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Some Results of Lambert Operators on L^p Spaces

Seyed Kamel Hosseini and Jahangir Cheshmavar*

Department of Mathematics, Payame Noor University, P.O.Box 19395-3697 Tehran, Iran
e-mail : kamelhosseini@chmail.ir (S.K. Hosseini); j_cheshmavar@pnu.ac.ir (J. Cheshmavar)

Abstract In this paper we provide a necessary and sufficient conditions for the Lambert operators to be invertible. Also, some properties of these type of operators will be investigated.

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1. Introduction

Operator in function spaces defined by conditional expectations have been studied since the work of Chen and Moy [1] and Sidak [2], see for example Brunk [3], in the setting of L^p spaces. This class of operators was further studied by Lambert [4, 5] to study hyponormal composition operators, the concept of L^p multipliers and Herron [6], to present the basic properties of the class of bounded weighted conditional expectation operators defined on the L^p spaces. Lambert showed that the relationship between a chain of sigma algebras and the set of multiplication operators which are the contractive idempotent, the so-called conditional expectation operators. Further work in this direction can be found in [7, 8]. Later, Jabbarzadeh and Sarbaz in [9] have characterized the Lambert multipliers acting between two L^p-spaces, by using some properties of conditional expectation operator. In this paper, we provide a necessary and sufficient conditions for the Lambert operators to be invertible. Also we show that they can be characterized in terms of the conditional expectation induced by an associated σ-finite subalgebra. Some new properties of these type of operators will be investigated. Our exposition regarding Lambert multipliers follows [5, 6, 10].

Let (X, Σ, μ) be a complete σ-finite measure space, $\mathcal{A} \subseteq \Sigma$ be complete σ-finite subalgebra and $1 \leq p \leq \infty$. We view $L^p(\mathcal{A}) = L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$ as a Banach subspace of $L^p(\Sigma)$. Denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$.

For each nonnegative $f \in L^0(\Sigma)$ or $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique measurable function $E(f)$ with the following conditions:

*Corresponding author.

(i) $E(f)$ is $\mathcal{A}$-measurable;
(ii) If A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ converges, we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$

For every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $f \longmapsto E(f)$, from $L^p(\Sigma)$ to $L^p(\mathcal{A}), 1 \leq p \leq \infty$, is called the *conditional expectation operator with respect to* $\mathcal{A}$. If $E(f)$ exists for a function $f \in L^0(\Sigma)$, then we say that f is *conditionable*. We will need the following standard facts concerning $E(f)$, for more details of the properties of E, we refer the interested reader to [6, 7, 11]:

- If g is $\mathcal{A}$-measurable then $E(fg) = E(f)g$;
- $|E(f)|^p \leq E(|f|^p)$;
- $\|E(f)\|_p \leq \|f\|_p$;
- If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$;
- $E(1) = 1$.

As an operator on $L^p(\Sigma)$, $E(\cdot)$ is the contractive idempotent and

$$E(L^p(\Sigma)) = L^p(\mathcal{A})$$

.

Let $f, g \in L^0(\Sigma)$ be conditionable. We define

$$f \star g = fE(g) + gE(f) - E(f)E(g). \tag{1.1}$$

Let $1 \leq p \leq \infty$. A conditionale function $u \in L^0(\Sigma)$ for which $u \star f \in L^p(\Sigma)$ for each $f \in L^p(\Sigma)$, is called *Lamber multiplier*. In other words, a conditionable function $u \in L^0(\Sigma)$ is Lambert multiplier if and only if the corresponding $\star$-multiplication operator $T_u : L^p(\Sigma) \to L^p(\Sigma)$ defined as $T_u f = u \star f$ is bounded. In this case T_u is called *Lambert operator*.

Define $K_p^\star, (1 \leq p \leq \infty)$, the set of all Lambert multipliers from $L^p(\Sigma)$ into $L^p(\Sigma)$, as follows:

$$K_p^\star = \{u \in L^0(\Sigma) : \text{u is conditionable}, \ u \star L^p(\Sigma) \subseteq L^p(\Sigma)\} \tag{1.2}$$

As that shown in [9] that if we define

$$\|u\|_{K_p^\star} = \|E(|u|^p)\|_\infty^{1/p}, \tag{1.3}$$

for every $u \in K_p^\star$, then $K_p^\star$ is a Banach space with the norm $\|\cdot\|_{K_p^\star}$ and

$$\|u\|_{K_p^\star} \leq \|T_u\| \leq 3\|u\|_{K_p^\star}. \tag{1.4}$$

In the following section, we provide a necessary and sufficient condition for the Lambert operators to be invertible, where $\mathcal{A}$ is a non-atomic measure space. We show that they can be characterized in terms of the conditional expectation induced by an associated σ-finite subalgebra $\mathcal{A}$. Some new properties of these type of operators will be investigated. We use some ideas of [12].

