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On the k-Hop Domination Numbers of Spanning Trees of Unicyclic Graphs

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Abstract The work of Kundu and Majumder (Kundu and Majumder, 2016) leads to an approach to determine the k-hop domination number of a connected graph by examining the k-hop domination numbers of its spanning trees. Given this approach, a quadratic-time algorithm to compute the k-hop domination number of a unicyclic graph can be derived. In this article, we prove that the k-hop domination numbers of a unicyclic graph and its spanning trees differ by at most one, thus yielding a linear-time algorithm for finding a near-optimal k-hop dominating set with the tightly bounded error of 1.

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1. Introduction

In wireless sensor networks or mobile ad hoc networks, the successful efficient communication between nodes is often found through *cluster formation* in which certain nodes are elected as cluster heads [1–3]. A cluster head functions as a gateway for inter-cluster communication while messages from and to sensor nodes within a cluster are distributed via it. The communication between nodes in the cluster is adequately determined by the cluster head's *coverage*. That is, some nodes might not be able to communicate if they are over some *distance* from its cluster head. In graph theory, the concept of cluster formation corresponds to the notion of *domination* in *graphs* that represent the networks. A basic problem in graph domination is required to find a small subset of graph vertices called *dominating set* such that every vertex not in the set is adjacent to some vertex in the set [4]. A more generalized notion of graph domination, called *k-hop domination*,

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considers a *k-hop dominating set* in which the distance from every vertex not in the set to some vertex in the set is required to be at most some constant [5].

The mathematical formulation of domination and k-hop domination in graphs can be described as follows: Let $G = (V, E)$ be an undirected graph, where V is a set of n vertices and E is a set of m edges. A *dominating set* of G is a subset $S \subseteq V$ such that each vertex $v \in V$ is either in S or adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set S of G is *optimal* if and only if $|S| = \gamma(G)$. In this article, we consider a generalization of dominating set called *k-hop dominating set* [5]. A *k-hop dominating set* of G is a subset $D \subseteq V$ such that each vertex in V is either in D or at distance k or less from some vertex in D. Likewise, the *k-hop domination number* of G, denoted by $\gamma_k(G)$, is the smallest cardinality of a k-hop dominating set of G. A k-hop dominating set D of G is *optimal* if and only if $|D| = \gamma_k(G)$.

Applications of domination in graphs are known in several areas such as wireless sensor networks [1], mobile ad hoc networks [2, 3], warehouse and station placement [4], viral marketing in social networks [6], etc. From the algorithmic and complexity standpoint, most variants of the problem attempting to find a dominating set of smallest cardinality are known to be intractable in general graphs. In particular, for a given arbitrary graph G and integer B, the problem to decide whether $\gamma(G) \leq B$ is a classical *NP*-complete problem called Dominating Set Problem [7]. The problem to decide whether $\gamma_k(G) \leq B$ is at least as hard as the Dominating Set Problem, since every dominating set of G is a k-hop dominating set of G, for $k = 1$. Recently, there have been a number of works devoted to finding efficient algorithms for determining the k-hop domination number as well as an optimal k-hop dominating set on some restricted graph classes. In [8], Demaine et al. gave an algorithm to test whether a given planar or map graph has a k-hop dominating set of size at most p in time $O((2k+1)^{O(k\sqrt{p})}n + n^4)$. In [9], Borradaile and Le gave an algorithm that finds an optimal k-hop dominating set of a graph with bounded treewidth in time $O((2k+1)^{tw}n)$, where tw is the treewidth of the graph. In [10], Kundu and Majumder proposed a linear-time algorithm for constructing an optimal k-hop dominating set of a tree.

In addition to the linear-time algorithm given in [10], Kundu and Majumder showed that for a connected graph G, there exists a spanning tree T of G such that $\gamma_k(G) = \gamma_k(T)$. Their results imply an approach to determine the k-hop domination number of a connected graph by examining the k-hop domination numbers of its spanning trees. For this work, we extend the results to the class of *unicyclic* graphs, that is, the class of connected graphs containing exactly one cycle. Given the aforementioned approach, a quadratic-time algorithm for computing the k-hop domination number of a unicyclic graph can be derived. In this article, we consider the k-hop domination numbers of spanning trees of a unicyclic graph G. For any spanning tree T of a unicyclic graph G, we show that $\gamma_k(G) \leq \gamma_k(T) \leq \gamma_k(G)+1$. This yields a linear-time algorithm for finding a near-optimal k-hop dominating set of G with the tightly bounded error of 1.

