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The Hermite-Hadamard Inequality for s-Convex Functions in the Second Sense via Conformable Fractional Integrals

Erhan Set*, **Abdurrahman Gözpınar and İlker Mumcu**

Department of Mathematics, Faculty of Arts and Sciences, Ordu University, 52200 Ordu, Turkey
e-mail : erhanset@yahoo.com (E. Set); abdurrahmangozpinar79@gmail.com (A. Gözpınar); mumcuilker@msn.com (I. Mumcu)

Abstract In this study, we established a new identity via conformable fractional integral. By using that identity we get new results for s-convex functions in the second sense. Our results have some relations with the Hermite Hadamard type inequalities in the literature.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

One of the most famous inequality for convex functions is so called Hermite-Hadamard inequality as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Since its discovery in 1983, Hermite-Hadamard's inequality has been considered the most useful inequality in mathematical analysis. A number of the papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see and references therein ([1],[2]). s-convex function in the second sense was introduced in Breckner's paper [3] and a number of properties and connections with s-convexity in the first sense are discussed in paper [4]. For more study, see([5],[6],[7])

*Corresponding author.

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Definition 1.1. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in \mathbb{R}_+$ and all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

We denote this by K_s^2. It is obvious that the s-convexity means just the convexity when $s = 1$.

In [7] Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense as follows:

Theorem 1.2. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0,\infty), a < b$. If $f \in L^1[a,b]$ then the following inequality hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

Theory of convex functions has great importance in various fields of pure and applied sciences. It is known that theory of convex functions is closely related to theory of inequalities. Many interesting convex functions inequalities established via Riemann-Liouville fractional integrals. To understand these studies, let's give some necessary definition and mathematical preliminaries of fractional calculus theory as follows, which are used further in this study. For more details, one can consult ([8–12]).

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Lioville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < b$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1} dx$. It is to be noted that $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

The beta function is defined as follows:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \qquad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function. The Incomplete beta function is defined by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

In [13], Sarıkaya et al. gave a remarkable integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals as follows:

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in [a, b]$. If f is convex function on $[a, b]$, then the following inequality for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[(J^{\alpha}_{a^+}f)(b) + (J^{\alpha}_{b^-}f)(a)] \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

It is obviously seen that, if we take $\alpha = 1$ in Theorem 1.4, then the inequality (1.3) reduces to well known Hermite-Hadamard's inequality as (1.1).

Hermite Hadamard type inequality for s-convex functions on Riemann-Liouville fractional integral is given in [14] as follows:

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is s-convex mapping in the second sense on $[a, b]$, then the following inequality for fractional integral with $\alpha > 0$ and $s \in (0, 1]$ hold:*

$$\begin{aligned} 2^{s-1}f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[(J^{\alpha}_{a^+}f)(b) + (J^{\alpha}_{b^-}f)(a)] \\ &\le [\frac{1}{\alpha+s} + B(\alpha, s+1)]\frac{f(a)+f(b)}{2} \end{aligned} \tag{1.4}$$

where B(a,b) is beta function.

In [15], Zhu et al. established a new identity for differentiable convex mappings via Riemann-Liouville fractional integral. In this work we will generalize this identity for conformable fractional integral in section 2.

Lemma 1.6. [15] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f^{'} \in [a, b]$, then the following equality for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[(J^{a}_{\alpha^-}f)(b) + (J^{b}_{\alpha^+}f)(a)] - f\left(\frac{a+b}{2}\right) \qquad (1.5) \\ &= \frac{b-a}{2}\left[\int_0^1 k(t)f^{'}(ta+(1-t)b)dt - \int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f^{'}(ta+(1-t)b)dt\right. \end{aligned}$$

where

$$k(t) = \begin{cases} 1 & , 0 < t \le \frac{1}{2} \\ -1 & , \frac{1}{2} < t \le 1. \end{cases}$$

Using the above identity, they gave the following result on Riemann-Liouville fractional intagral.

Theorem 1.7. [15] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f^{'}|$ is convex on $[a, b]$, then the following fractional inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[(J^{a}_{\alpha^-}f)(b) + (J^{b}_{\alpha^+}f)(a)] - f\left(\frac{a+b}{2}\right)\right| \qquad (1.6) \\ &\le \frac{b-a}{4(\alpha+1)}(\alpha+3-\frac{1}{2^{\alpha-1}})[|f^{'}(a)| + |f^{'}(b)|]. \end{aligned}$$

In this work we will study on conformable fractional integrals. In the following, we give some definitions and properties of conformable fractional integrals which helps obtain main identity and Theorems in section 2. Recently, some authors started to study on conformable fractional integral. In the paper numbered with [16], Khalil et al. defined the fractional integral of order $0<\alpha\leq 1$ only. In [17], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha>0$.

