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On the Spectrum of Weakly Prime Submodule

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Abstract A proper submodule P of an R-module M is called a weakly prime submodule, if for each submodule K of M and elements a, b of R, $abK \subseteq P$ implies that $aK \subseteq P$ or $bK \subseteq P$. Let $WSpec(M)$ be the set of all weakly prime submodules of M. In this paper, a topology on $WSpec(M)$ is introduced. We investigate some basic properties of the open and closed sets in that topology and establish their relationships with weakly prime radical and Flat Module. We also investigate some topological properties in $WSpec(M)$ such as connectedness, separation axioms etc. Finally we try to characterize the spectrum of weakly prime submodule with the help of quasi multiplication module. we prove that if M is a finitely generated quasi multiplication R-module then $WSpec(M)$ is compact.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a ring and M a unitary R-module. Let P be a proper submodule of M. It is said that P is a prime submodule of M, if the condition $ra \in P$, $r \in R$ and $a \in M$ implies that $a \in P$ or $rM \subseteq P$. In this case, if $L = (P : M) = \{r \in R | rM \subseteq P\}$, we say that P is a L-prime submodule of M. It is easy to see that L is a prime ideal of R.

A proper submodule P of an R-module M is called a weakly prime submodule, if for each submodule K of M and elements a, b of R, $abK \subseteq P$ implies that $aK \subseteq P$ or $bK \subseteq P$.

Weakly prime submodules have been introduced by Behboodi and Koohi [1] and these have been studied by several authors [1–5]. If we consider R as an R-module, then prime submodules and weakly prime submodules are exactly prime ideals of R. More generally for every multiplication module any submodule is a prime submodule if and only if it is a weakly prime submodule. For every R-module, it is easy to see that any prime submodule is a weakly prime submodule, but the converse is not always correct. For example let R

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be a ring with $dimR \neq 0$, and $P \subset Q$ a chain of prime ideals of R. Then it is easy to see that for the free R-module $R \oplus R$, the submodule $P \oplus Q$ is a weakly prime submodule which is not a prime submodule [3]. Let $WSpec(M)$ to be the set of all weakly prime submodules of M. We call $WSpec(M)$,the weakly prime spectrum of M.

Let N be a proper submodule of M. The intersection of all weakly prime submodules of M containing N is called weakly prime radical of N and is dented by $wrad(N)$. If there does not exist any weakly prime submodule of M containing N, then we say $wrad(N) = M$. Weakly prime radical has been studied by Azizi [2].

An R-module M is called quasi multiplication module if for every weakly prime submodule P of M, we have $P = IM$, where I is an ideal of R (see [5]). One can easily show that if M is a quasi multiplication module, then $P = (P : M)M$ for every weakly prime submodule P of M.

In Section 2, a topology on $WSpec(M)$ is introduced. We investigate some basic properties of the open and closed sets in that topology and establish their relationships with weakly prime radical and Flat Module. We show that $WSpec(M)$ is T_0 and T_1 space but not a Hausdorff or T_2 space. Also, it has been proved that $WSpec(M)$ is a disconnected space if and only if M is a direct sum of any two non-zero submodules of M.

In Section 3, we try to characterize $WSpec(M)$ with the help of quasi multiplication module. It has been proved that if M is a finitely generated quasi multiplication R-module then $WSpec(M)$ is compact.

2. Spectrum of Weakly Prime Submodule

Let M be an R-module,we consider $WSpec(M)$ to be the set of all weakly prime submodules of M. We call $WSpec(M)$, the weakly prime spectrum of M. In this section, a topology on $WSpec(M)$ is introduced.This topology is defined exactly similar to the zariski topology on the spectrum of prime submodules,and the set of prime submodules $Spec(M)$ of M is a toplogical subspace of $WSpec(M)$.

For each submodule N of M, we define the variety of N, denoted by $V(N)$, as

$$V(N) = \{P \in WSpec(M) | N \subseteq P\}.$$

Also for each submodule N of M, let $W(N) = \{P \in WSpec(M) | N \nsubseteq P\}$.