The rest of the article is organized as follows: First, the related notations and terminology are given in Section 2. Section 3 discusses the preliminaries which includes the quadratic-time algorithm. In Section 4, we demonstrate bounds on the k-hop domination numbers of spanning trees of unicyclic graphs. Finally, the conclusions and discussions of this work are given in Section 5.

2. Notations and Terminology

Throughout this article, unless otherwise mentioned, let G be an undirected graph with n vertices and m edges. Let the vertex set of G be denoted by $V(G)$ and the edge set be denoted by $E(G)$.

2.1. Unicyclic Graph

A *path* in a graph G is a sequence of vertices $(v_1, v_2, \ldots, v_p) \in V^p$ such that $\{v_i, v_{i+1}\} \in E$, for $1 \leq i \leq p-1$. Let P be a path. The length of path P, denoted by $|P|$, is the number of vertices in the sequence minus one. Let $P(x, y, G)$ denote a path that starts from vertex $x \in V$ and ends at vertex $y \in V$ in graph G. $P(x, y, G)$ is called a *simple* path if and only if all the intermediate vertices between x and y are *distinct*. A *cycle* is a simple path that starts and ends at the same vertex. It is required that a cycle must comprise at least three vertices. A graph G is called *acyclic* if and only if it has no cycle. A graph G is called *connected* if and only if there is a path from x to y in G for all $x, y \in V$. Note that a path from x to y is not necessarily unique. A graph G is called a *tree* if and only if G is connected, and $P(x, y, G)$ is unique for all $x, y \in V$. In other words, a tree is a connected acyclic graph. A graph G is called a *unicyclic graph* if and only if G is connected and contains exactly one cycle.

2.2. k-Hop Domination Number

Let $dist(x, y, G)$ denote the *distance* or the length of a shortest path from vertex x to vertex y in graph G, that is, $dist(x, y, G) = min\{|P(x, y, G)|\}$. For a vertex $x \in V$ and a subset $U \subseteq V$, let $dist(x, U, G)$ denote the minimum distance from vertex x to some vertex $y \in U$ in G, that is, $dist(x, U, G) = min_{y \in U}\{dist(x, y, G)\}$. A *$k$-hop dominating set* of G is a subset $D \subseteq V$ such that each vertex $u \in V$ is either in D or $dist(u, D, G) \leq k$. Let D be a k-hop dominating set of G. A vertex in D is called a *dominator* of G. For a vertex $x \in V$ and a subset $S \subseteq D$, let $dom(x, S, G)$ denote the set of dominator(s) in S for x in graph G, that is, $dom(x, S, G) = \{y \in S \mid dist(x, y, G) \leq k\}$. It is possible that $x \in dom(x, S, G)$. If $y \in dom(x, S, G)$, we say that the vertex x is *dominated* by node $y \in S$. Let $dom(S, G)$ denote the set of vertices dominated by nodes in S, that is, $dom(S, G) = \{u \in V \mid dom(u, S, G) \neq \emptyset\}$. A subset $S \subseteq V$ is a k-hop dominating set if and only if $dom(S, G) = V$, and $dom(x, S, G) \neq \emptyset$ for each $x \in V$. The *k-hop domination number* of G, denoted by $\gamma_k(G)$, is the number of vertices in a k-hop dominating set of smallest cardinality of G. A k-hop dominating set D of G is *optimal* if and only if $|D| = \gamma_k(G)$. An example of an optimal k-hop dominating set of a unicyclic graph is illustrated in Figure 1.