Definition 1.8. Let $\alpha\in(n,n+1]$ and set $\beta=\alpha-n$ then the left conformable fractional integral starting at a if order α is defined by

$$(I_{\alpha}^{a}f)(t)=\frac{1}{n!}\int_{a}^{t}(t-x)^{n}(x-a)^{\beta-1}f(x)dx$$

Analogously, the right conformable fractional integral is defined by

$$({}^{b}I_{\alpha}f)(t)=\frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha=n+1$ then $\beta=\alpha-n=n+1-n=1$ where $n=0,1,2...$ and hence $(I_{\alpha}^{a}f)(t)=(J_{n+1}^{a}f)(t)$. In [18] Set et.al. gave Hermite Hadamard's inequality for conformable fractional integral as follows:

Theorem 1.9. *Let $f:[a,b]\to\mathbb{R}$ be a function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]\leq\frac{f(a)+f(b)}{2}\tag{1.7}$$

with $\alpha\in(n,n+1]$, where Γ is Euler Gamma function.

For some new studies on conformable fractional integral, see ([19],[20],[21]). In papers ([22]-[26]) Set et.al obtained some new Hermite- Hadamard, Ostrowski , Chebyshev type inequalities by using conformable fractional integrals for various classes of functions. The aim of this study is to established new Hermite-Hadamard's inequalities for conformable fractional integral related to fractional integral.

In [22], Set et al. established a generalization of Hermite-Hadamard type inequality for s-convex functions via conformable fractional integrals as follows:

Theorem 1.10. *Let $f:[a,b]\to\mathbb{R}$ be a function with $0\leq a<b$, $s\in(0,1]$ and $f\in L_1[a,b]$. If f is s-convex function in the second sense on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$\begin{aligned}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)&\leq\frac{1}{(b-a)^{\alpha}2^{s}}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]\\&\leq\left[\frac{B(n+s+1,\alpha-n)+B(n+1,\alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^{s}}\end{aligned}\tag{1.8}$$

with $\alpha\in(n,n+1]$, $n\in\mathbb{N}$, $n=0,1,2...$where Γ is Euler Gamma function and $B(a,b)$ is a beta function.

They also noticed the relation with fractional and classical Hermite Hadamard type integral inequalities as follows;

Remark 1.11. If we choose $s = 1$ in Theorem 1.10, using relation between Γ and B functions properties, the inequality (1.8) reduced to inequality (1.7).

Remark 1.12. If we choose $\alpha = n + 1$ in Theorem 1.10, the inequality (1.8) reduced to inequality (1.4). And also if we choose $\alpha, s = 1$ in the inequality (1.8), then we get well-known Hermite Hadamard's inequality as (1.1).

2. Main Results

In order to achieve our aim, we will have a important conformable fractional identity for differentiable functions involving conformable fractional integrals as follows;

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f^{'} \in L[a, b]$, then the following equality for conformable fractional integrals holds:*

$$\begin{aligned} & \frac{n!}{2(b-a)^\alpha}[I_\alpha^a f(b) +^b I_\alpha f(a)] - B(n+1, \alpha - n) f(\frac{a+b}{2}) \qquad (2.1) \\ = & \frac{b-a}{2}\Big\{ \int_0^1 k(t) f^{'}(ta + (1-t)b)dt \\ & - \int_0^1 \big[B_{1-t}(n+1, \alpha - n) - B_t(n+1, \alpha - n)\big] f^{'}((ta + (1-t)b)dt \Big\}, \end{aligned}$$

where

$$k(t) = \begin{cases} B(n+1, \alpha - n) & , 0 < t \leq \frac{1}{2} \\ -B(n+1, \alpha - n) & , \frac{1}{2} < t \leq 1 \end{cases}$$

and $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2...$where Γ is Euler Gamma function and $B(a, b)$ is a beta function, $B_t(a, b)$ is an incompleted beta function.

