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Three-step iterative convergence theorems with errors for set-valued mappings in Banach spaces [1]

Y. C. Lin

Abstract : Let $q > 1$ and E be a real q-uniformly smooth Banach space, K be a nonempty closed convex subset of E and $T : K \to 2^K$ be a set-valued mapping. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying some restrictions. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iteration process with errors: $x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n Ty_n + u_n$, $y_n \in (1 - \beta_n)x_n + \beta_n Tz_n + v_n$, $z_n \in (1 - \gamma_n)x_n + \gamma_n Tx_n + w_n, n \geq 1$. Sufficient and necessary conditions for the strong convergence $\{x_n\}$ to a fixed point of T are established. We also derive the corresponding new results on the strong convergence of the three-step iterative process with set-valued version and application to approximate the solution of the inclusion $f \in Tx$ in K.

Keywords : Fixed point, Lipschitzian, (strongly) accretive, (strictly, strongly) pseudocontractive, three-steps iteration process (with errors), Ishikawa (Mann) iteration process (with errors), $q-$uniformly smooth Banach space.
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1 Introduction and Preliminaries

Let E be an arbitrary real Banach space and let $J_q(q > 1)$ denote the generalized duality mapping from E into $2^{E^\star}$ given by

$$J_q(x) = \{f \in E^\star : \langle x, f\rangle = \|x\|^q = \|x\|\|f\|\},$$

where $E^\star$ denote the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and $E^\star$. In particular, J_2 is called the normalized duality mapping and it is usually denote by J. It is well known (see [12]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if $E^\star$ is strictly convex then J_q is single-valued. The single-valued generalized duality mapping will be denoted by j_q in the sequel.

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Definition 1.1. Let E be a normed space, K be a nonempty subset of E. $T : K \to 2^E$ is a set-valued mapping.

(i) T is said to be Lipschitzian mapping with constant L if $\forall x,\ y \in K$, $\forall u \in Tx$, $\forall v \in Ty$,

$$\|u - v\| \leq L\|x - y\|. \tag{1.1}$$

(ii) T is said to be accretive [6] if $\forall x,\ y \in K$, $\forall u \in Tx$ and $\forall v \in Ty$, there exists $j_2(x - y) \in J_2(x - y)$ such that

$$\langle u - v, j_2(x - y)\rangle \geq 0,$$

or, equivalently, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0. \tag{1.2}$$

(iii) T is said to be strongly accretive [6] if $\forall x,\ y \in K$, $\forall u \in Tx$ and $\forall v \in Ty$, there exists $j_2(x - y) \in J_2(x - y)$ such that

$$\langle u - v, j_2(x - y)\rangle \geq k\|x - y\|^2,$$

or, equivalently, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq k\|x - y\|^q, \tag{1.3}$$

for some $k > 0$. Without loss of generality, we can assume that $k \in (0, 1)$ and such a number k is called the strong accretive constant of T.

(iv) T is said to be (strongly) pseudocontractive [6] if $I - T$ (where I denotes the identity mapping) is a (strongly) accretive mapping. That is, $\forall x,\ y \in K$, $\forall u \in (I - T)x$ and $\forall v \in (I - T)y$, there exists $j_2(x - y) \in J_2(x - y)$ such that

$$\langle u - v, j_2(x - y)\rangle \geq 0 \text{ (resp.}, \langle u - v, j_2(x - y)\rangle \geq k\|x - y\|^2),$$

or, equivalently, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0 \text{ (resp.}, \langle u - v, j_q(x - y)\rangle \geq k\|x - y\|^q). \tag{1.4}$$

The constant k is said to be a strongly accretive constant with respect to $I - T$.

(v) T is said to be strictly pseudocontractive if $\forall x,\ y \in K$, $\forall u \in Tx$ and $\forall v \in Ty$, there exists $\lambda > 0$ and $j_2(x - y) \in J_2(x - y)$ such that

$$\langle u - v, j_2(x - y)\rangle \leq \|x - y\|^2 - \lambda\|(x - u) - (y - v)\|^2, \tag{1.5}$$

or, equivalently, there exists $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \leq \|x - y\|^q - \lambda\|(x - u) - (y - v)\|^2\|x - y\|^{q-2}.$$

We note that the strictly pseudocontractive single-valued mapping has been discussed in [1, 13]. Without loss of generality we may assume $\lambda \in (0,1)$, then (1.5) can be written in the form

$$\langle (x-u)-(y-v), j(x-y)\rangle \geq \lambda \|(x-u)-(y-v)\|^2. \tag{1.6}$$

From (1.6), we have

$$\|x-y\| \geq \lambda \|x-y-(u-v)\| \geq \lambda(\|u-v\| - \|x-y\|),$$

so that

$$\|u-v\| \leq \frac{(1+\lambda)}{\lambda}\|x-y\|,$$

$\forall x,\ y \in K$ and $\forall u \in Tx,\ \forall v \in Ty$. Hence a strictly pseudocontractive mapping is also a Lipschitzian mapping.

