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Some Results on Fuzzy Cosets of Fuzzy Subgroups of a Group

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Abstract In this paper, proofs of some results relating to fuzzy subgroup have been provided. Some new theorems on fuzzy cosets of fuzzy subgroup of a group are also stated and proved.

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1. Introduction

The concept of fuzzy sets was initiated by L. A. Zadeh [1] in 1965. The fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Since then it has become a vigorous area of research in engineering, medical science, social science and graph theory etc. Several researchers have done extensive works in the field of fuzzy subgroup of a group. In 1971, Rosenfeld [2] introduced the concept of fuzzy subgroup of a group. P. Bhattacharya [3] introduced fuzzy left and right coset of a fuzzy subgroup. W. B. Vasantha Kandasamy [4] discussed the concepts of fuzzy middle coset of fuzzy subgroup. The aim of this paper is to obtain some theorems on left, right and middle cosets of a fuzzy subgroup of any group.

2. Priliminary Concepts

In this section, we site the fundamental definitions that will be used in the sequel.

Definition 2.1. [1, 5] Let X be any non empty set. A fuzzy subset μ of X is just a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.2. [1] Let λ, μ be two fuzzy subsets of X. Then $\lambda = \mu$ if and only if $\lambda(x) = \mu(x)$, for all $x \in X$.

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Definition 2.3. [5] A fuzzy subset μ of a group G is called a fuzzy subgroup of G if
(i) $\mu(xy) \geq \min(\mu(x), \mu(y))$
(ii) $\mu(x^{-1}) \geq \mu(x)$, for all $x, y \in G$.

Definition 2.4. [6] A fuzzy subgroup μ of a group G is called normal fuzzy subgroup if and only if for all $x, y \in G$, $\mu(xy) = \mu(yx)$.

Definition 2.5. [5, 7, 8] Let μ be a fuzzy subgroup of a group G and $a, b \in G$. Then
(i) The left coset $a\mu$ of μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$, for all $x \in G$.
(ii) The right coset μa of μ is defined by $(\mu a)(x) = \mu(xa^{-1})$, for all $x \in G$.
(iii) The middle coset $a\mu b$ of μ is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$, for all $x \in G$.

Definition 2.6. [9] Let μ and λ be fuzzy subgroups of G. Then they are conjugate if for some $a \in G$, we have $\mu(a^{-1}xa) = \lambda(x)$, for all $x \in G$.

Proposition 2.7. [5] *A fuzzy subset μ of a group G is a fuzzy subgroup of G if and only if for all $x, y \in G$ we have $\mu(xy^{-1}) \geq \min(\mu(x), \mu(y))$.*

In the following theorem, a necessary condition for middle coset of a group to be fuzzy subgroup is given.

Theorem 2.8. [9] *Every middle coset $a\mu b$ of a fuzzy subgroup μ of a group G is a fuzzy subgroup if μ is fuzzy conjugate to some fuzzy subgroup λ of G.*

Here we recall two lemmas which will be required in the sequel (see [5]).

Lemma 2.9. *If μ is a fuzzy subgroup of a group G and e being identity in G, then $\mu(x) \leq \mu(e)$, for all $x \in G$.*

Lemma 2.10. *If μ is a fuzzy subgroup of a group G, then $\mu(x) = \mu(x^{-1})$, for all $x \in G$.*

3. Main Results

In this section, some theorems related to fuzzy cosets of a given group are established.