Proof. It sufficies to note that

$$\begin{aligned} I = & \int_0^{\frac{1}{2}} B(n+1, \alpha - n) f^{'}(ta + (1-t)b)dt \\ & - \int_{\frac{1}{2}}^1 B(n+1, \alpha - n) f^{'}(ta + (1-t)b)dt \\ & - \int_0^1 B_{1-t}(n+1, \alpha - n) f^{'}(ta + (1-t)b)dt \\ & + \int_0^1 B_t(n+1, \alpha - n) f^{'}(ta + (1-t)b)dt. \qquad (2.2) \end{aligned}$$

Integrating by parts, we have ;

$$\begin{aligned} I_1 = & B(n+1, \alpha - n) \int_0^{\frac{1}{2}} f^{'}(ta + (1-t)b)dt \\ = & \frac{B(n+1, \alpha - n)}{b-a} \left[f(b) - f(\frac{a+b}{2}) \right], \qquad (2.3) \end{aligned}$$

$$\begin{aligned} I_2 &= -B(n+1,\alpha-n)\int_{\frac{1}{2}}^{1} f^{'}(ta+(1-t)b)dt \\ &= \frac{B(n+1,\alpha-n)}{b-a}\left[f(a)-f(\frac{a+b}{2})\right], \end{aligned} \tag{2.4}$$

Changing variables with $x = ta + (1-t)b$ in following integration, we get

$$\begin{aligned} I_3 &= -\int_0^1 B_{1-t}(n+1,\alpha-n)f^{'}(ta+(1-t)b)dt \\ &= -\int_0^1 \left(\int_0^{1-t} x^n(1-x)^{\alpha-n-1}dx\right)f^{'}(ta+(1-t)b)dt \\ &= -\left(\int_0^{1-t} x^n(1-x)^{\alpha-n-1}dx\right)\frac{f(ta+(1-t)b)dt}{a-b}\Big|_0^1 \\ &\quad -\int_0^1 (1-t)^n t^{\alpha-n-1} f(ta+(1-t)b)\frac{dt}{a-b} \\ &= \left(\int_0^1 x^n(1-x)^{\alpha-n-1}dx\right)\frac{f(b)}{a-b} \\ &\quad +\frac{1}{a-b}\int_a^b (\frac{x-a}{b-a})^n(\frac{b-x}{b-a})^{\alpha-n-1}f(x)\frac{dx}{a-b} \\ &= -B(n+1,\alpha-n)\frac{f(b)}{b-a}+\frac{n!}{(b-a)^{\alpha+1}}({}^bI_\alpha f)(a) \end{aligned} \tag{2.5}$$

$$\begin{aligned} I_4 &= \int_0^1 B_t(n+1,\alpha-n)f^{'}((ta+(1-t)b)dt \\ &= B_t(n+1,\alpha-n)\frac{f(ta+(1-t)b)}{a-b}\Big|_0^1 \\ &\quad -\int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)\frac{dt}{a-b} \\ &= -B(n+1,\alpha-n)\frac{f(a)}{b-a}+\frac{1}{b-a}\int_a^b (\frac{b-x}{b-a})^n(\frac{x-a}{b-a})^{\alpha-n-1}f(x)\frac{dx}{b-a} \\ &= -B(n+1,\alpha-n)\frac{f(a)}{b-a}+\frac{n!}{(b-a)^{\alpha+1}}(I_\alpha^a f)(b) \end{aligned} \tag{2.6}$$

Submitting (2.3), (2.4), (2.5), (2.6) in (2.2) we get,

$$I = \frac{-2B(n+1,\alpha-n)}{b-a}f(\frac{a+b}{2})+\frac{n!}{(b-a)^{\alpha+1}}[(I_\alpha^b f)(a)+({}^bI_\alpha f)(a)].$$

Thus, by multiplying both sides by $\frac{b-a}{2}$, we have desired result. ∎

Remark 2.2. If we choose $\alpha = n+1$ in Lemma 2.1, the equality (2.1) becomes the equality (1.5)

Now, using the obtained identity, we will establish some inequalities connected with the left part of the inequality (1.8). We shall offer some results, which embodied in the following two theorems.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$ and $|f'|$ is s-convex in the second sense, then the following inequality for conformable fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]-B(n+1,\alpha-n)f(\frac{a+b}{2})\right| \\
\leq\ & \frac{(b-a)}{2(s+1)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \\
&\times\Big[2B(n+1,\alpha-n)+B_{\frac{1}{2}}(\alpha-n+s+1,n+1)+B_{\frac{1}{2}}(n+s+2,\alpha-n) \\
&-B_{\frac{1}{2}}(\alpha-n,n+s+2)-B_{\frac{1}{2}}(n+1,\alpha-n+s+1)\Big],
\end{aligned}$$