2. The Results

Recall that the Lambert operator T_u is said to be a Fredholm operator if $\mathcal{R}(T_u)$ is closed, $dim\mathcal{N}(T_u) < \infty$ and $codim\mathcal{R}(T_u) < \infty$, where $\mathcal{R}(T_u)$ and $\mathcal{N}(T_u)$ denotes the kernel and the range of T_u, respectively. Also recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atoms is called non-atomic. It is a well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu|_{\mathcal{A}})$ can be uniquely decomposed as

$$X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B,$$

where $\{A_n\}_{n\in\mathbb{N}}$ is a collection of disjoint $\mathcal{A}$-atoms and B is non-atomic part of X see e.g. [8, 13].

The necessary and sufficient condition for the operator T_u to be Fredholm operator is the following:

Theorem 2.1. [9] *Suppose that $u \in K_p^\star$ and $\mathcal{A}$ is a non-atomic measure space. Then the operator T_u is Fredholm on $L^p(\Sigma), (1 \le p < \infty)$ if and only if $|E(u)| \ge \delta$ almost everywhere on X for some $\delta > 0$.*

Let now $\mathcal{R}_p = \{T_u : u \in K_p^\star\}$. The following elementary lemmas, we show that the set of all Lambert operators $T_u, u \in K_p^\star$ is a commutative Banach algebra and also, we characterize the operator T_u in terms of the conditional expectation induced by $\mathcal{A}$:

Lemma 2.2. *For every $u, v \in K_p^\star$ and $\lambda \in \mathbb{C}$, the following statements hold:*

(i) $T_{u+v} = T_u + T_v$;
(ii) $T_{\lambda u} = \lambda T_u$;
(iii) $T_u T_v = T_{u*v}$;
(iv) *$\mathcal{R}_p$ is commutative and has an identity;*
(v) *$\mathcal{R}_p$ is a closed subalgebra of all bounded linear operators on $L^p(\Sigma)$.*

Proof. The proof of (i) and (ii) are obvious. For (iii),

$$\begin{aligned} T_u T_v f &= T_u(vE(f) + fE(v) - E(v)E(f)) \\ &= (u \star v)E(f) + fE(u \star v) - E(u \star v)E(f) \\ &= T_{u\star v} f. \end{aligned}$$

Since $u \star v \in K_p^\star$, $\mathcal{R}_p$ is closed under composition operators.
(iv)

$$T_u T_v = T_{u\star v} = T_{v\star u} = T_v T_u.$$

Therefore, $\mathcal{R}_p$ is commutative. Let $T_u \in \mathcal{R}_p$, then

$$T_1 T_u = T_{1\star u} = T_u = T_{u\star 1} = T_u T_1,$$

that is, T_1 is an identity of $\mathcal{R}_p$.

Consequently $\mathcal{R}_p$ is a commutative subalgebra of all bounded linear operators on $L^p(\Sigma)$.
(v) Let $\{T_{u_n}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{R}_p$. By the first inequality of (1.4) and (i), $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $K_p^\star$, it follows that there is $u \in K_p^\star$ such that $\{u_n\}_{n\in\mathbb{N}}$ converges to u. Now by the second inequality of (1.4) and (i), $\{T_{u_n}\}_{n\in\mathbb{N}}$ converges to T_u. ∎

Corollary 2.3. *$\mathcal{R}_p$ is a commutative Banach algebra with an identity.*

Lemma 2.4. *Let $u \in K_p^\star$. Then the following holds:*

(i) If T_u is invertible operator, then there is $w \in K_p^\star$ such that $(T_u)^{-1} = T_w$ and $E(w) = \frac{1}{E(u)}$;

(ii) If $u \in L^\infty(\mathcal{A})$ and T_u is invertible, then $(T_u)^{-1} = T_{\frac{1}{u}}$;

(iii) If $|E(u)| \geq \delta$ almost everywhere on X for some $\delta > 0$, then T_u is injective.