Theorem 3.1. *Let μ be a fuzzy subgroup of a group G, then $e\mu = \mu = \mu e$, where e being identity in G.*

Proof. Let μ be a fuzzy subgroup of a group G and e being identity in G. Then $e\mu$ and μe are respectively left and right cosets of μ in G.
For all $x \in G$, we have

$$\begin{aligned}(e\mu)(x) &= \mu(e^{-1}x) \\ &= \mu(x) = \mu(xe^{-1}) = (\mu e)(x).\end{aligned}$$

From the equality of fuzzy subsets, $e\mu = \mu = \mu e$. ∎

Theorem 3.2. *Let μ be a fuzzy subgroup of a group G and $a \in G$. Then $\mu(a) = \mu(e)$ if and only if $a\mu = \mu = \mu a$, where e being identity in G.*

Proof. Let $a \in G$ with $\mu(a) = \mu(e)$.
For all $x \in G$, we have

$$\begin{aligned} (a\mu)(x) &= \mu(a^{-1}x) \\ &\geq \min(\mu(a), \mu(x)) \\ &= \min(\mu(e), \mu(x)) \\ &= \mu(x), \text{by Lemma 2.9.} \end{aligned} \tag{3.1}$$

Again, for all $x \in G$,

$$\begin{aligned} \mu(x) &= \mu(aa^{-1}x) \\ &\geq \min(\mu(a), \mu(a^{-1}x)) \\ &= \min(\mu(e), \mu(a^{-1}x)) \\ &= \mu(a^{-1}x), \text{by Lemma 2.9.} \\ &= (a\mu)(x). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we conclude that for all $x \in G$, $(a\mu)(x) = \mu(x)$. This implies that $a\mu = \mu$.
Similarly, we can prove that $\mu = \mu a$.
Conversely, let $a\mu = \mu = \mu a$. Then

$$\mu(a) = (\mu a)(a) = \mu(aa^{-1}) = \mu(e).$$

■

Theorem 3.3. *Let μ be a fuzzy subgroup of an abelian group G, then every left coset of μ is a right coset of μ in G, i.e., $a\mu = \mu a$, for all $a \in G$.*

Proof. Let μ be a fuzzy subgroup of an abelian group G and $a \in G$ be any element. Then $a\mu$ and μa are respectively left and right cosets of μ in G.
Now, for all $x \in G$, we have

$$\begin{aligned} (a\mu)(x) &= \mu(a^{-1}x) \\ &= \mu(xa^{-1}) \\ &= (\mu a)(x). \end{aligned}$$

From the equality of fuzzy subsets, we have $a\mu = \mu a$, for all $a \in G$. ■

Theorem 3.4. *Let μ be a normal fuzzy subgroup of a group G, then every left coset of μ is a right coset of μ in G, i.e., $a\mu = \mu a$, for all $a \in G$.*

Proof. The proof is straight forward. The proof of the Theorem 3.3 can be followed. ■

Theorem 3.5. *Let λ and μ be two fuzzy subgroups of a group G and $a \in G$. Then*

$$\begin{aligned} (i)\ a\lambda &= a\mu \textit{ if and only if } \lambda = \mu. \\ (ii)\ \lambda a &= \mu a \textit{ if and only if } \lambda = \mu. \end{aligned}$$

Proof. Proof of first part (i):
Let $a\lambda = a\mu$, where $a \in G$ and let e being identity in G.

From the definition of left coset, we have for all $x \in G$,

$$\begin{aligned}\lambda(x) &= \lambda(ex)\\ &= \lambda(a^{-1}ax)\\ &= (a\lambda)(ax)\\ &= (a\mu)(ax)\\ &= \mu(a^{-1}ax) = \mu(ex) = \mu(x).\end{aligned}$$

The equality of fuzzy subsets implies that $\lambda = \mu$.
Conversely, suppose that $\lambda = \mu$. Let $a \in G$.
Now, for all $x \in G$,

$$\begin{aligned}(a\lambda)(x) &= \lambda(a^{-1}x)\\ &= \mu(a^{-1}x)\\ &= (a\mu)(x).\end{aligned}$$

Therefore $a\lambda = a\mu$.
By the similar arguments, we can prove the second part (ii). ■

The following corollary is the direct consequence of the Theorem 3.3 and the Theorem 3.5.