$\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2...$*where $B(a,b)$ is a beta function, $B_t(a,b)$ is an incompleted beta function.*

Proof. Using Lemma 2.1 and the s-convexity of $|f'|$, we get the following inequalities;

$$\begin{aligned}
&\left|\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]-B(n+1,\alpha-n)f(\frac{a+b}{2})\right| \\
=\ & \frac{(b-a)}{2}\left\{\left|\int_0^1 k(t)f'(ta+(1-t)b)dt\right.\right. \\
&\left.\left.-\int_0^1\left[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right]f'(ta+(1-t)b)dt\right|\right\} \\
\leq\ & \frac{(b-a)}{2}\left\{\left|\int_0^{\frac{1}{2}} B(n+1,\alpha-n)f'(ta+(1-t)b)dt\right.\right. \\
&\left.-\int_{\frac{1}{2}}^1 B(n+1,\alpha-n)f'(ta+(1-t)b)\right| \\
&\left.+\left|\int_0^1\Big(B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\Big)f'(ta+(1-t)b)\right|\right\}dt \\
\leq\ & \frac{(b-a)}{2}\left\{\int_0^{\frac{1}{2}} B(n+1,\alpha-n)\left|f'(ta+(1-t)b)\right|dt\right. \\
&+\int_{\frac{1}{2}}^1 B(n+1,\alpha-n)\left|f'(ta+(1-t)b)\right|dt \\
&+\int_0^{\frac{1}{2}}\left[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right]\left|f'(ta+(1-t)b)\right|dt
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\frac{1}{2}}^{1} \left[B_t(n+1,\alpha-n) - B_{1-t}(n+1,\alpha-n) \right] \left| f^{'}(ta+(1-t)b) \right| dt \Bigg\} \\
\leq &\quad \frac{(b-a)}{2} \Bigg\{ \int_0^1 B(n+1,\alpha-n) \Big(t^s |f^{'}(a)| + (1-t)^s |f^{'}(b)| \Big) \\
&\quad + \int_0^{\frac{1}{2}} \left[B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n) \right] \Big(t^s |f^{'}(a)| + (1-t)^s |f^{'}(b)| \Big) dt \\
&\quad + \int_{\frac{1}{2}}^{1} \left[B_t(n+1,\alpha-n) - B_{1-t}(n+1,\alpha-n) \right] \Big(t^s |f^{'}(a)| + (1-t)^s |f^{'}(b)| \Big) dt \Bigg\} \\
:= &\quad \frac{(b-a)}{2} (\Phi_1 + \Phi_2 + \Phi_3) \qquad (2.7)
\end{aligned}
$$

In fact noticed that, well-known Newton Leibnits formula is as follow

$$
\frac{d}{dt} \int_{v(t)}^{u(t)} f(x)dx = f(u(t))u^{'}(t) - f(v(t))v^{'}(t).
$$

Since $|f^{'}|$ is s- convex and using the above equation, we can compute the following integrals;