Clearly, for any two submodules N and L of M such that $N \subseteq L$,

$$V(L) \subseteq V(N) \text{ and } W(N) \subseteq W(L).$$

Now, we define the family $\zeta(M) = \{W(N) | N \text{ is a submodule of } M\}$. We begin with the following Proposition.

Proposition 2.1. *Let M be an R-module. Then the following statements hold:*
(i) $\zeta(M)$ is a topology on $WSpec(M)$.
(ii) $WSpec(M)$ is a T_0 topological space.

Proof. (i) Note that $W(0) = \emptyset$ and $W(M) = Wspec(M)$.
Let $\{N_\alpha\}_{\alpha \in I}$ and $\{N_i\}_{i=1}^n$ be two families of submodules of M. We show that

$$\text{(a) } \bigcup_{\alpha \in I} W(N_\alpha) = W(\textstyle\sum_{\alpha \in I} N_\alpha).$$

$$\text{(b) } \bigcap_{i=1}^n W(N_i) = W(\textstyle\bigcap_{i=1}^n N_i).$$

(a) $P \in \bigcup_{\alpha\in I} W(N_\alpha) \iff N_{\alpha_0} \nsubseteq P$ for some $\alpha_0 \in I \iff \sum_{\alpha\in I} N_\alpha \nsubseteq P \iff P \in W(\sum_{\alpha\in I} N_\alpha)$.
(b) $P \in \bigcap_{i=1}^{n} W(N_i) \iff P \in W(N_i)$ for each $i, 1 \leq i \leq n \iff N_i \nsubseteq P$ for each $i, 1 \leq i \leq n \iff \bigcap_{i=1}^{n} N_i \nsubseteq P \iff P \in W(\bigcap_{i=1}^{n} N_i)$.
(a) and (b) show that $\zeta(M)$ is closed under arbitrary union and finite intersection. Hence, $\zeta(M)$ is a topology on $WSpec(M)$.
(ii) Let P_1 and P_2 be two distinct points of $WSpec(M)$. If $P_1 \nsubseteq P_2$,then obviously $P_2 \in W(P_1)$ and $P_1 \notin W(P_1)$ showing that $WSpec(M)$ is a T_0 topological space. ∎

We have, in general $WSpec(M)$ is not a T_1 space which can be observed in case of $\mathbb{Z}$ as a module over itself since $WSpec(\mathbb{Z})$ coincides with $Spec(\mathbb{Z})$ and $Spec(\mathbb{Z})$ with the zariski topology is not a T_1 space.

As $WSpec(M)$ is not a T_1 space, so it is also not a Hausdorff or T_2 space in general since $WSpec(\mathbb{Z})$ coincides with $Spec(\mathbb{Z})$ and $Spec(\mathbb{Z})$ with the zariski topology is not a Hausdorff space (see [6]).

Also,we know that a totally disconnected space is a Hausdorff space (see [7, 8]). So, by the above statement it is clear that $WSpec(M)$ is not a totally disconnected space.

Proposition 2.2. *$WSpec(M)$ is a disconnected space if and only if $M = N \oplus L$ for any two non-zero submodule N and L of M.*

Proof. Assume $M = N \oplus L$. Then $WSpec(M) = W(M) = W(N + L) = W(N) \cup W(L)$.Since N and L are non-zero, it follows that the corresponding open sets $W(N)$ and $W(L)$ are non-empty. Also, $N \cap L = 0$ implies that $W(N)$ and $W(L)$ are disjoint. Thus, $WSpec(M) = W(N) \cup W(L)$ is a disconnection of $WSpec(M)$, as required.
Conversely, assume that $WSpec(M)$ is a disconnected space and $WSpec(M) = A \cup B$ is a disconnection of $WSpec(M)$. Since A and B are open, we have $A = W(N)$ and $B = W(L)$ for submodules N and L of M. Also, A and B are non-empty implies that N and L are non-zero. As A and B are disjoint,it follows that $N \cap L = 0$.
Now, $W(M) = WSpec(M) = A \cup B = W(N) \cup W(L) = W(N + L)$ implies that $M = N + L$. Hence, $M = N \oplus L$. ∎

It is clear from the above Proposition that $WSpec(M)$ is a connected space if and only if it cannot be written as a direct sum of atleast one pair of non-zero submodules of M.

