition 2.1. We say that a function $S:\mathcal{A}\to\mathcal{X}$ is a *left σ-intertwining mapping* if for each $a\in\mathcal{A}$, the function $\varphi_a:\mathcal{A}\to\mathcal{X}$ defined by $\varphi_a(b)=\sigma(a)S(b)-S(ab)$ is continuous. In the same manner, S is a *right σ-intertwining mapping* if for each $a\in\mathcal{A}$, the function $\phi_a:\mathcal{A}\to\mathcal{X}$ defined by $\varphi_a(b)=S(ba)-S(b)\sigma(a)$ is continuous; we denote the set of all left σ-intertwining mappings (or right σ-intertwining mappings) of $\mathcal{A}$ to $\mathcal{X}$, by $LI_\sigma(\mathcal{A},\mathcal{X})$ (or $RI_\sigma(\mathcal{A},\mathcal{X})$).

Remark 2.3. In view of the uniform bounded theorem, $S:\mathcal{A}\to\mathcal{X}$ is σ-intertwining if and only if, it is both left σ-intertwining mapping and right σ-intertwining; the set of all σ-intertwining mappings of $\mathcal{A}$ to $\mathcal{X}$, denoted by $I_\sigma(\mathcal{A},\mathcal{X})$.

Remark 2.4. Let $\mathcal{A}$, $\mathcal{X}$, σ be as in Definition 2.1. Similar to the definition Δ^1, for each natural number n, we consider $\Delta^n : L^n(\mathcal{A},\mathcal{X})\to L^{n+1}(\mathcal{A},\mathcal{X})$ defined for each $S\in L^n(\mathcal{A},\mathcal{X})$ and each $(a_1,a_2,...a_n,a_{n+1})\in\mathcal{A}^{n+1}$ by

$$\begin{aligned}((\Delta^n)(S))(a_1,a_2,...a_n,a_{n+1}) &= (-1)^0\sigma(a_1)S(a_2,a_3,...a_{n+1})\\ &+ (-1)^1 S(a_1a_2,a_3,...a_{n+1})\\ &+ (-1)^2 S(a_1,a_2a_3,a_4,...a_{n+1})\\ &+ ... + (-1)^{n+1} S(a_1,a_2,...a_n)\sigma(a_{n+1}).\end{aligned}$$

Also for $n = 0$, we define $\Delta^0 : \mathcal{X} \to L^1(\mathcal{A}, \mathcal{X})$ by the rule

$$((\Delta^0)(x))(a) = \sigma(a)x - x\sigma(a), \quad \forall x \in \mathcal{X} \quad \text{and} \quad \forall a \in \mathcal{A}.$$

One can easily prove that $\Delta^1 \circ \Delta^0 = 0$ and so $Im(\Delta^0) \subseteq Kerl(\Delta^1)$. Also, in general, it can be proved that for each nonnegative integer n, $\Delta^n \circ \Delta^{n-1} = 0$ and so $Im(\Delta^{n-1}) \subseteq Kerl(\Delta^n)$. Thus the following complex is presentable:

$$\begin{aligned} 0 \xrightarrow{0} \mathcal{X} \ &\xrightarrow{\Delta^0} \ L^1(\mathcal{A}, \mathcal{X}) \xrightarrow{\Delta^1} L^2(\mathcal{A}, \mathcal{X}) \xrightarrow{\Delta^2} L^3(\mathcal{A}, \mathcal{X}) \\ &\xrightarrow{\Delta^3} \ \ldots \xrightarrow{\Delta^{n-1}} L^n(\mathcal{A}, \mathcal{X}) \xrightarrow{\Delta^n} L^{n+1}(\mathcal{A}, \mathcal{X}) \xrightarrow{\Delta^{n+1}} \ldots \end{aligned}$$

Definition 2.5. Let $\mathcal{A}$, $\mathcal{X}$, σ be as in Definition 2.1 and $x \in \mathcal{X}$. A linear mapping $d_x : \mathcal{A} \to \mathcal{X}$ is said to be an *inner σ-derivation* if for each $a \in \mathcal{A}$, $d_x(a) = \sigma(a)x - x\sigma(a)$. It can be easily prove that the function d_x is a σ-derivation.