$$
\begin{aligned}
\Phi_1 &= \int_0^1 B(n+1,\alpha-n) \left[t^s \left| f^{'}(a) \right| + (1-t)^s \left| f^{'}(b) \right| \right] dt \qquad (2.8) \\
&= \frac{B(n+1,\alpha-n)}{s+1} \left[\left| f^{'}(a) \right| + \left| f^{'}(b) \right| \right].
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2 &= \int_0^{\frac{1}{2}} \left[B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n) \right] \Big(t^s |f^{'}(a)| + (1-t)^s |f^{'}(b)| \Big) dt \\
&= \int_0^{\frac{1}{2}} \left(\int_t^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) \Big(t^s |f^{'}(a)| + (1-t)^s |f^{'}(b)| \Big) dt \\
&= |f^{'}(a)| \Bigg\{ \left(\int_t^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{t^{s+1}}{s+1} \Bigg|_0^{\frac{1}{2}} \\
&\quad - \frac{1}{s+1} \int_0^{\frac{1}{2}} \left[-1.(1-t)^n t^{\alpha-n-1} - 1.t^n (1-t)^{\alpha-n-1} \right] t^{s+1} dt \Bigg\} \\
&\quad + |f^{'}(b)| \Bigg\{ \left(\int_t^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{-(1-t)^{s+1}}{s+1} \Bigg|_0^{\frac{1}{2}} \\
&\quad + \frac{1}{s+1} \int_0^{\frac{1}{2}} \left[-1.(1-t)^n t^{\alpha-n-1} - 1.t^n (1-t)^{\alpha-n-1} \right] (1-t)^{s+1} dt \Bigg\} \\
&= |f^{'}(a)| \Bigg\{ \frac{1}{s+1} \int_0^{\frac{1}{2}} (1-t)^n t^{\alpha-n+s} dt + \frac{1}{s+1} \int_0^{\frac{1}{2}} t^{n+s+1} (1-t)^{\alpha-n-1} dt \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+|f'(b)|\left\{\frac{1}{s+1}\int_0^1 x^n(1-x)^{\alpha-n-1}dx - \frac{1}{s+1}\int_0^{\frac{1}{2}} t^{\alpha-n-1}(1-t)^{n+s+1}dt\right.\\
&\left.-\frac{1}{s+1}\int_0^{\frac{1}{2}} t^n(1-t)^{\alpha-n+s}dt\right\}\\
=\;& \frac{|f'(a)|}{s+1}\Big[B_{\frac{1}{2}}(\alpha-n+s+1,n+1)+B_{\frac{1}{2}}(n+s+2,\alpha-n)\Big] \qquad (2.9)\\
&+\frac{|f'(b)|}{s+1}\Big[B(n+1,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+s+2)-B_{\frac{1}{2}}(n+1,\alpha-n+s+1)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 =\;& \int_{\frac{1}{2}}^1 \big[B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\big]\Big(t^s|f'(a)|+(1-t)^s|f'(b)|\Big)dt\\
=\;& \int_{\frac{1}{2}}^1 \left(\int_{1-t}^t x^n(1-x)^{\alpha-n-1}dx\right)\Big(t^s|f'(a)|+(1-t)^s|f'(b)|\Big)dt\\
=\;& |f'(a)|\left\{\left(\int_{1-t}^t x^n(1-x)^{\alpha-n-1}dx\right)\frac{t^{s+1}}{s+1}\bigg|_{\frac{1}{2}}^1\right.\\
&\left.-\frac{1}{s+1}\int_{\frac{1}{2}}^1 \big[1.t^n(1-t)^{\alpha-n-1}+1.(1-t)^n t^{\alpha-n-1}\big]t^{s+1}dt\right\}\\
&+|f'(b)|\left\{\left(\int_{1-t}^t x^n(1-x)^{\alpha-n-1}dx\right)\frac{-(1-t)^{s+1}}{s+1}\bigg|_{\frac{1}{2}}^1\right.\\
&\left.+\frac{1}{s+1}\int_{\frac{1}{2}}^1 \big[1.t^n(1-t)^{\alpha-n-1}+1.(1-t)^n t^{\alpha-n-1}\big](1-t)^{s+1}dt\right\}\\
=\;& |f'(a)|\left\{\frac{1}{s+1}\int_0^1 x^n(1-x)^{\alpha-n-1}dx-\frac{1}{s+1}\int_{\frac{1}{2}}^1 t^{n+s+1}(1-t)^{\alpha-n-1}dt\right.\\
&\left.-\frac{1}{s+1}\int_{\frac{1}{2}}^1 t^{\alpha-n+s}(1-t)^n dt\right\}\\
&+|f'(b)|\left\{\frac{1}{s+1}\int_{\frac{1}{2}}^1 t^n(1-t)^{\alpha-n+s}dt+\frac{1}{s+1}\int_{\frac{1}{2}}^1 t^{\alpha-n-1}(1-t)^{n+s+1}dt\right\}\\
=\;& \frac{|f'(a)|}{s+1}\Big[B(n+1,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+s+2)-B_{\frac{1}{2}}(n+1,\alpha-n+s+1)\Big]\\
&+\frac{|f'(b)|}{s+1}\Big[B_{\frac{1}{2}}(n+s+2,\alpha-n)+B_{\frac{1}{2}}(\alpha-n+s+1,n+1)\Big] \qquad (2.10)
\end{aligned}
$$