3. Preliminaries

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$. A *spanning* subgraph of G is a subgraph $G' = (V, E')$ such that $E' \subseteq E$. If D is a k-hop dominating set of a spanning subgraph G', then D is also a k-hop dominating set of G. Thus, $\gamma_k(G) \leq \gamma_k(G')$. A spanning subgraph T of G is a *spanning tree* if and only if T is a tree. For any spanning tree T of G, it holds that $\gamma_k(G) \leq \gamma_k(T)$. Kundu and Majumder demonstrated the following results in [10].

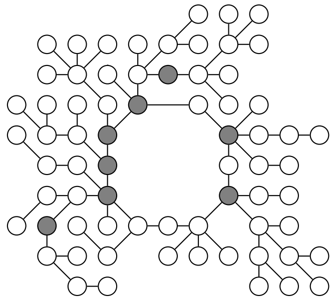

FIGURE 1. Illustration of a unicyclic graph G of 69 vertices with $\gamma_3(G) = 8$. The gray vertices are the dominators in an optimal 3-hop dominating set of G.

Theorem 3.1. ([10, Theorem 6]) *The algorithm of Kundu and Majumder can compute an optimal k-hop dominating set of a tree of n vertinces in $O(n)$ time.*

Theorem 3.2. ([10, Theorem 1]) *For each optimal k-hop dominating set D of a connected graph G, there exists a spanning tree T of G such that D is also an optimal k-hop dominating set D of T.*

The following corollary immediately follows from Theorem 3.2.

Corollary 3.3. *For a connected graph G, there exists a spanning tree T of G such that $\gamma_k(T) = \gamma_k(G)$.*

Figure 2 gives an illustration of Corollary 3.3. By Theorem 3.2, one can determine the k-hop domination number of a connected graph by examining the k-hop domination numbers of its spanning trees.

Theorem 3.4. *Let G be a connected graph of n vertices. Let $l(n)$ denote the number of unique spanning trees of G. We can compute the k-hop domination number of G in $\Omega(l(n)n)$ time.*

Typically, the number of unique spanning trees of a graph may vary that it can even grow beyond polynomial for some graphs (for instance, complete graphs). Thus, any algorithm whose computation is based on examining the k-hop domination numbers of spanning trees would have worst-case exponential running time in general. For the case of a unicyclic graph of n vertices, the computation requires just quadratic time.

Proposition 3.5. *A unicyclic graph of n vertices has at most n possible unique spanning trees.*

Lemma 3.6. *Let G be a unicyclic graph of n vertices. At most n possible unique spanning trees of G can be constructed in $O(n^2)$ time; each unique spanning tree can be constructed in $O(n)$ time.*

Proof. Since G contains exactly one cycle, at most $n-1$ edges of G can be tree edges. Let C be a cycle in G. We can always find an edge in $E(C)$ in $O(n)$ time. Let $\{x, y\} \in E(C)$ be such an edge. Since C is a cycle, there are two paths that starts from x and ends at y; the edge $\{x, y\}$ itself is a path of length 1. The edge $\{x, y\}$ and all of the edges along the longer path from x to y are all the edges of $E(C)$. To find the longer path from x to y, we can use Depth-First-Search (DFS). On the graph of n vertices and n edges, the DFS runs in $O(n)$ time.

To construct all of the possible unique spanning trees of G, we make at most n copies of graph G, one copy to construct each unique spanning tree. Let T_i be the i-th copy of G. For each $e_i \in E(C)$, we remove e_i from $E(T_i)$, each edge removal makes T_i a unique spanning tree of G. The time taken for making each copy is $O(n)$ time. Hence, the time taken by the entire construction is $O(n^2)$. ∎

Theorem 3.7 (Quadratic-time Algorithm for Unicyclic Graphs)**.** *There exists an algorithm that can compute the k-hop domination number of a unicyclic graph G of n vertices in $O(n^2)$ time.*

4. Bounds on the k-Hop Domination Numbers of Spanning Trees of Unicyclic Graphs

Let G be a unicyclic graph of n vertices. The goal of this section is to show that for any spanning tree T of G, $\gamma_k(G) \leq \gamma_k(T) \leq \gamma_k(G) + 1$. The lower bound of the inequalities is due to the fact that any spanning tree T of G is a spanning subgraph of G, thus $\gamma_k(G) \leq \gamma_k(T)$. Before discussing the upper bound, it is worthwhile to observe the k-hop domination numbers of spanning trees of G for the case when G is a cycle.