Definition 2.6. Let $\mathcal{A}$, $\mathcal{X}$, σ be as in Definition 2.1. For each nonnegative integer n, we denote $Kerl(\Delta^n)$ by $Z^n_\sigma(\mathcal{A}, \mathcal{X})$ and call each of itself elements a *n-σ-cocycle*. Also we denote $Im(\Delta^n)$ by $N^{n+1}_\sigma(\mathcal{A}, \mathcal{X})$ and call each of itself elements a *n-σ-coboundary*. Clearly $Z^n_\sigma(\mathcal{A}, \mathcal{X}) \subseteq L^n(\mathcal{A}, \mathcal{X})$ and $N^n_\sigma(\mathcal{A}, \mathcal{X}) \subseteq L^n(\mathcal{A}, \mathcal{X})$.

Remark 2.7. In view of the fact stated recently, $N^n_\sigma(\mathcal{A}, \mathcal{X}) \subseteq Z^n_\sigma(\mathcal{A}, \mathcal{X})$.

Remark 2.8. Suppose $T \in N^1_\sigma(\mathcal{A}, \mathcal{X}) = Im(\Delta^0)$. It turn out that there exists $x \in \mathcal{X}$ such that $T = (\Delta^0)(x)$. Then for each $a \in \mathcal{A}$ we have $T(a) = \big((\Delta^0)(x)\big)(a) = \sigma(a)x - x\sigma(a)$; i.e. T is an inner σ-derivation. Also it is well known that if $S \in Z^1_\sigma(\mathcal{A}, \mathcal{X}) = Kerl(\Delta^1)$ then

$$\sigma(a)S(b) - S(ab) + S(a)\sigma(b) = 0, \quad \forall a, b \in \mathcal{A};$$

i.e. S is a σ-drivation.

Definition 2.9. As it was mentioned above, $N^1_\sigma(\mathcal{A}, \mathcal{X}) \subseteq Z^1_\sigma(\mathcal{A}, \mathcal{X})$. We set

$$H^1_\sigma(\mathcal{A}, \mathcal{X}) = N^1_\sigma(\mathcal{A}, \mathcal{X}) - Z^1_\sigma(\mathcal{A}, \mathcal{X})$$

and define it the σ-cohomology space of $\mathcal{A}$ with coefficients in $\mathcal{X}$. In the same manner, we define $H^n_\sigma(\mathcal{A}, \mathcal{X}) = N^n_\sigma(\mathcal{A}, \mathcal{X}) - Z^n_\sigma(\mathcal{A}, \mathcal{X})$ and call it the *n-σ-cohomology space* of $\mathcal{A}$ with coefficients in $\mathcal{X}$.

Remark 2.10. Let $\mathcal{A}$, $\mathcal{X}$, σ be as in Definition 2.1. It is clear that the unitization of $\mathcal{A}$, i.e. $\mathcal{A}_+ = C \oplus \mathcal{A}$, is a $\mathcal{A}$-bimodule too. The vector space $B(\mathcal{A}_+, \mathcal{X})$ with the definition

$$(\sigma(a)f)(x) = \sigma(a)f(x), \quad (f\sigma(a))(x) = f(ax), \quad \forall a \in \mathcal{A}, \quad \forall f \in B(\mathcal{A}_+, \mathcal{X}), \quad \forall x \in \mathcal{X}$$

is a $\sigma(\mathcal{A})$-bimodule.