Corollary 3.6. *Let λ and μ be two fuzzy subgroups of an abelian group G and $a \in G$. Then $a\lambda = \mu a$ if and only if $\lambda = \mu$.*

Theorem 3.7. *Let μ be a fuzzy subgroup of a group G and $a \in G$. If $\mu(x) = \mu(a^{-1}xa)$, for all $x \in G$, then $a\mu = \mu a$.*

Proof. Let μ be a fuzzy subgroup of a group G. Let $a \in G$ and e being identity in G.
Let $\mu(x) = \mu(a^{-1}xa)$, for all $x \in G$.
From the definition of right coset, we have for all $x \in G$,

$$\begin{aligned}(\mu a)(x) &= \mu(xa^{-1})\\ &= \mu(a^{-1}(xa^{-1})a)\\ &= \mu(a^{-1}xa^{-1}a)\\ &= \mu(a^{-1}xe)\\ &= \mu(a^{-1}x)\\ &= (a\mu)(x).\end{aligned}$$

Hence $a\mu = \mu a$. ■

The necessary condition for a middle coset $a\mu b$ to be a fuzzy subgroup is given in the Theorem 2.8. In view of the Theorem 3.1, we can conclude from the Theorem 2.8 that the left and right fuzzy cosets are also fuzzy subgroups with the same condition.

Theorem 3.8. *Let μ be a fuzzy subgroup of a group G and $a, b, c, d \in G$. Then*
(i) $a(b\mu) = (ab)\mu$, if $b\mu$ is a fuzzy subgroup of G.
(ii) $(\mu a)b = \mu(ab)$, if μa is a fuzzy subgroup of G.
(iii) $(ab)\mu(cd) = a(b\mu c)d$, if $b\mu c$ is a fuzzy subgroup of G.

Proof. Proof of (i): Now, for all $x \in G$, we have

$$\begin{aligned}
[a(b\mu)](x) &= (b\mu)(a^{-1}x) \\
&= \mu(b^{-1}(a^{-1}x)) \\
&= \mu(b^{-1}a^{-1}x) \\
&= \mu((ab)^{-1}x) \\
&= [(ab)\mu](x).
\end{aligned}$$

The equality of fuzzy subsets implies $a(b\mu) = (ab)\mu$.
The proof of (ii) is same as the proof of (i).
Proof of (iii): For all $x \in G$, we have

$$\begin{aligned}
[(ab)\mu(cd)](x) &= \mu[(ab)^{-1}x(cd)^{-1}] \\
&= \mu(b^{-1}a^{-1}xd^{-1}c^{-1}) \\
&= \mu(b^{-1}(a^{-1}xd^{-1})c^{-1}) \\
&= (b\mu c)(a^{-1}xd^{-1}) \\
&= [a(b\mu c)d](x).
\end{aligned}$$

Therefore $(ab)\mu(cd) = a(b\mu c)d$. ∎

Remark 3.9. In the Theorem 3.8, the symbols $(ab)\mu$, $\mu(ab)$ and $(ab)\mu(cd)$ represent the fuzzy cosets, not the functional values of μ.

Theorem 3.10. *Let λ and μ be two fuzzy subgroups of a group G and $a, b \in G$. Then*
(i) $a\lambda = b\mu$ if and only if $\lambda = a^{-1}b\mu$.
(ii) $\lambda a = \mu b$ if and only if $\lambda = \mu b a^{-1}$.

Proof. Proof of (i): Let λ and μ be two fuzzy subgroups of a group G. Let $a, b \in G$ and e being identity in G. Suppose that $a\lambda = b\mu$.
Now, for all $x \in G$, we have

$$\begin{aligned}
(a^{-1}b\mu)(x) &= \mu((a^{-1}b)^{-1}x) \\
&= \mu(b^{-1}(ax)) \\
&= (b\mu)(ax) \\
&= (a\lambda)(ax) \\
&= \lambda(a^{-1}ax) \\
&= \lambda(ex) \\
&= \lambda(x).
\end{aligned}$$

This implies that $\lambda = a^{-1}b\mu$.