Combining (2.8), (2.9), (2.10) with (2.7), we get desired result. ∎

Corollary 2.4. *Taking $s = 1$ in Theorem 2.3 i.e. $|f^{'}|$ is convex, we get the following result:*

$$\left|\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]-B(n+1,\alpha-n)f(\frac{a+b}{2})\right| \tag{2.11}$$
$$\leq \frac{(b-a)}{4}\left(|f^{'}(a)|+|f^{'}(b)|\right)\Big[2B(n+1,\alpha-n)+B_{\frac{1}{2}}(\alpha-n+2,n+1)$$
$$+B_{\frac{1}{2}}(n+3,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+3)-B_{\frac{1}{2}}(n+1,\alpha-n+2)\Big].$$

Remark 2.5. Taking $\alpha = n + 1$ in Corollary 2.4, the inequality (2.11) reduces the inequality (1.6).

Theorem 2.6. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $s \in (0, 1]$. If $f^{'} \in L[a, b]$ and $|f^{'}|^{q}$ is s-convex in the second sense, then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]-B(n+1,\alpha-n)f(\frac{a+b}{2})\right| \tag{2.12}$$
$$\leq \frac{b-a}{2}\left(B(n+1,\alpha-n)(\frac{1}{2})^{\frac{1}{p}}+\delta^{\frac{1}{p}}\right)\left\{\left[\frac{(\frac{1}{2})^{s+1}|f^{'}(a)|^{q}+(1-(\frac{1}{2})^{s+1})|f^{'}(b)|^{q}]}{s+1}\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{(1-(\frac{1}{2})^{s+1})|f^{'}(a)|^{q}+(\frac{1}{2})^{s+1}|f^{'}(b)|^{q}]}{s+1}\right]^{\frac{1}{q}}\right\}$$

where $\delta = \int_0^{\frac{1}{2}}\left(\int_t^{1-t}x^{n}(1-x)^{\alpha-n-1}dx\right)^{p}dt$, $\alpha \in (n, n+1]$, $n = 0, 1, 2...$, $B(a, b)$, $B_t(a, b)$ *are beta and incompleted beta functions with* $\frac{1}{p}+\frac{1}{q}=1, q>1$.

Proof. Using Lemma 2.1, s-convexity of $|f^{'}|^{q}$ and well-known Hölder inequality, we obtain

$$\left|\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]-B(n+1,\alpha-n)f(\frac{a+b}{2})\right|$$
$$\leq \frac{(b-a)}{2}\left\{\left|\int_0^{\frac{1}{2}}B(n+1,\alpha-n)f^{'}(ta+(1-t)b)dt\right.\right.$$
$$\left.-\int_{\frac{1}{2}}^{1}B(n+1,\alpha-n)f^{'}(ta+(1-t)b)\right|$$
$$\left.+\left|\int_0^{1}\Big(B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\Big)f^{'}(ta+(1-t)b)\right|\right\}dt$$

$$\leq \frac{(b-a)}{2}\Bigg\{\int_0^{\frac{1}{2}} B(n+1,\alpha-n)\big|f^{'}(ta+(1-t)b)\big|dt$$
$$+\int_{\frac{1}{2}}^1 B(n+1,\alpha-n)\big|f^{'}(ta+(1-t)b)\big|dt$$
$$+\int_0^{\frac{1}{2}} \big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big]\big|f^{'}(ta+(1-t)b)\big|dt$$
$$+\int_{\frac{1}{2}}^1 \big[B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\big]\big|f^{'}(ta+(1-t)b)\big|dt\Bigg\}$$
$$\leq \frac{(b-a)}{2}\Bigg\{\Big(\int_0^{\frac{1}{2}} B(n+1,\alpha-n)^p dt\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}\big|f^{'}(ta+(1-t)b)\big|^q dt\Big]^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{1}{2}}^1 B(n+1,\alpha-n)^p dt\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^1\big|f^{'}(ta+(1-t)b)\big|^q dt\Big]^{\frac{1}{q}}$$
$$+\Big(\int_0^{\frac{1}{2}}\Big[\int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx\Big]^p dt\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}\big|f^{'}(ta+(1-t)b)\big|^q dt\Big]^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{1}{2}}^1\Big[\int_{1-t}^{t} x^n(1-x)^{\alpha-n-1}dx\Big]^p dt\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^1\big|f^{'}(ta+(1-t)b)\big|^q dt\Big]^{\frac{1}{q}}\Bigg\}, \tag{2.13}$$