We now investigate some of the basic properties of the open sets $W(N)$ and closed sets $V(N)$ in $\zeta(M)$.

Proposition 2.3. *For each submodule K of M and elements a, b of R*
(i) $V(aK \cup bK) = V(aK) \cup V(bK)$.
(ii) $W(aK \cap bK) = W(aK) \cap W(bK)$.
(iii) $V(abK) = V(aK) \cup V(bK)$.
(iv) $W(abK) = W(aK) \cap W(bK)$.

Proof. (i) it is obvious from the definition
(ii) it is clear from part (b) in the proof of proposition 2.1(i).
(iii) We have $P \in V(abK) \implies abK \subseteq P \implies aK \subseteq P$ or $bK \subseteq P \implies P \in V(aK)$ or $P \in V(bK) \implies P \in V(aK) \cup V(bK)$. Thus $V(abK) \subseteq V(aK) \cup V(bK)$.
Conversely, it is clear that $abK \subseteq aK \cup bK$ which implies $V(abK) \supseteq V(aK \cup bK) = V(aK) \cup V(bK)$ by part(i), as required.
(iv) It is obvious that $abK \subseteq aK \cap bK$ which implies $W(abK) \subseteq W(aK \cap bK) = W(aK) \cap$

$W(bK)$ by part(ii).
Conversely, $P \in W(aK) \cap W(bK) \Longrightarrow P \in W(aK)$ and $P \in W(bK) \Longrightarrow aK \nsubseteq P$ and $bK \nsubseteq P \Longrightarrow abK \nsubseteq P \Longrightarrow P \in W(abK)$. It follows that $W(aK) \cap W(bK) \subseteq W(abK)$ as desired. ∎

Proposition 2.4. *Let M be an R-module.Then the following statements hold:*
(i) If $\{N_\alpha\}_{\alpha\in I}$ is a family of submodules of M,then $V(\sum_{\alpha\in I} N_\alpha) = \bigcap_{\alpha\in I} V(N_\alpha)$.
(ii) If N is a submodule of M, then $V(N) = V(wrad(N))$.
(iii) For submodules N and L of M, if $V(N) \subseteq V(L)$, then $L \subseteq wrad(N)$.
(iv) For submodules N and L of M, $V(N) = V(L)$ if and only if $wrad(N) = wrad(L)$.

Proof. (i) Let $P \in \bigcap_{\alpha\in I} V(N_\alpha)$. Then $N_\alpha \subseteq P$ for every $\alpha \in I$, so $\sum_{\alpha\in I} N_\alpha \subseteq P$, which implies that $\bigcap_{\alpha\in I} V(N_\alpha) \subseteq V(\sum_{\alpha\in I} N_\alpha)$. The reverse inclusion is similar.
(ii) Since $N \subseteq wrad(N)$, we have $V(wrad(N)) \subseteq V(N)$. For the reverse inclusion, assume that $P \in V(N)$. Then $N \subseteq P$; hence $wrad(N) \subseteq P$ which implies $P \in V(wrad(N))$ which gives $V(N) \subseteq V(wrad(N))$, and so we have equality.
(iii) It is obvious.
(iv) Let $V(N) = V(L)$. By (ii), we have $V(N) \subseteq V(wrad(L))$; hence $wrad(L) \subseteq wrad(N)$ by (iii). Similarly, $wrad(N) \subseteq wrad(L)$, and so we have equality. The other implication is similar. ∎

Lemma 2.5 ([3], Lemma 2.1)**.** *Let M be an R-module and N be a proper submodule of M.*
(i) N is a weakly prime submodule if and only if for every submodule K of M not contained in N, $(N : K)$ is a prime ideal of R.
In particular $(N : M)$ is a prime ideal of R.
(ii) Let N be a weakly prime submodule of M. Then for all submodules K and L of M not contained in N, $(N : K) \subseteq (N : L)$ or $(N : L) \subseteq (N : K)$.