In 1967, the concept of a single-valued accretive mapping was introduced by Browder and Kato independently (we refer to [6]). Browder stated that the following initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0,\ u(0) = u_0, \tag{1.7}$$

is solvable if T is locally Lipschitzian and accretive on E.

In Hilbert spaces, (1.5) (hence,(1.6)) is equivalent to the following inequality

$$\|u-v\|^2 \leq \|x-y\|^2 + k\|(x-u)-(y-v)\|^2,\ k = (1-\lambda) < 1.$$

It is well known that the Mann iterative process (with errors) and the Ishikawa iterative process (with errors) have been extensively applied to approximate the solutions of nonlinear operator equations or fixed points of nonlinear mappings in Hilbert spaces or Banach spaces in the literature. See, e.g.,[4-11]. In 1974, Rhoades [10, Theorem 1.1] proved the iterative convergence theorem using the Mann iterative process with single-valued operator T. Noor[2] has suggested and analyzed three-step iterative methods for finding the approximate solutions of the variational inequalities in a Hilbert space.

Recently, Osilike and Udomene [13] improved, unified and developed the results [10, Theorem 1.1] and Browder and Petryshyn's corresponding results [1] in two aspects: (i) Hilbert spaces are extended to the setting of $q-$uniformly smooth Banach spaces ($q > 1$); (ii) Mann iterative process is extended to the case of Ishikawa iterative process with single-valued operator T.

Let E be a real $q-$uniformly smooth Banach space with $q > 1$, K be a nonempty closed convex (not necessarily bounded) subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a set-valued mapping with $F(T) \neq \emptyset$. We note that if $K + K \subseteq K$ and K contains a point $x \neq 0$ then K is unbounded. Let $\{u_n\},\ \{v_n\}$ and $\{w_n\}$ be three sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$be real sequences in $[0,1]$ satisfying certain restrictions. Let $\{x_n\}$ be the sequence generated from $x_1 \in K$ by the three-step iterative process with errors:

$$\begin{cases} z_n \in (1-\gamma_n)x_n + \gamma_n T x_n + w_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T z_n + v_n, \\ x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \ n \geq 1. \end{cases} \tag{1.8}$$

Especially, if $u_n = 0$, $v_n = 0$ and $w_n = 0$ for $n \in N$, then $\{x_n\}$ is called the three-step iterative sequence which was suggested and analyzed by Noor [3] with T is a single-valued mapping and by Noor[2] and Lin[14] with T is a set-valued mapping, if $\gamma_n = 0$ and $w_n = 0$ for $n \in N$, then $\{x_n\}$ is called the Ishikawa iterative sequence with error which discussed by Liu[4], Zeng et. al.[16] and Lin et. al.[15] with T is a single-valued mapping; if $\gamma_n = \beta_n = 0$ and $v_n = w_n = 0$ for $n \in N$, then $\{x_n\}$ is called the Mann iterative sequence with error.

In this paper, by using Jensen's inequality and new approximation methods, we construct some simplify conditions to establish the sufficient and necessary conditions for the strong convergence of $\{x_n\}$ to a fixed point of T. The uniqueness of the fixed point of T is also discussed.

It is worth noting that comparing with [3, Theorem 2.1 and Theorem 2.2] our results have the following features: (i) The domain of the mapping T is restricted to a nonempty convex subset K with $K + K \subset K$ but not the whole space E. (ii) The single-version of the three iterative process is replaced by the set-version of the three-step iterative process with errors. (iii) Our restrictions imposed on $\{\alpha_n\}$ are different from those in [3, Theorem 2.1 and Theorem 2.2]. (iv) We establish the sufficient and necessary conditions on the strong convergence of the three-step iterative process with errors. Furthermore, our results also improve and extend the corresponding results in [1, 10, 15, 16]. As an application, we also derive the strongly convergent results to approximate the solution of the inclusion $f \in Tx$ in K.