Conversely, suppose that $\lambda = a^{-1}b\mu$. Then, for all $x \in G$, we have

$$\begin{aligned} (a\lambda)(x) &= \lambda(a^{-1}x) \\ &= (a^{-1}b\mu)(a^{-1}x) \\ &= \mu((a^{-1}b)^{-1}(a^{-1}x)) \\ &= \mu(b^{-1}aa^{-1}x) \\ &= \mu(b^{-1}ex) \\ &= \mu(b^{-1}x) \\ &= (b\mu)(x). \end{aligned}$$

This implies that $a\lambda = b\mu$.
The proof of (ii) is same as the proof of (i). ■

Theorem 3.11. *Let λ and μ be two fuzzy subgroups of a group G and $a, b \in G$. Then*
(i) $a\lambda = \mu b$ if and only if $\lambda = a^{-1}\mu b$.
(ii) $a\lambda = \mu b$ if and only if $\mu = a\lambda b^{-1}$.

Proof. Proof of (i): Let λ and μ be two fuzzy subgroups of a group G. Let $a, b \in G$ and e being identity in G. Suppose that $a\lambda = \mu b$.
Now, for all $x \in G$, we have

$$\begin{aligned} (a^{-1}\mu b)(x) &= \mu((a^{-1})^{-1}xb^{-1}) \\ &= \mu((ax)b^{-1}) \\ &= (\mu b)(ax) \\ &= (a\lambda)(ax) \\ &= \lambda(a^{-1}ax) \\ &= \lambda(ex) \\ &= \lambda(x). \end{aligned}$$

Therefore $\lambda = a^{-1}\mu b$.
Conversely, let $\lambda = a^{-1}\mu b$. Then, for all $x \in G$, we have

$$\begin{aligned} (a\lambda)(x) &= \lambda(a^{-1}x) \\ &= (a^{-1}\mu b)(a^{-1}x) \\ &= \mu((a^{-1})^{-1}(a^{-1}x)b^{-1}) \\ &= \mu(aa^{-1}xb^{-1}) \\ &= \mu(exb^{-1}) \\ &= \mu(xb^{-1}) \\ &= (\mu b)(x). \end{aligned}$$

This implies that $a\lambda = \mu b$.
Similar arguments proves the second one. ■

Theorem 3.12. *Let μ be a fuzzy subgroup of a group G and $a, b, c, d \in G$. Then*

$$a\mu b = c\mu d \text{ if and only if } b^{-1}\mu a^{-1} = d^{-1}\mu c^{-1}.$$

Proof. Let $a\mu b = c\mu d$. Using the Lemma 2.10, we have, for all $x \in G$

$$\begin{aligned}(b^{-1}\mu a^{-1})(x) &= \mu(bxa)\\ &= \mu((bxa)^{-1})\\ &= \mu((a^{-1}x^{-1}b^{-1})\\ &= (a\mu b)(x^{-1})\\ &= (c\mu d)(x^{-1})\\ &= \mu(c^{-1}x^{-1}d^{-1})\\ &= \mu((dxc)^{-1})\\ &= \mu(dxc)\\ &= (d^{-1}\mu c^{-1})(x).\end{aligned}$$

This implies that $b^{-1}\mu a^{-1} = d^{-1}\mu c^{-1}$.
Conversely, suppose that $b^{-1}\mu a^{-1} = d^{-1}\mu c^{-1}$. Then by the Lemma 2.10, we have, for all $x \in G$

$$\begin{aligned}(a\mu b)(x) &= \mu(a^{-1}xb^{-1})\\ &= \mu((bx^{-1}a)^{-1})\\ &= \mu(bx^{-1}a)\\ &= (b^{-1}\mu a^{-1})(x^{-1})\\ &= (d^{-1}\mu c^{-1})(x^{-1})\\ &= \mu(dx^{-1}c)\\ &= \mu((dx^{-1}c)^{-1})\\ &= \mu(c^{-1}xd^{-1})\\ &= (c\mu d)(x).\end{aligned}$$

Therefore $a\mu b = c\mu d$. This completes the theorem. ∎

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