Let's calculate the following simple integral equations;

$$\int_0^{\frac{1}{2}} B(n+1,\alpha-n)^p dt = \frac{B(n+1,\alpha-n)^p}{2} = \int_{\frac{1}{2}}^1 B(n+1,\alpha-n)^p dt. \tag{2.14}$$

Since $|f^{'}|^q$ is s-convex, we get the following simple computation;

$$\begin{aligned}\int_0^{\frac{1}{2}}\big|f^{'}(ta+(1-t)b)\big|^q &= \int_0^{\frac{1}{2}}\Big(t^s|f^{'}(a)|^q+(1-t)^s|f^{'}(b)|^q\Big)dt \\ &= \frac{[(\frac{1}{2})^{s+1}|f^{'}(a)|^q+(1-(\frac{1}{2})^{s+1})|f^{'}(b)|^q]}{s+1},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\int_{\frac{1}{2}}^1\big|f^{'}(ta+(1-t)b)\big|^q &= \int_{\frac{1}{2}}^1\Big(t^s|f^{'}(a)|^q+(1-t)^s|f^{'}(b)|^q\Big)dt \\ &= \frac{[(1-(\frac{1}{2})^{s+1})|f^{'}(a)|^q+(\frac{1}{2})^{s+1}|f^{'}(b)|^q]}{s+1}.\end{aligned} \tag{2.16}$$

In fact noticed that, changing variables with $u=1-t$ we can write the following equality;

$$\delta=\int_0^{\frac{1}{2}}\Big[\int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx\Big]^p dt=\int_{\frac{1}{2}}^1\Big[\int_{1-u}^{u} x^n(1-x)^{\alpha-n-1}dx\Big]^p du. \tag{2.17}$$

■

Thus combining the equalities (2.14)-(2.17) with (2.13) we get desired result as (2.12).

Corollary 2.7. *For $s=1$ in Theorem 2.6, we have;*

$$\begin{aligned} &\Big|\frac{n!}{2(b-a)^\alpha}[I^a_\alpha f(b)+^b I_\alpha f(a)]-B(n+1,\alpha-n)f(\frac{a+b}{2})\Big| \\ \leq\ & \frac{b-a}{2}\bigg(B(n+1,\alpha-n)(\frac{1}{2})^{\frac{1}{p}}+\delta^{\frac{1}{p}}\bigg) \\ &\times\Bigg\{\Bigg[\frac{|f^{'}(a)|^q+3|f^{'}(b)|^q}{8}\Bigg]^{\frac{1}{q}}+\Big[\frac{3|f^{'}(a)|^q+|f^{'}(b)|^q}{8}\Big]^{\frac{1}{q}}\Bigg\}. \end{aligned}$$

Corollary 2.8. *If we take $\alpha=n+1$ in Corollary 2.7, we get*

$$\begin{aligned} \delta &= \int_0^{\frac{1}{2}}\Big(\int_t^{1-t}x^n(1-x)^{\alpha-n-1}dx\Big)^p dt \\ &= \int_0^{\frac{1}{2}}\Big(\frac{(1-t)^\alpha-t^\alpha}{\alpha}\Big)^p \\ &\leq \frac{1}{\alpha^p}\int_0^{\frac{1}{2}}(1-2t)^{\alpha p}dt \\ &= \frac{1}{\alpha^p}\frac{1}{2(\alpha p+1)} \end{aligned}$$

where we used the fact that, for $\alpha\in(0,1]$ and $\forall t_1,t_2\in[0,1]$ then $|t_1^\alpha-t_2^\alpha|\leq|t_1-t_2|^\alpha$ holds. Thus,

$$\begin{aligned} &\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[I^a_\alpha f(b)+^b I_\alpha f(a)]-f(\frac{a+b}{2})\Big| \\ \leq\ & \frac{b-a}{2}\bigg[(\frac{1}{2})^{\frac{1}{p}}\big[1+(\frac{1}{\alpha p+1})^{\frac{1}{p}}\big]\bigg] \\ &\times\Bigg\{\Bigg[\frac{|f^{'}(a)|^q+3|f^{'}(b)|^q}{8}\Bigg]^{\frac{1}{q}}+\Big[\frac{3|f^{'}(a)|^q+|f^{'}(b)|^q}{8}\Big]^{\frac{1}{q}}\Bigg\}. \end{aligned}$$

and if we take $\alpha=1$ in the above inequality then

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\Big| \\ \leq\ & \frac{b-a}{2}\bigg[(\frac{1}{2})^{\frac{1}{p}}\big[1+(\frac{1}{p+1})^{\frac{1}{p}}\big]\bigg] \\ &\times\Bigg\{\Bigg[\frac{|f^{'}(a)|^q+3|f^{'}(b)|^q}{8}\Bigg]^{\frac{1}{q}}+\Big[\frac{3|f^{'}(a)|^q+|f^{'}(b)|^q}{8}\Big]^{\frac{1}{q}}\Bigg\} \end{aligned}$$

holds.