Proof. (i) Let T_u be invertible, there is $w \in K_p^\star$ such that

$$T_{u\star w} = T_u T_w = T_1,$$

that is, $u \star w = 1$, $(T_u)^{-1} = T_w$ and

$$1 = E(1) = E(u \star w) = E(uE(w)) + wE(u) - E(u)E(w) = E(u)E(w).$$

(ii) Suppose that $u \in L^\infty(\mathcal{A})$ and T_u is invertible, then $|u| \geq \delta$ for some $\delta > 0$. Therefore, we have $\frac{1}{u} \in K_p^\star$. Let $f \in L^p(\Sigma)$. Then, we have

$$\begin{aligned} T_u T_{\frac{1}{u}} f &= T_u(\frac{1}{u}Ef + fE\frac{1}{u} - E\frac{1}{u}Ef) \\ &= T_u(f\frac{1}{u}) \\ &= uE(f\frac{1}{u}) + f\frac{1}{u}Eu - EuE(f\frac{1}{u}) = f, \end{aligned}$$

similarly, $T_{\frac{1}{u}} T_u f = f$. Therefore, $(T_u)^{-1} = T_{\frac{1}{u}}$.

(iii) For $f \in L^p(\Sigma)$, let $T_u f = 0$. Then,

$$\begin{aligned} 0 = E(T_u f) &= E(uEf + fEu - EfEu) \\ &= EuEf + EuEf - EuEf \\ &= EuEf. \end{aligned}$$

Now by the hypothesis, $Ef = 0$. Thus,

$$0 = T_u f = uEf + fEu - EfEu = fEu.$$

Therefore, $f = 0$. ∎

Let us consider the case that $\mathcal{A}$ is a non-atomic measure space. We prove an analogue of Proposition (3) in [12] in the $K_p^\star$-setting.

Theorem 2.5. *Suppose that $\mathcal{A}$ is a non-atomic measure space. Let T_u be a Lambert operator on $L^p(\Sigma), (1 \leq p < \infty)$, where $u \in K_p^\star$. Then the following are equivalent:*

(i) T_u is an invertible operator;

(ii) T_u is an Fredholm operator;

(iii) $\mathcal{R}(T_u)$ is closed and $codim\mathcal{R}(T_u) < \infty$;

(iv) $|E(u)| \geq \delta$ almost everywhere on X for some $\delta > 0$.

Proof. We here show that (iii) implies (iv), because the implications $(iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

Suppose that $\mathcal{R}(T_u)$ is closed and $codim\mathcal{R}(T_u) < \infty$. We claim that T_u is onto. If it were false, pick $g \in L^p(\Sigma)\backslash\mathcal{R}(T_u)$. Since $\mathcal{R}(T_u)$ is closed, we can find a function $g^* \in L^q(\Sigma)$, the dual space of $L^p(\Sigma)$, such that

$$\int_X \overline{g} g^* d\mu = 1 \quad \text{and} \quad \int_X \overline{T_u f} g^* d\mu = 0,$$

for all $f \in L^p(\Sigma)$. From the first equality, $\int_X E(\overline{g}g^*)d\mu = 1$. Hence the set $E_\delta = \{x \in X : |E(\overline{g}g^*)(x)| > \delta\}$ must have positive measure for some $\delta > 0$. As $\mathcal{A}$ is non-atomic, we can chose a sequence $\{E_n\}$ of subsets of E_δ with $0 < \mu(E_n) < \mu(E_\delta)$ and $E_m \bigcap E_n = \emptyset$ for some $m \neq n$. Let $g_n^* = \chi_{E_n} g^*$, where the symbol χ_E is the characteristic function of $E \in \Sigma$. Then, $0 \neq g_n^* \in L^q(\Sigma)$, because

$$\int_X |\overline{f_0} g_n| d\mu \geq \int_{E_n} |\overline{g} g_n^*| d\mu = \int_{E_n} E(|\overline{g} g_n^*|) d\mu \geq \int_{E_n} |E(\overline{g} g_n^*)| d\mu \geq \delta\mu(E_n) > 0$$

for each n. Now for any $f \in L^p(\Sigma)$, $\chi_{E_n} f \in L^p(\Sigma)$ and so

$$(T_u^* g_n^*, f) = (g_n^*, T_u f) = \int_{E_n} \overline{T_u f} g^* d\mu = \int_X \overline{T_u(\chi_{E_n} f)} g^* d\mu = (g^*, T_u(\chi_{E_n} f)),$$

which implies that $T_u^* g_n^* = 0$ and so $g_n^* \in \mathcal{N}(T_u^*)$. Thus sequence $\{g_n^*\}$ forms a linearly independent subset of $\mathcal{N}(T_u^*)$. This contradicts the fact that $dim\mathcal{N}(T_u^*) = codim\mathcal{R}(T_u) < \infty$. Hence T_u is onto and the result follows from the lemma (2.4).