Proposition 2.6. *Let M be an R-module, then the following statements hold:*
(i) Let N be submodule of M.Then for any $P \in W(N)$, $(P : N)$ is a prime ideal of R.
In particular $(P : M)$ is a prime ideal of R.
(ii) For all submodules K and L of M, if $P \in W(K \cap L)$ then $(P : K)$ and $(P : L)$ are comparable.

Proof. (i) Assume $P \in W(N)$. It follows that P is a weakly prime submodule of M such that $N \nsubseteq P$. Then by Lemma 2.5(i), $(P : N)$ is a prime ideal of R.
In particular, since by definition $P \in WSpec(M) = W(M)$, by the result above $(P : M)$ is a prime ideal of R.
(ii)Assume $P \in W(K \cap L)$. Then $K \cap L \nsubseteq P$ which implies $K \nsubseteq P$ and $L \nsubseteq P$. By Lemma 2.5(ii), $(P : K) \subseteq (P : L)$ or $(P : L) \subseteq (P : K)$, as required. ∎

The following result shows the connection between weakly prime submodule and flat module. We try to characterize the spectrum of weakly prime submodule with the help of flat module.

Lemma 2.7 ([3], Theorem 3.3)**.** *Let M be an R-module.*
(i) If F is a flat R-module and P a weakly prime submodule of M such that $F \otimes P \neq F \otimes M$, then $F \otimes P$ is a weakly prime submodule of $F \otimes M$.

(ii) Let F be a faithfully flat R-module. Then P is weakly prime submodule of M if and only if $F \otimes P$ is a weakly prime submodule of $F \otimes M$.

Proposition 2.8. *Let M be an R-module. Then the following statements hold:*
(i) If F is a flat R-module and $P \in V(N)$ for submodule N of M, then $F \otimes P \in V(F \otimes N)$.
(ii) Let F be a flat R-module. For any two submodules N and L of M such that $N \subseteq L$, $V(F \otimes L) \subseteq V(F \otimes N)$.
(iii) Let F be a faithfully flat R-module.Then for submodule N of M, $P \in V(N)$ if and only if $F \otimes P \in V(F \otimes N)$.

Proof. (i) Assume $P \in V(N)$. It follows that $N \subseteq P$. Therefore, $0 \longrightarrow N \longrightarrow P \longrightarrow 0$ is an exact sequence. Since F is flat, so we have the exact sequence $0 \longrightarrow F \otimes N \longrightarrow F \otimes P \longrightarrow 0$ which yields $F \otimes N \subseteq F \otimes P$. By Lemma 2.7(i) we have $F \otimes P$ is a weakly prime submodule of $F \otimes M$. Hence, $F \otimes P \in V(F \otimes N)$.
(ii) Since $N \subseteq L$, so $0 \longrightarrow N \longrightarrow L \longrightarrow 0$ is an exact sequence. As F is flat, so we have the exact sequence $0 \longrightarrow F \otimes N \longrightarrow F \otimes L \longrightarrow 0$ which yields $F \otimes N \subseteq F \otimes L$ which implies $V(F \otimes L) \subseteq V(F \otimes N)$ as required.
(iii) Assume $F \otimes P \in V(F \otimes N)$. It follows that $F \otimes N \subseteq F \otimes P$. Therefore, $0 \longrightarrow F \otimes N \longrightarrow F \otimes P \longrightarrow 0$. Since F is faithfully flat, so we have the exact sequence $0 \longrightarrow N \longrightarrow P \longrightarrow 0$ which yields $N \subseteq P$. Thus, $P \in V(N)$. The reverse implication is clear by part (i) above. ∎

3. Quasi Multiplication Module

An R-module M is called quasi multiplication module if for every weakly prime submodule P of M, we have $P = IM$, where I is an ideal of R (see [5]). One can easily show that if M is a quasi multiplication module, then $P = (P : M)M$ for every weakly prime submodule P of M.

Clearly, every multiplication module is quasi multiplication and every quasi multiplication module is weak multiplication. But the converse of the above may not be true. We have $\mathbb{Q}$ is both weak multiplication and quasi multiplication $\mathbb{Z}$-module which is not a multiplication module as can be seen in [5, 9].

In this section We try to characterize the spectrum of weakly prime submodule with the help of quasi multiplication module.