Theorem 2.11. *Let $\mathcal{A}, \mathcal{X}, \sigma$ be as in Definition 2.1. Let $S : \mathcal{A} \to \mathcal{X}$ be a left σ-intertwining mapping. Then there exists a module valued σ-derivation $D : \mathcal{A} \to B(\mathcal{A}_+, \mathcal{X})$ and a continuous left $\sigma(\mathcal{A})$-module homomorphism $U : B(\mathcal{A}_+, \mathcal{X}) \to \mathcal{X}$ such that $U \circ D = S$.*

Proof. As it was mentioned already, we consider $B(\mathcal{A}_+, \mathcal{X})$ as a $\sigma(\mathcal{A})$-bimodule by

$$(\sigma(a)f)(x) = \sigma(a)f(x), \quad (f\sigma(a))(x) = f(ax), \quad \forall a \in \mathcal{A}, \quad \forall f \in B(\mathcal{A}_+, \mathcal{X}), \quad \forall x \in \mathcal{X}.$$

We define $U : B(\mathcal{A}_+, \mathcal{X}) \to \mathcal{X}$ by

$$U(T) = T(1), \quad \forall T \in B(\mathcal{A}_+, \mathcal{X})$$

and $D : \mathcal{A} \to B(\mathcal{A}_+, \mathcal{X})$ by

$$D(a)(\beta, b) = \beta S(a) - \sigma(a)S(b) + S(ab).$$

On the one hand,

$$D(a_1a_2)(\beta, b) \quad = \quad \beta S(a_1a_2) + S(a_1a_2b) - \sigma(a_1a_2)S(b).$$

On the other hand

$$\begin{aligned}(D(a_1)\sigma(a_2))(\beta, b) &= D(a_1)((0, a_2)(\beta, b)) \\ &= D(a_1)((0, \beta a_2 + a_2b)(\beta, b)) \\ &= \beta S(a_1a_2) + S(a_1a_2b) - \beta\sigma(a_1)S(a_2) - \sigma(a_1)S(a_2b)\end{aligned}$$

and

$$\begin{aligned}(\sigma(a_1)D(a_2))(\beta, b) &= \sigma(a_1)(D(a_2)(\beta, b)) \\ &= \sigma(a_1)(\beta S(a_2) + S(a_2b) - \sigma(a_2)S(b)) \\ &= \beta\sigma(a_1)S(a_2) + \sigma(a_1)S(a_2b) - \sigma(a_1)\sigma(a_2)S(b).\end{aligned}$$

Thus we have

$$D(a_1a_2) \quad = \quad D(a_1)\sigma(a_2) + \sigma(a_1)D(a_2).$$

To prove $S = U \circ D$, we arrive at

$$\begin{aligned}(U \circ D)(a) &= U(D(a)) = (D(a))(1) = D(a)(1, 0) \\ &= 1S(a) + S(a, 0) - \sigma(a)S(0) = S(a).\end{aligned}$$

It is necessary to mention that clearly $D(a) \in B(\mathcal{A}_+, \mathcal{X})$. ∎

Theorem 2.12. *Let $D : \mathcal{A} \to B(\mathcal{A}_+, \mathcal{X})$ be a module valued σ-derivation. The mapping $S = U \circ D : \mathcal{A} \to B(\mathcal{A}_+, \mathcal{X}) \to \mathcal{X}$ is left σ-intertwining.*

Proof. To prove, suppose $a \in \mathcal{A}$ is arbitrary. We define $\varphi_a : \mathcal{A} \to \mathcal{X}$ by

$$\varphi_a(b) = \sigma(a)S(b) - S(ab), \quad \forall b \in \mathcal{A}.$$

Thus we get

$$\begin{aligned}\varphi_a(b) &= \sigma(a)(U \circ D)(b) - (U \circ D)(ab) \\ &= \sigma(a)(D(b)(1)) - (D(ab)(1)) \\ &= \sigma(a)(D(b)(1)) - (\sigma(a)D(b)(1) + D(a)\sigma(b))(1) \\ &= \sigma(a)D(b)(1) - \sigma(a)D(b)(1) - D(a)(b) = -D(a)(b).\end{aligned}$$