■

Recall that the spectrum of an operator T is the set

$$sp(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

and recall that the essential range of a function $f : X \longrightarrow \mathbb{C}$ is the set of all $\lambda \in \mathbb{C}$ such that $f^{-1}(\mathcal{O})$ has positive measure for every open neighborhood $\mathcal{O}$ of λ, that is

$$\text{ess range}(f) = \{\lambda \in \mathbb{C} : \mu\{x \in X : |f(x) - \lambda| < \varepsilon\} \neq 0, \forall \varepsilon > 0\}.$$

Proposition 2.6. *Let $u \in K_p^\star$ and let $\mathcal{A}$ be a non-atomic measure space. Then $sp(T_u) =$ ess range(Eu).*

Proof. Suppose that $\lambda \notin$ ess range(Eu), then there exists $\varepsilon_0 > 0$ such that $|Eu - \lambda| \geq \varepsilon_0$ almost everywhere, but $Eu - \lambda = E(u - \lambda)$ and $u - \lambda \in K_p^\star$, then by theorem (2.5) $T_u - \lambda I = T_{u-\lambda}$ is invertible, that is, $\lambda \notin sp(T_u)$.
Conversely, let $\lambda \notin sp(T_u)$. Then, $T_u - \lambda I = T_{u-\lambda}$ is invertible, that is, there exists $\varepsilon_0 > 0$ such that $|E(u - \lambda)| = |Eu - \lambda| \geq \varepsilon_0$ almost everywhere. Then, $\lambda \notin$ ess range(Eu). ■

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called unitary if $TT^* = T^*T = I$, T is called projection if $T = T^*$ and $T^2 = T$, T is called positive if $\langle Tx, x\rangle \geq 0$ holds for every $x \in \mathcal{H}$ and T is a partial isometry if $\|Th\| = \|h\|$ for h orthogonal to the kernel of T. It is known that an operator T on a Hilbert space is partial isometry if and only if $TT^*T = T$. In the following proposition we consider particular case $p = 2$ to characterizing projection, unitary and positive operators on $L^2(\Sigma)$. First, we have the following Proposition stated in [9]:

Proposition 2.7. *Let $u \in K_2^\star$. Then the following claims are true:*

(i) T_u is a normal operator if and only if $u \in L^\infty(\mathcal{A})$,
(ii) T_u is a self adjoint operator if and only if $u \in L^\infty(\mathcal{A})$ is real valued.

Proposition 2.8. *Let $u \in K_2^\star$. Then the following holds:*

(i) T_u is a unitary operator if and only if $u \in L^\infty(\mathcal{A})$ and $|u| = 1$;
(ii) T_u is a projection operator if and only if $u \in L^\infty(\mathcal{A})$ is real valued and $u^2 = u$;
(iii) T_u is a positive operator if and only if $u \in L^\infty(\mathcal{A})$ and $u \geq 0$;

(iv) If T_u is partial isometry, then $E(|u|^2) + |Eu|^2 = 1$ on $\sigma(E(u))$.

Proof. (i) Assume T_u is unitary. Then its normal operator and $u \in L^\infty(\mathcal{A})$. Moreover, for each $f \in L^2(\Sigma)$, apply Prop. (Prop. 3.1 in [9]) with $n = 1$, we have $T_u^* f = E(\bar{u}f) + \bar{E}u(f - Ef)$, and then

$$E(T_u^* f) = E(E(\bar{u}f) + \bar{E}u(f - Ef)) = E(E(\bar{u}f)) + \bar{E}u(Ef - Ef) = E(\bar{u}f)$$

$$\begin{aligned} T_u(T_u^* f) &= uE(T_u^* f) + T_u^* f Eu - EuE(T_u^* f) \\ &= uE(\bar{u}f) + (E(\bar{u}f) + \bar{E}u(f - Ef))Eu - Eu\bar{E}u \\ &= u\bar{u}Ef + |u|^2 f - u\bar{u}Ef = |u|^2 f. \end{aligned}$$

That is, $|u|^2 f = f$ and then $|u| = 1$.