Recall that if M be an R-module and N be a submodule of M such that $N = IM$ for some ideal I of R, then we say that I is a presentation ideal of N. Note that it is possible that for a submodule N, no such presentation ideal exist. For example, assume that M is a vector space over an arbitrary field F with $dim_F M \geq 2$ and let N be a proper subspace of M such that $N \neq 0$. Then M is of finite length (so M is noetherian, artinian and injective), but M is not multiplication and N has not any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module (see [10]).

Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by ([10], Theorem 3.4), NK is independent of presentations of N and K. It is easy to see that NK is a submodule of M and $NK \subseteq N \cap K$. Following lemma shows the relationship between multiplication and quasi multiplication module.

Lemma 3.1 ([5], Corollary 3.5)**.** *Every finitely generated quasi multiplication module is a multiplication module.*

In view of above lemma we shall say that if M be a finitely generated quasi multiplication R-module and P be a weakly prime submodule of M such that $P = IM$ for some ideal I of R, then this I is a presentation ideal of P.

Lemma 3.2 ([4], Theorem 2.6)**.** *Let R be a commutative ring, M a finitely generated multiplication R-module and P be a proper submodule of M. Then the following statements are equivalent:*
(i) P is a weakly prime submodule of M.
(ii) For submodules N, K of M with $0 \neq NK \subseteq P$, either $N \subseteq P$ or $K \subseteq P$.

Proposition 3.3. *Let R be a commutative ring and M be a finitely generated quasi multiplication R-module. Then a proper submodule P of M is weakly prime if and only if for submodules N, K of M with $0 \neq NK \subseteq P$, either $N \subseteq P$ or $K \subseteq P$.*

Proof. This is a direct consequence of Lemmas 3.1 and 3.2 above. ∎

Proposition 3.4. *Let M be a finitely generated quasi multiplication R-module. Then the following statements hold:*
(i) If S is a subset of M, $V(S) = V(< S >)$.
(ii) $V(N) \cup V(K) = V(NK) = V(N \cap K)$ for submodules N and K of M with $NK \neq 0$.
(iii) $V(N) \cup V(IM) = V(IN) = V(N \cap IM)$ for every ideal I of R and for every submodule N of M with $IN \neq 0$.
(iv) $V(IM) \cup V(JM) = V(IJM) = V(IM \cap JM)$ for every ideals I and J of R with $IJM \neq 0$.

Proof. (i) Obvious.
(ii) Since $NK \subseteq N$ and $NK \subseteq K$, then clearly $V(N) \cup V(K) \subseteq V(NK)$.
Conversely, suppose that $P \in V(NK)$ which implies that $NK \subseteq P$. Since $NK \neq 0$, by Proposition 3.3 above we have either $N \subseteq P$ or $K \subseteq P$, that is $P \in V(N) \cup V(K)$. Therefore, $V(NK) \subseteq V(N) \cup V(K)$, and hence $V(N) \cup V(K) = V(NK)$. The other part immediately follows from Proposition 2.4(i) as a particular case.
(iii) Follows directly from Lemma 3.1 and part(ii)above.
(iv) Follows directly from Lemma 3.1 and part(iii)above. ∎

Lemma 3.5 ([11], Proposition 3.2)**.** *Let M be a multiplication R-module. Then $u \in M$ is unit if and only if $< u >= M$.*

Now, assume that M is an R-module and let $X = WSpec(M)$. For each subset S of M, by X_S we mean $X - V(S) = \{P \in X | S \nsubseteq P\}$. If $S = \{m\}$, we denote by $X_m = \{P \in X | Rm \nsubseteq P\} = \{P \in X | m \nsubseteq P\}$. Clearly, the sets X_m are open, and they are called basic open sets.

Proposition 3.6. *Let M be a finitely generated quasi multiplication R-module. Then the following statements hold:*
(i) $X_{IM} \cap X_{JM} = X_{IJM}$ for every ideals I and J of R with $IJM \neq 0$.
(ii) $X_m \cap X_n = X_{mn}$ for every $m, n \in M$ with $mn \neq 0$.
(iii) $X_m = \emptyset \Longleftrightarrow m$ is a nilpotent element of M.
(iv) $X_m = X \Longleftrightarrow m$ is a unit in M.