It is clear that $D(a)(b) \in B(\mathcal{A}_+, \mathcal{X})$; hence $\varphi_a : \mathcal{A} \to \mathcal{X}$ is continuous and consequently $U \circ D : \mathcal{A} \to B(\mathcal{A}_+, \mathcal{X}) \to \mathcal{X}$ is left σ-intertwining. ∎

Corollary 2.13. *By considering $F = B(\mathcal{A}_+, \mathcal{X})$, the function*

$$\phi : Z^1_\sigma(\mathcal{A}, \mathcal{F}) \to LI_\sigma(\mathcal{A}, \mathcal{X})$$
$$\phi(D) = U \circ D, \quad \forall\ D \in Z^1_\sigma(\mathcal{A}, \mathcal{F})$$

is an onto linear mapping.

Theorem 2.14. *ϕ is continuous.*

Proof.

$$\begin{aligned} ||\phi|| &= \sup_{||D||\leq 1} ||\phi(D)|| \\ &= \sup_{||D||\leq 1} ||U \circ D|| \end{aligned}$$

and

$$\begin{aligned} ||U \circ D|| &= \sup_{||a||\leq 1} ||(U \circ D)(a)|| \\ &= \sup_{||a||\leq 1} ||(D(a))(1)|| \\ &\leq ||D(a)|| \leq ||D||. \end{aligned}$$

Then $||\phi|| \leq 1$; i.e. ϕ is continuous. ■

Theorem 2.15. *ϕ is one-to-one.*

Proof. We claim that the σ-derivation D is unique for each left σ-intertwining S. It turn out that

$$\begin{aligned} \beta D(a)(1) + (D(ab)(1)) - \sigma(a)(D(b)(1)) &= \beta D(a)(1) + (\sigma(a)D(b) \\ &+ D(a)\sigma(b))(1) - \sigma(a)(D(b)(1)) \\ &= \beta D(a)(1) + D(a)b \\ &= D(a)(\beta, b). \end{aligned}$$

Thus if $\phi(D_1) = \phi(D_2)$, then $D_1(a)(1) = D_2(a)(1)$, for each $a \in \mathcal{A}$ and hence $D_1 = D_2$. ■

Remark 2.16. If in Remark 2.4, we replace the spaces $L^n(\mathcal{A}, \mathcal{X})$ by $B^n(\mathcal{A}, \mathcal{X})$, then we will denote $Z^n_\sigma(\mathcal{A}, \mathcal{X})$ and $N^n_\sigma(\mathcal{A}, \mathcal{X})$ by $\mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{X})$ and $\mathfrak{N}^n_\sigma(\mathcal{A}, \mathcal{X})$, respectively and call each of theirs elemants the continuous n-σ-cocycle and the continuous n-σ-coboundary, respectively; in this case,

$$\mathfrak{H}^1_\sigma(\mathcal{A}, \mathcal{X}) = \mathfrak{N}^1_\sigma(\mathcal{A}, \mathcal{X}) - \mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{X})$$

is called the continuous σ-cohomology of $\mathcal{A}$ with coefficients in $\mathcal{X}$. In the same manner,

$$\mathfrak{H}^n_\sigma(\mathcal{A}, \mathcal{X}) = \mathfrak{N}^n_\sigma(\mathcal{A}, \mathcal{X}) - \mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{X})$$

is called the continuous n-σ-cohomology of $\mathcal{A}$ with coefficients in $\mathcal{X}$.

Theorem 2.17. *The mapping*

$$\phi : \mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{F}) \to B(\mathcal{A}, \mathcal{X})$$
$$\phi(D) = U \circ D, \quad \forall\ D \in \mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{F})$$

is a linear homeomorphism.