Conversely, suppose that $u \in L^\infty(\mathcal{A})$ and $|u| = 1$. Therefore, T_u is normal. We take $f \in L^2(\Sigma)$. Then $f = |u|^2 f = T_u T_u^* f$, and hence T_u is unitary.

(ii) Assume T_u is projection. Then its self-adjoint operator and $u \in L^\infty(\mathcal{A})$ is real valued. Moreover, for each $f \in L^2(\Sigma)$;

$$\begin{aligned} u^2 f &= u^2 Ef + u^2 f - u^2 Ef \\ &= uE(uf) + ufEu - EuE(uf) \\ &= T_u(T_u f). \end{aligned}$$

On the other hand,

$$\begin{aligned} T_u^2 f &= T_u f \\ &= uEf + fEu - EuEf \\ &= uEf + uf - uEf = uf. \end{aligned}$$

Therefore, $u^2 = u$.

Conversely, suppose that $u \in L^\infty(\mathcal{A})$ is real valued and $u^2 = u$. Then T_u is self-adjoint. We take $f \in L^2(\Sigma)$. Then
$T_u^2 f = u^2 f = uf = T_u f$, and hence T_u is projection.

(iii) Assume T_u is positive. Then $u \in L^\infty(\mathcal{A})$ is real valued.
Let $F = \{x \in X : u(x) < 0\}$. We take $f \in L^2(\Sigma)$. Then $\int_F u|f|^2 d\mu = 0$. Therefore $u \geq 0$.

Conversely, suppose that $u \in L^\infty(\mathcal{A})$ and $u \geq 0$.
Hence, for each $f \in L^2(\Sigma)$ we have

$$\langle T_u f, f \rangle = \langle uf, f \rangle = \int_X uf\overline{f} d\mu = \int_X u|f|^2 d\mu \geq 0.$$

Therefore, T_u is positive.
(iv) Let T_u is partial isometry. Then $T_u T_u^* T_u = T_u$, that is,

$$E(u)E(f) = E(u)E(|u|^2)E(f) + |E(u)|^2 E(f)E(u),$$

and hence $E(u)(E(|u|^2) + |E(u)|^2 - 1)E(f) = 0$ for all $f \in L^2(\Sigma)$. We get that $E(|u|^2) + |E(u)|^2 = 1$ on $\sigma(E(u))$. ∎

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References

[1] S.T. Chen, Moy, Characterizations of conditional expectation as a transformation on function spaces, Pacific J. Math. 4 (1954) 47–63.

[2] S. Sidak, On relations between strick sense and wide sense conditional expectation, Theory of Prob. Appl. 2 (1957) 267–271.

[3] H.D. Brunk, On an extension of the concept conditional expectation, Proc. Amer. Math. Soc. 14 (1963) 298–304.

[4] A. Lambert, Hyponormal composition operators, Bull. London Math. Soc. 18 (4) (1986) 395–400.

[5] A. Lambert, L^p Multipliers and nestedsigma-algebras, Oper. Theory Adv. Appl. 104 (1998) 147–153.

[6] J.D. Herron, Weighted Conditional Expectation Operators of L^p Space, ProQuest LLC, Ann Arbor, MI, 2004.

[7] A. Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Roy. Soc. Edinburgh Ser. A 118 (1991) 111–118.

[8] H. Takagi, K. Yokouchi, Multiplication and composition operators between two L^p-spaces, Contemp. Math. 232 (1999) 321–338.

[9] M.R. Jabbarzadeh, S. Khalil Sarbaz, Lambert multipliers between L^p-spaces, Czech. Math. J. 60 (135) (2010) 31–43.

[10] J. Cheshmavar, S.K. Hosseini S. Khalil Sarbaz, A note on Lambert Multipliers between L^p-spaces, In Review.

[11] M.M. Rao, Conditional Measure and Applications, Marcel Dekker, New York, 1993.

[12] H. Takagi, Fredholm weighted composition operators, Integr. Equat. Oper. Th. 16 (1993) 267–276.

[13] A.C. Zaanen, Integration, 2nd ed. North-Holland, Amsterdam, 1967.