Proof. (i) Immediately follows from Proposition 3.4(iv) by taking complement.
(ii) It follows from Proposition 3.4(ii)by choosing $N = \{m\}$ and $K = \{n\}$ and then by

taking complement.
(iii) We have $X_m = \emptyset \iff m$ is in every weakly prime submodule. Since M is a finitely generated quasi multiplication module,so by Lemma 3.1, M is a multiplication module.Therefore,m is in every weakly prime submodule $\iff m$ is in every prime submodule. Hence, m is a nilpotent element of M.
(iv) We have $X_m = X \iff V(m) = \emptyset$(taking complement)$\iff V(< m >) = \emptyset$, by Proposition 3.4(i) on choosing $S = \{m\} \iff < m >= M \iff m$ is a unit in M, by Lemma 3.5. ■

Proposition 3.7. *Let M be a finitely generated quasi multiplication R-module. Then the set $\mathbb{B} = \{X_m | m \in M\}$ forms a base for the topology $\zeta(M)$ on X.*

Proof. Suppose that U is an open set in $X = WSpec(M)$. Then $U = X - V(N)$ for some submodule N of M. Let $N =< \{m_i | i \in I\} >$ where $\{m_i | i \in I\}$ is a generator set of N. Then $V(N) = V(\sum_{i\in I} Rm_i) = \bigcap_{i\in I} V(Rm_i)$ by Proposition 2.4(i). It follows that $U = X - V(N) = X - \bigcap_{i\in I} V(Rm_i) = \bigcup_{i\in I} X_(m_i)$.
Thus, $\mathbb{B}$ is a base for the topology $\zeta(M)$ on X. ■

Proposition 3.8. *Let M be a finitely generated quasi multiplication R-module. Then every basic open set of X is compact.*

Proof. By Proposition 3.7, it is enough to show that every cover of basic open sets has a finite subcover.
Suppose that $X_m \subseteq \bigcup_{t\in I} X_{m_t}$, and let N be the submodule of M generated by $\{m_t | t \in I\}$. It follows that $\bigcap_{t\in I} V(Rm_t) = V(N) \subseteq V(Rm)$, so $V(wrad(N)) \subseteq V(wrad(< m >))$ by Proposition 2.4(ii); hence $wrad(< m >) \subseteq wrad(N)$ by Proposition 2.4(iii). Moreover, since M is a finitely generated quasi multiplication module, it can be shown that $wrad(N) = wrad(A)M$ where $A = (N : M)$. By assumption, there exists a finite subset J of I and $r_i \in wrad(A)(i \in J)$ such that $m = \sum_{t\in J} r_t m_t$ (1). For $r_i \in wrad(A)$, there is a positive integer s_i such that $r_i^{s_i} \in A$. If $s = \sum_{i\in J} s_i$, then $r_i^s \in A$ for every $i \in J$. For each $i \in J$, there exists a presentation ideal I_i of R such that $Rm_i = I_i M$; so by (i) it can be conclude that $m \in \sum_{i\in J} r_i I_i M = (\sum_{i\in J}(r_i I_i))M$. Thus $m^s \subseteq (\sum_{i\in J}(r_i I_i)^s)M \subseteq AM$. Therefore by Proposition 3.4(i), $V(N) = \bigcap_{i\in I} V(Rm_i) \subseteq \bigcap_{i\in J} V(Rm_i) \subseteq V(m) = V(Rm) = V(m^s)$. Taking complements, we have $X_m \subseteq \bigcup_{i\in J} X_{m_i}$. This complete the proof. ■

Proposition 3.9. *Let M be a finitely generated quasi multiplication R-module. Then an open set of X is compact if and only if it is a finite union of basic open sets.*

Proof. It is a consequence of Propositions 3.7 and 3.8 above. ■

Proposition 3.10. *Let M be a finitely generated quasi multiplication R-module. Then X is compact.*

Proof. Let $M = \sum_{i=1}^n Rm_i$. Then $V(M) = \emptyset$; hence $X_M = X$, that is, $X = \bigcup_{i=1}^n X_{m_i}$. Thus, X is compact. ■

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