Proof. Suppose $D \in \mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{F})$. Then

$$\sup_{||a||\leq 1} ||(U \circ D)(a)|| \leq ||D||$$

and hence $S = U \circ D \in B(\mathcal{A}, \mathcal{X})$. Now assume that $T \in B(\mathcal{A}, \mathcal{X})$. Since for each $a \in \mathcal{A}$, the mapping $\varphi_a : \mathcal{A} \to \mathcal{X}$ defined by

$$\varphi_a(b) = \sigma(a)T(b) - T(ab), \quad \forall\ b \in \mathcal{A},$$

is continuous, then $T \in LI_\sigma(\mathcal{A}, \mathcal{X})$; in fact, $B(\mathcal{A}, \mathcal{X}) = LI_\sigma(\mathcal{A}, \mathcal{X})$. Thus there exists $D \in Z^1_\sigma(\mathcal{A}, \mathcal{F})$ such that $\phi(D) = T$. Since

$$||D|| = \sup_{||a||\leq 1} ||D(a)||$$

and

$$||D(a)|| \sup_{||(\beta,b)||\leq 1} ||D(a)(\beta, b)||$$

and

$$D(a)(\beta, b) = \beta S(a) - \sigma(a)S(b) + S(ab);$$

then $D \in \mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{F})$. Thus ϕ is bounded, one-to-one and onto and since $\mathfrak{Z}^1_\sigma(\mathcal{A}, \mathcal{F})$ is a Banach space, from the open mapping theorem, ϕ is homeomorphism. ■

Corollary 2.18. *Let $\mathcal{A}$, $\mathcal{X}$, σ be as in Definition 2.1. Let all of module valued σ-derivation from $\mathcal{A}$ to $B(\mathcal{A}_+, \mathcal{X})$ be continuous. Then every left σ-intertwining mapping is continuous.*

Remark 2.19. Similar to left σ-intertwinings, the concept of the right σ-intertwinings is presentable. Moreove that, all of theorems and corollary 2.18 until 2.17 hold for right σ-intertwinings too; the only pointable fact is that, for this case, we define the function $D : \mathcal{A} \to B(\mathcal{A}_+, \mathcal{X})$ by

$$D(a)(\beta, b) = \beta S(a) + S(ba) - S(b)\sigma(a)$$

and consider $B(\mathcal{A}_+, \mathcal{X})$ as a $\sigma(\mathcal{A})$-bimodule by

$$(\sigma(a)f)(x) = f(xa), \quad (f\sigma(a))(x) = f(x)\sigma(a), \quad \forall a \in \mathcal{A}, \quad \forall f \in B(\mathcal{A}_+, \mathcal{X}), \quad \forall x \in \mathcal{X}.$$

Remark 2.20. The facts such as left intertwinings and right intertwinings stated by now, can be extended as the following.

Definition 2.21. The mapping $S \in L^n(\mathcal{A}, \mathcal{X})$ is said to be *σ-intertwining in the last variable* if for each $a_1, a_2, a_3, ...a_n \in \mathcal{A}$, the linear mapping

$$\varphi : \mathcal{A} \to \mathcal{X}$$
$$\varphi(a) = (\delta^n(S)(a_1, a_2, a_3, ...a_n, a), \quad \forall a \in \mathcal{A}$$

is continuous.

Theorem 2.22. *Let $S : \mathcal{A}^n \to \mathcal{X}$ be a σ-intertwining mapping in the last variable. There exists a continuous left $\sigma(\mathcal{A})$-module homomorphism $U : B(\mathcal{A}_+, \mathcal{X}) \to \mathcal{X}$ and a n-σ-cocycle $D : \mathcal{A}^n \to B(\mathcal{A}_+, \mathcal{X})$ such that $U \circ D = S$.*

Proof. It need be noted that we consider $B(\mathcal{A}_+, \mathcal{X})$ as a $\sigma(\mathcal{A})$-bimodule by

$$(\sigma(a)f)(x) = \sigma(a)f(x), \quad (f\sigma(a))(x) = f(ax), \quad \forall a \in \mathcal{A}, \quad \forall f \in B(\mathcal{A}_+, \mathcal{X}), \quad \forall x \in \mathcal{X}.$$