Proposition 4.1. *If G is a cycle of n vertices, then for any spanning tree T of G, $\gamma_k(T) = \gamma_k(G) = \lceil n/(2k+1) \rceil$.*

To provide the upper bound, we assume a spanning tree T of a unicyclic graph G such that $\gamma_k(T) = \gamma_k(G)$ by Corollary 3.3. Then, there exists an optimal k-hop dominating set D of T such that $|D| = \gamma_k(T)$. Let T' be an arbitrary spanning tree of G, $T' \neq T$. Since $\gamma_k(T) = \gamma_k(G)$, $|D| = \gamma_k(G) \leq \gamma_k(T')$. Let $b = \gamma_k(T') - \gamma_k(G)$. The extreme case where $b \leq n$ is trivial. To give a tighter upper bound, we will demonstrate the construction of a k-hop dominating set D' of T' such that $D' \supseteq D$ and $|D'| \leq |D| + 2$. This will imply $\gamma_k(T') \leq |D'| \leq \gamma_k(T) + 2$, thus $b \leq 2$. Nevertheless, our objective is to prove that $b \leq 1$. Assume $|D'| = |D| + 2$, we will subsequently show that $|D'|$ is not optimal. Therefore, it must be the case that $|D'| \leq |D| + 1$. This gives us $\gamma_k(T') \leq |D'| \leq \gamma_k(T) + 1$, and so $b \leq 1$. Hence, $\gamma_k(G) \leq \gamma_k(T') \leq \gamma_k(G) + 1$.

For the sake of the following discussions, we shall give an explicit construction of T' from T. Let us assume an optimal k-hop dominating set D of T. Since G is unicyclic, T' differs from T in one edge, and vice versa. Let $\{x, y\}$ be an edge in $E(T)$, but not in $E(T')$; and $\{x', y'\}$ an edge in $E(T')$, but not in $E(T)$. Clearly, both $\{x, y\}$ and $\{x', y'\}$ must be the edges of the cycle in G. Removing edge $\{x, y\}$ from T would result in disconnecting T into two subtrees. Let T_x denote such the resulting subtree which includes vertex x, and T_y the resulting subtree that includes vertex y. The vertex sets $V(T_x)$ and $V(T_y)$ are disjoint and the pair $(V(T_x), V(T_y))$ forms a partition of V. The spanning tree T' is the result of connecting the subtrees T_x and T_y with the edge $\{x', y'\}$. So, the construction of T' is done by $E(T') = (E(T) \setminus \{\{x, y\}\}) \cup \{\{x', y'\}\}$. For additional notations, for a vertex u, we will use $D(u, T \cap T_x) = dom(u, D, T) \cap V(T_x)$ to denote a set of dominator(s) for u in D that are also the vertices of subtree T_x. A set of dominator(s) in D that are also the vertices of subtree T_x is denoted by $D(T \cap T_x) = D \cap V(T_x) = \bigcup_{u \in V} D(u, T \cap T_x)$. The pair $(D(u, T \cap T_x), (D(u, T \cap T_y))$ forms a partition of $dom(u, D, T)$, and the pair $(D(T \cap T_y), (D(T \cap T_x))$ forms a partition of D.

Lemma 4.2. *There exists a k-hop dominating set D' of T' such that $D' \supseteq D$ and $|D'| \leq |D| + 2$. Furthermore, $|D'| = |D| + 2$ if and only if $x, y \notin D$.*

Proof. To prove the lemma, it suffices to show that for each $u \in V$, if $u \notin D$, then $dist(u, D, T') \leq k$ or $dist(u, x, T') \leq k$ or $dist(u, y, T') \leq k$. This implies the existence of a k-hop dominating set D' of T' with $D' = D \cup \{x, y\} \supseteq D$, thus $|D'| \leq |D| + 2$.