References

[1] M. Alomari, M. Darus, On the Hadamards inequality for log-convex functions on the coordinates, J. Inequal. Appl. (2009) Article ID 283147.

[2] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000 [ONLINE: http://www.staff.vu.edu.au/RGMIA/monographs/hermite-hadamard.html].

[3] W.W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, Pupl. Inst. Math. 23 (1978) 13–20.

[4] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, Aequationes Math. 48 (1994) 100–111.

[5] M. Alomari, M. Darus, S.S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. 23 (9) 1071–1076.

[6] M. Avci, H. Kavurmaci, M.E. Özdemir, New inequalities of Hermite–Hadamard type via s-convex functions in the second sense with applications, Appl. Math. Comput. 217 (12) (2011) 5171–5176.

[7] S.S. Dragomir, S. Fitzpatrik, The Hadamard's inequality for s-convex functions in the second sense, Demonstr. Math. 32 (4) (1999) 687–696.

[8] R. Gorenflo, F. Mainardi, Fractional Calculus: Integral and Differential Equations of Fractional Order, arXiv:0805.3823v1, 2008.

[9] İ. İşcan, Generalization of different type integral inequalities for s-convex functions via fractional integrals, Appl. Anal. 93 (9) (2014) 1846–1862.

[10] İ. İşcan, Hermite Hadamard type inequalities for harmonically convex functions via fractional integral, Appl. Math. Comput. 238 (2014) 237–244.

[11] I. Podlubni, Fractional Differential Equations, Academic Press, San Diego, 1999.

[12] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Comput. Math. Appl. 63 (7) (2012) 1147–1154.

[13] M.Z. Sarıkaya, E. Set, H. Yaldız, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model. 57 (2013) 2403–2407.

[14] E. Set, M.Z. Sarıkaya, M.E. Özdemir, H. Yıldırım, The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results, J. Appl. Math., Stat. Inform. 10 (2) (2014) 69–83.

[15] C. Zhu, M. Feckan, J. Wang, Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula, J. Appl. Math. Stat. Inform. 8 (2) (2012).

[16] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014) 65–70.

[17] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math. 279 (2015) 57–66.

[18] E. Set, A.O. Akdemir, İ. Mumcu, Hadamard's inequality and its extensions for conformable fractioanal integrals of any order $\alpha > 0$, Creat. Math. Inform. 27 (2) (2018) 197–206.

[19] T. Abdeljawad, On multiplicative fractional calculus, arXiv:1510.04176v1 (2018).

[20] O.R. Anderson Taylor's formula and integral inequalities for conformable fractional derivatives, arXiv:1409.5888v1 (2014).

[21] N. Benkhettou, S. Hassani, D.E.M. Torres, A conformable fractional calculus on arbitrary time scales, Journal of King Saud University-Science 28 (1) (2016) 93–98.

[22] E. Set, A. Gözpınar, A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals, Stud. Univ. Babeş-Bolyai Math. 62 (3) (2017) 309–323.

[23] E. Set, M.Z. Sarıkaya, A. Gözpınar, Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities, Creat. Math. Inform. 26 (2) (2017) 223–231.

[24] E. Set, A. Gözpınar, A. Ekinci, Hermite-Hadamard type inequalities via conformable fractional integrals, Acta Math. Univ. Comenianae. LXXXVI (2) (2017) 309–320.

[25] E. Set, A.O. Akdemir, İ. Mumcu, Ostrowski type inequalities for functions whose derivatives are convex via conformable fractional integrals, J. Adv. Math. Stud. 10 (3) (2017) 386–395.

[26] E. Set, A.O. Akdemir, İ. Mumcu, Chebyshev type inequalities for conformable fractional integrals, Miskolc Mathematical Notes 20 (2) (2019) 1227–1236.