Suppose $U : B(\mathcal{A}_+, \mathcal{X}) \to \mathcal{X}$ is the same one that mentioned earlier. We consider the mapping

$$D : \mathcal{A}^n \to B(\mathcal{A}_+, \mathcal{X})$$

defined by

$$\begin{aligned} D(a_1, a_2, a_3, ...a_n)(\beta, b) &= \beta S(a_1, a_2, a_3, ...a_n) \\ &+ S(a_1, a_2, a_3, ...a_n)\sigma(b) \\ &+ (-1)^n \Delta^n(S) S(a_1, a_2, a_3, ...a_n, b). \end{aligned}$$

Since S is σ-intertwining in the last variable, one can conclude that $D(a) \in B(\mathcal{A}_+, \mathcal{X})$.

Clearly $U \circ D = S$; because

$$\begin{aligned} (U \circ D)(a_1, a_2, a_3, ...a_n) &= (D(a_1, a_2, a_3, ...a_n))(1) \\ &= (D(a_1, a_2, a_3, ...a_n))(1, 0) \\ &= 1S(a_1, a_2, a_3, ...a_n) \\ &+ S(a_1, a_2, a_3, ...a_n)0 \\ &+ (-1)^n \Delta^n(a_1, a_2, a_3, ...a_n, 0) \\ &= S(a_1, a_2, a_3, ...a_n) \end{aligned}$$

Now it should be showed that $\Delta^n(D) = 0$. It turn out that

$$\begin{aligned} (\Delta^n(D))(a_1, a_2, a_3, ...a_n, a) &= \sigma(a_1) D(a_2, a_3, ...a_n, a) \\ &- D(a_1 a_2, a_3, ...a_n, a) \\ &+ D(a_1, a_2 a_3, ...a_n, a) \\ &- ... \\ &+ (-1)^n D(a_1, a_2, a_3, ..., a_n a) \\ &+ (-1)^{n+1} D(a_1, a_2, a_3, ..., a_n)\sigma(a); \end{aligned}$$

on the one hand

$$\begin{aligned} \sigma(a_1) D(a_2, a_3, ...a_n, a)(\beta, b) &= \sigma(a_1)\beta S(a_2, a_3, ..., a_n, a) \\ &+ \sigma(a_1) S(a_2, a_3, ..., a_n, a)\sigma(b) \\ &+ (-1)^n \sigma(a_1)\Delta^n(S)(a_2, a_3, ...a_n, a, b) \end{aligned}$$

and

$$\begin{aligned}
-D(a_1a_2, a_3, ...a_n, a)(\beta, b) &= -\beta S(a_1a_2, a_3, ...a_n, a) \\
&- S(a_1a_2, a_3, ..., a_n, a)\sigma(b) \\
&+ (-1)^{n+1}\Delta^n(S)(a_1a_2, a_3, ...a_n, a, b)
\end{aligned}$$

and so on, until that

$$\begin{aligned}
(-1)^n D(a_1, a_2, a_3, ..., a_n a)(\beta, b) &= (-1)^n \beta S(a_1, a_2, a_3, ..., a_n a) \\
&+ (-1)^n S(a_1, a_2, a_3, ..., a_n a)\sigma(b) \\
&+ (-1)^n(-1)^n\Delta^n(S)(a_1a_2, a_3, ...a_n a, b)
\end{aligned}$$

and

$$\begin{aligned}
& (-1)^{n+1}(D(a_1, a_2, a_3, ..., a_n)\sigma(a))(\beta, b) \\
= & (-1)^{n+1}(D(a_1, a_2, a_3, ..., a_n)\sigma(a))(0, \beta a + ab) \\
= & (-1)^{n+1}\beta S(a_1, a_2, a_3, ..., a_n)\sigma(a) \\
+ & (-1)^{n+1}\beta S(a_1, a_2, a_3, ..., a_n)\sigma(a)\sigma(b) \\
+ & (-1)^{2n+1}\Delta^n(S)(a_1, a_2, a_3, ...a_n, ab) \\
+ & (-1)^{2n+1}\beta\Delta^n(S)(a_1, a_2, a_3, ...a_n, a).
\end{aligned}$$