Let u be an arbitrary vertex in V. Then, either $u \in V(T_x)$ or $u \in V(T_y)$. Since D is a k-hop dominating set of T, there exists a dominator $d \in D$ such that $dist(u, d, T) \leq k$, and either $d \in D(u, T \cap T_x)$ or $d \in D(u, T \cap T_y)$. Suppose $u \notin D$. Consider the following cases:

Case 1: If $u \in V(T_x)$ and $d \in D(u, T \cap T_x)$, then there exists path $P(u, d, T)$ with $|P(u, d, T)| = dist(u, d, T) \leq k$. Since $P(u, d, T)$ is a path from $u \in V(T_x)$ to $d \in V(T_x)$, $P(u, d, T_x) = P(u, d, T)$. Moreover, since T_x is a subtree of T', $P(u, d, T') = P(u, d, T_x)$. It follows that $|P(u, d, T')| = |P(u, d, T_x)| = |P(u, d, T)| \leq k$. Therefore, $dist(u, D, T') \leq dist(u, d, T') \leq k$.

Case 2: If $u \in V(T_x)$ and $d \in D(u, T \cap T_y)$, then there exists path $P(u, d, T)$ with $|P(u, d, T)| = dist(u, d, T) \leq k$. Since $P(u, d, T)$ is a path from $u \in V(T_x)$ to $d \in V(T_y)$, it must include both x and y. So, $P(u, d, T) = P(u, x, T) \cdot (x, y) \cdot P(y, d, T)$ with $|P(u, d, T)| = |P(u, x, T)| + 1 + |P(y, d, T)| \leq k$, thus $|P(u, x, T)| < k$. Since $P(u, x, T)$ is a path from $u \in V(T_x)$ to $x \in V(T_x)$, $P(u, x, T_x) = P(u, x, T)$. Moreover, since T_x is a subtree of T', $|P(u, x, T')| = |P(u, x, T_x)| = |P(u, x, T)| < k$. Therefore, $dist(u, x, T') \leq |P(u, x, T')| < k$.

Case 3: If $u \in V(T_y)$ and $d \in D(u, T \cap T_x)$, then this case is similar to Case 2. We have that $dist(u, y, T') \leq |P(u, y, T')| < k$.

Case 4: If $u \in V(T_y)$ and $d \in D(u, T \cap T_y)$, then this case is similar to Case 1. We have that $dist(u, D, T') \leq dist(u, d, T') \leq k$.

Given all the four cases above, we conclude that for each $u \in V$, if $u \notin D$, then $dist(u, D, T') \leq k$ or $dist(u, x, T') \leq k$ or $dist(u, y, T') \leq k$. Hence, $D' = D \cup \{x, y\}$ is a k-hop dominating set D' of T'. Furthermore, $|D'| = |D|$ if and only if $x \in D$ and $y \in D$; $|D'| = |D| + 1$ if and only if either $x \in D$ or $y \in D$; and $|D'| = |D| + 2$ if and only if $x \notin D$ and $y \notin D$. The lemma holds. ∎

Observe that if $\gamma_k(T_x) + \gamma_k(T_y) = \gamma_k(T)$, then $\gamma_k(T') = \gamma_k(T)$. Moreover, if $D(T \cap T_x)$ is a k-hop dominating set of T_x and $D(T \cap T_y)$ is a k-hop dominating set of T_y, then $|D(T \cap T_x)| \geq \gamma_k(T_x)$ and $|D(T \cap T_y)| \geq \gamma_k(T_y)$, and so $|D(T \cap T_x)| + |D(T \cap T_y)| = |D| \geq \gamma_k(T')$. In the following lemma, we show that it holds that if $|D(T \cap T_x)| < \gamma_k(T_x)$, then $|D(T \cap T_y)| \geq \gamma_k(T_y)$, and vice versa.

Lemma 4.3. *If $D(T \cap T_x)$ is not a k-hop dominating set of T_x, then $D(T \cap T_y)$ is a k-hop dominating set of T_y.*

Proof. Suppose $D(T \cap T_x)$ is not a k-hop dominating set of T_x. Then, there exists a vertex $u \in V(T_x)$ such that $u \notin D(T \cap T_x)$ and $dist(u, D(T \cap T_x), T_x) > k$. We now prove the lemma by contradiction.

Suppose, to the contrary, that $D(T \cap T_y)$ is not a k-hop dominating set of T_y. Then, there exists a vertex $v \in V(T_y)$ such that $v \notin D(T \cap T_y)$ and $dist(v, D(T \cap T_y), T_y) > k$. Note that $dist(u, D(T \cap T_x), T_x) + dist(v, D(T \cap T_y), T_y) > 2k$. Since D is a k-hop dominating set of T, there exist a dominator $d_u \in D(u, T \cap T_y)$ with $dist(u, d_u, T) \leq k$, and a dominator $d_v \in D(u, T \cap T_x)$ with $dist(v, d_v, T) \leq k$. Accordingly, there are paths

$P(u, d_u, T)$ with $|P(u, d_u, T)| = dist(u, d_u, T) \leq k$, and $P(v, d_v, T)$ with $|P(v, d_v, T)| = dist(v, d_v, T) \leq k$. Furthermore, $P(u, d_u, T)$ is a path from $u \in V(T_x)$ to $d_u \in V(T_y)$, and $P(v, d_v, T)$ is a path from $v \in V(T_y)$ to $d_v \in V(T_x)$, both paths must include x and y. So, $P(u, d_u, T) = P(u, x, T_x) \cdot (x, y) \cdot P(y, d_u, T_y)$ with $|P(u, d_u, T)| = |P(u, x, T_x)| + 1 + |P(y, d_u, T_y)| \leq k$, and $P(v, d_v, T) = P(v, y, T_y) \cdot (y, x) \cdot P(x, d_v, T_x)$ with $|P(v, d_v, T)| = |P(v, y, T_y)| + 1 + |P(x, d_v, T_x)| \leq k$. Let $q = |P(u, x, T_x)|$ and $r = |P(v, y, T_y)|$. Then, $|P(y, d_u, T_y)| \leq k - 1 - q < k - q$ and $|P(x, d_v, T_x)| \leq k - 1 - r < k - r$. Consider paths $P(u, d_v, T_x) = P(u, x, T_x) \cdot P(x, d_v, T_x)$ and $P(v, d_u, T_y) = P(v, y, T_y) \cdot P(y, d_u, T_y)$. We have that $|P(u, d_v, T_x)| = |P(u, x, T_x)| + |P(x, d_v, T_x)| < q + k - r$, and $|P(v, d_u, T_y)| = |P(v, y, T_y)| + |P(y, d_u, T_y)| < r + k - q$. It follows that $dist(u, D(T \cap T_x), T_x) + dist(v, D(T \cap T_y), T_y) \leq dist(u, d_v, T_x) + dist(v, d_u, T_y) < (q+k-r) + (r+k-q) < 2k$. This contradics the fact that $dist(u, d_v, T_x) + dist(v, d_u, T_y) \geq dist(u, D(T \cap T_x), T_x) + dist(v, D(T \cap T_y), T_y) > 2k$. Therefore, $D(T \cap T_y)$ is a k-hop dominating set of T_y. ■