Adding the first parts of the relations stated above, we get

$$\beta\Delta^n(S)(a_1, a_2, a_3, ...a_n, a) \tag{2.2}$$

where (2.2) is deleted with the last part of the last relation. Adding the second parts, we get

$$\Delta^n(S)(a_1, a_2, a_3, ...a_n, a)\sigma(b) \tag{2.3}$$

Adding the third parts and (2.3), we get

$$\Delta^{n+1}(\Delta^n(a_1, a_2, a_3, ...a_n, b)) \tag{2.4}$$

whose produce is equal to 0; in fact $\Delta^n(D) = 0$. ■

Theorem 2.23. *Let $D \in Z^n_\sigma(\mathcal{A}, \mathcal{F})$. There exists a σ-intertwining maping in the last variable $S \in L^n(\mathcal{A}, \mathcal{X})$ such that $S = U \circ D$.*

Proof. Define $S = U \circ D$. We should show that for each $a_1, a_2, a_3, ...a_n \in \mathcal{A}$, the linear mapping

$$\begin{aligned}
&\varphi : \mathcal{A} \to \mathcal{X} \\
&\varphi(a) = (\delta^n(S)(a_1, a_2, a_3, ...a_n, a), \quad \forall a \in \mathcal{A}
\end{aligned}$$

is continuous. Since

$$\begin{aligned}
(\delta^n(S)(a_1, a_2, a_3, ...a_n, a) &= \sigma(a_1)S(a_2, a_3, ..., a_n, a) \\
&- S(a_1a_2, a_3, ..., a_n, a) \\
&+ S(a_1, a_2a_3, ..., a_n, a) \\
&+ ...(-1)^n S(a_1, a_2, a_3, ..., a_na) \\
&+ (-1)^{n+1} S(a_1, a_2, a_3, ..., a_n)\sigma(a) \\
&= \sigma(a_1)\Big(D(a_2, a_3, ..., a_n, a)(1)\Big) \\
&- \Big(D(a_1a_2, a_3, ..., a_n, a)\Big)(1) \\
&+ \Big(D(a_1, a_2a_3, ..., a_n, a)\Big)(1) \\
&+ ...(-1)^n \Big(D(a_1, a_2, a_3, ..., a_na\Big)(1) \\
&+ (-1)^{n+1}\Big(D(a_1, a_2, a_3, ..., a_n)\Big)(1)\sigma(a) \\
&= \sigma(a_1)\Big(D(a_2, a_3, ..., a_n, a)(1)\Big) \\
&- \sigma(a_1)\Big(D(a_2, a_3, ..., a_n, a)(1)\Big) \\
&+ (-1)^n\Big(D(a_1, a_2, a_3, ..., a_n)\sigma(a)\Big)(1) \\
&+ (-1)^{n+1}\Big(D(a_1, a_2, a_3, ..., a_n)\Big)(1)\sigma(a) \\
&= (-1)^n\Big(D(a_1, a_2, a_3, ..., a_n)(0, a) \\
&+ (-1)^{n+1}\Big(D(a_1, a_2, a_3, ..., a_n)\Big)(1)\sigma(a),
\end{aligned}$$

clearly φ is continuous. Then S is σ-intertwining in the last variable. ■

Corollary 2.24. *By considering $F = B(\mathcal{A}_+, \mathcal{X})$, the function*

$$\phi : Z^n_\sigma(\mathcal{A}, \mathcal{F}) \to LI^n_\sigma(\mathcal{A}, \mathcal{X})$$
$$\phi(D) = U \circ D, \quad \forall\ D \in Z^n_\sigma(\mathcal{A}, \mathcal{F})$$

is an onto, continuous and linear mapping.