Lemma 4.4. *If $|D'| = |D| + 2$, then D' is not optimal.*

Proof. Suppose $|D'| = |D| + 2$. By Lemma 4.2, $x \in D'$ and $y \in D'$; but $x \notin D$ and $y \notin D$. Consider $D(T \cap T_x)$ and $D(T \cap T_y)$. Clearly, if $D(T \cap T_x)$ is a k-hop dominating set of T_x and $D(T \cap T_y)$ is a k-hop dominating set of T_y, then $D(T \cap T_x) \cup D(T \cap T_y) = D$ is a k-hop dominating set of T'. If this is the case, then D' is not optimal, since $|D| < |D'|$. Without loss of generality, we assume that $D(T \cap T_x)$ is not a k-hop dominating set of T_x. Then, by Lemma 4.3, $D(T \cap T_y)$ is a k-hop dominating set of T_y. Let u be an arbitrary vertex in $V(T_x)$ such that $u \notin D \cap V(T_x)$ and $dist(u, D(T \cap T_x), T_x) > k$. Since D is a k-hop dominating set of T, there exists a dominator $d \in D(u, T \cap T_y)$ with $dist(u, d, T) \leq k$. By Case 2 in the proof of Lemma 4.2, we have that $dist(u, x, T_x) = |P(u, x, T_x)| < k$. This implies $D(T \cap T_x) \cup \{x\}$ is a k-hop dominating set of T_x. Furthermore, $(D(T \cap T_x) \cup \{x\}) \cup D(T \cap T_y) = D \cup \{x\}$ is a k-hop dominating set of T'. Hence, D' is not optimal, since $|D \cup \{x\}| = |D + 1| < |D'|$. ■

Lemma 4.5. *There exists a k-hop dominating set D' of T' such that $D' \supseteq D$ and $|D'| \leq |D| + 1$. Furthermore, $|D'| = |D| + 1$ if and only if either $x \in D$ or $y \in D$.*

Theorem 4.6. *For any spanning tree T of a unicyclic graph G, $\gamma_k(G) \leq \gamma_k(T) \leq \gamma_k(G) + 1$.*

An example of a spanning tree T' of a unicyclic graph G with$\gamma_k(T') = \gamma_k(G) + 1$ is shown in Figure 2. Note that the existence of such spanning tree has provided that the bound $b \leq 1$ is definitely tight.

Theorem 4.7. *There exists an algorithm that finds a k-hop dominating set D of a unicyclic graph G of n vertices with $|D| \leq \gamma_k(G) + 1$ in $O(n)$ time.*

Proof. By Lemma 3.6, the algorithm constructs a spanning tree T of G in $O(n)$ time. It then uses the Kundu-Majumder algorithm as subroutine to compute an optimal k-hop dominating set D of T in $O(n)$ time. The error bound is guaranteed by Theorem 4.6. ■

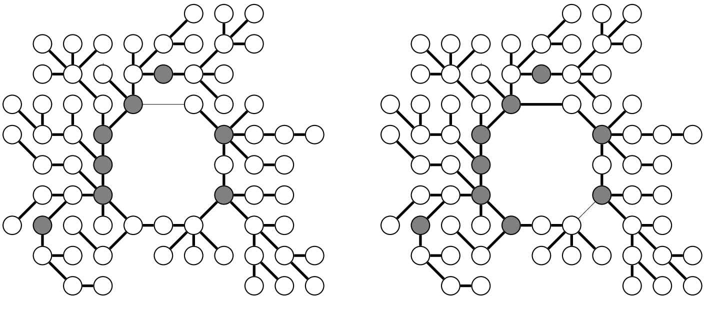

FIGURE 2. Illustration of two spanning trees T and T' of the uncyclic graph G corresponding to Figure 1. with $\gamma_3(T) = 8 = \gamma_3(G)$ and $\gamma_3(T') = 9 = \gamma_3(G) + 1$. The dominators in an optimal 3-hop dominating set of both trees are represented by the gray vertices. The bold edges are the edges of the trees.

5. Conclusions and Discussions

This article extends the work of Kundu and Majumder [10] to the class of unicyclic graphs. Our main contribution presents the relationship among the k-hop domination numbers of a unicyclic graph and its spanning trees. Indeed, we have shown that the numbers can differ by at most one. This in turn yields a linear-time algorithm for determining a near-optimal k-hop dominating set of a unicyclic graph with the tightly bounded error of 1. We notice that the algorithm by Borradaile and Le [9] that gives an optimal k-hop dominating set can run in linear time in the case where the input graph is unicyclic. Nonetheless, their algorithm is based on dynamic programming over tree decompositions, which is quite complicated to implement. The linear-time algorithm by Theorem 4.7, though produces a near-optimal k-hop dominating set, can be suggested as an alternative because of its simple implementation (see [10, p. 201] for implementation details). For future work, we plan to investigate possibilities to further extend the results to more interesting graph classes such as cactus graphs, block graphs, etc.

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