Theorem 2.25. *ϕ is one-to-one.*

Proof. We claim that D is unique for each σ-intertwining in the last variable S. It turn out that

$$\begin{aligned}
\beta D(a_1, a_2, a_3, ...a_n)(1) &+ \Big(D(a_1, a_2, a_3, ...a_n)(1)\Big)\sigma(a) \\
&+ (-1)^n\Big((-1)^n D(a_1, a_2, a_3, ..., a_n)(0, a) \\
&+ (-1)^{n+1}[D(a_1, a_2, a_3, ..., a_n)](1)\sigma(a)\Big) \\
&= D(a_1, a_2, a_3, ...a_n)(\beta, 0) \\
&+ D(a_1, a_2, a_3, ...a_n)(0, a) \\
&= D(a_1, a_2, a_3, ...a_n)(\beta, a)
\end{aligned}$$

Thus if $\phi(D_1) = \phi(D_2)$, then $D_1(a)(1) = D_2(a)(1)$ for each $a \in \mathcal{A}$ and hence $D_1 = D_2$. ∎

Theorem 2.26. *The mapping*

$$\phi : \mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{F}) \to B^n_\sigma(\mathcal{A}, \mathcal{X})$$
$$\phi(D) = U \circ D, \quad \forall\ D \in \mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{F})$$

is a linear homeomorphism.

Proof. If $D \in \mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{F})$, then $\phi(D) = U \circ D$ is continuous. Now if $T \in B^n(\mathcal{A}, \mathcal{X})$, then

$$\begin{aligned}(\delta^n(T)(a_1, a_2, a_3, ...a_n, a) &= \sigma(a_1)T(a_2, a_3, ..., a_n, a)\\ &- T(a_1a_2, a_3, ..., a_n, a)\\ &+ T(a_1, a_2a_3, ..., a_n, a)\\ &+ ...(-1)^nT(a_1, a_2, a_3, ..., a_na)\\ &+ (-1)^{n+1}T(a_1, a_2, a_3, ..., a_n)\sigma(a)\end{aligned}$$

is continuous. So $T \in LI^n_\sigma(\mathcal{A}, \mathcal{X})$. Then there exists $D \in Z^n_\sigma(\mathcal{A}, \mathcal{F})$ such that $\phi(D) = T$. Since T is continuous, then so is D. Then $D \in \mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{F})$. Since $\mathfrak{Z}^n_\sigma(\mathcal{A}, \mathcal{F})$ is Banach space, in view of the open mapping theorem, ϕ is homeomorphism. ∎

Definition 2.27. The mapping $S \in L^n(\mathcal{A}, \mathcal{X})$ is said to be *σ-intertwining in the first variable* if for each $a_1, a_2, a_3, ...a_n \in \mathcal{A}$, the linear mapping

$$\varphi : \mathcal{A} \to \mathcal{X}$$
$$\varphi(a) = (\delta^n(S)(a, a_1, a_2, a_3, ...a_n), \quad \forall a \in \mathcal{A}$$

is continuous.

Remark 2.28. Similar to σ-intertwining in the last variables, one can define a analogous statement called σ-intertwining in the first variable. Moreover, all of theorems and corollary 2.22 until 2.26 hold too; the only pointable fact is that, for this case, we define the function

$$D : \mathcal{A}^n \to B(\mathcal{A}_+, \mathcal{X})$$

by

$$\begin{aligned}D(a_1, a_2, a_3, ...a_n)(\beta, b) &= \beta S(a_1, a_2, a_3, ...a_n)\\ &+ \sigma(b)S(a_1, a_2, a_3, ...a_n)\\ &+ (-1)^n\Delta^n(S)(b, a_1, a_2, a_3, ...a_n).\end{aligned}$$

and consider $B(\mathcal{A}_+, \mathcal{X})$ as a $\sigma(\mathcal{A})$-bimodule by

$$(\sigma(a)f)(x) = f(xa), \quad (f\sigma(a))(x) = f(x)\sigma(a), \quad \forall a \in \mathcal{A}, \quad \forall f \in B(\mathcal{A}_+, \mathcal{X}), \quad \forall x \in \mathcal{X}.$$

References

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