Now, we give some preliminaries which will be used in the sequel. Let E be a real Banach space. The modulus of smoothness of E is defined as the function $\rho_E : [0,\infty) \to [0,\infty)$:

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}.$$

E is said to be uniformly smooth if and only if $\lim_{\tau \to 0_+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be $q-$uniformly smooth (or to have a modulus of smoothness of power type $q > 1$), if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. It is well known that Hilbert spaces, L_p and l_p spaces, $1 < p < \infty$, as well as the Sobolev spaces, $W^p_m, 1 < p < \infty$, are $p-$uniformly smooth. Hilbert spaces are 2-uniformly smooth while if $1 < p \leq 2$, L_p, l_p and W^p_m are $p-$uniformly smooth. If $p \geq 2$, L_p, l_p and W^p_m are 2-uniformly smooth.

Theorem 1.1 [12]. Let $q > 1$ and E be a real Banach space. Then the following are equivalent:

(1) E is $q-$uniformly smooth.

(2) There exists a constant $c_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{1.9}$$

Lemma 1.1 [11]. Let$\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $a_{n+1} \leq a_n + b_n, \forall n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

2 Main Results

Throughout this section, L stands for the Lipschitzian constant of T, λ and c_q are the constants appearing in inequalities (1.5), (1.6), (1.9), respectively.

Lemma 2.1. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then

$$\|x_{n+1} - x^\star\|^q \leq (1 + \delta_n)\|x_n - x^\star\|^q + \theta_n, \forall n \geq 1, \forall x^\star \in F(T), \tag{2.1}$$

where

$$\delta_n = -\alpha_n + \alpha_n(1 - \beta_n)L^q + \alpha_n\beta_n(1 - \gamma_n)L^{2q} + \alpha_n\beta_n\gamma_n L^{3q},$$

and

$$\begin{aligned}\theta_n \;=\; & q\alpha_n\beta_n L^{2q}\|w_n\|\|z_n - x^\star - w_n\|^{q-1} + \alpha_n\beta_n L^{2q}c_q\|w_n\|^q \\ & +q\alpha_n L^q\|v_n\|\|y_n - v_n - x^\star\|^{q-1} + \alpha_n L^q c_q\|v_n\|^q + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q.\end{aligned}$$

Proof. Let $x^\star$ be an arbitrary element in $F(T)$. Then it follows from (1.8) and (1.9) that, for some $\sigma_n \in Ty_n$,

$$\begin{aligned}\|x_{n+1} - x^\star\|^q &= \|(1 - \alpha_n)x_n + \alpha_n\sigma_n + u_n - x^\star\|^q \\ &\leq \|(1 - \alpha_n)x_n + \alpha_n\sigma_n - x^\star\|^q + q\langle u_n, j_q(x_{n+1} - u_n - x^\star)\rangle + c_q\|u_n\|^q \\ &\leq \|(1 - \alpha_n)x_n + \alpha_n\sigma_n - x^\star\|^q + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q. \end{aligned} \tag{2.2}$$

By Jensen's inequality, we have

$$\begin{aligned}\|(1-\alpha_n)x_n+\alpha_n\sigma_n-x^\star\|^q &= \|(1-\alpha_n)(x_n-x^\star)+\alpha_n(\sigma_n-x^\star)\|^q\\ &\leq (1-\alpha_n)\|x_n-x^\star\|^q+\alpha_n\|\sigma_n-x^\star\|^q\\ &\leq (1-\alpha_n)\|x_n-x^\star\|^q+\alpha_n L^q\|y_n-x^\star\|^q, \qquad (2.3)\end{aligned}$$

and, by (1.9) and Jensen's inequality, for some $\eta_n \in Tz_n$, we have

$$\begin{aligned}\|y_n-x^\star\|^q &= \|(1-\beta_n)x_n+\beta_n\eta_n+v_n-x^\star\|^q\\ &= \|(1-\beta_n)(x_n-x^\star)+\beta_n(\eta_n-x^\star)+v_n\|^q\\ &\leq \|(1-\beta_n)(x_n-x^\star)+\beta_n(\eta_n-x^\star)\|^q+q\langle v_n, j_q(y_n-v_n-x^\star)\rangle+c_q\|v_n\|^q\\ &\leq (1-\beta_n)\|x_n-x^\star\|^q+\beta_n L^q\|z_n-x^\star\|^q+q\|v_n\|\|y_n-v_n-x^\star\|^{q-1}\\ &\quad +c_q\|v_n\|^q, \qquad (2.4)\end{aligned}$$

and, for some $\nu_n \in Tx_n$, we have

$$\begin{aligned}\|z_n-x^\star\|^q &= \|(1-\gamma_n)x_n+\gamma_n\nu_n+w_n-x^\star\|^q\\ &= \|(1-\gamma_n)(x_n-x^\star)+\gamma_n(\nu_n-x^\star)+w_n\|^q\\ &\leq \|(1-\gamma_n)(x_n-x^\star)+\gamma_n(\nu_n-x^\star)\|^q+q\langle w_n, j_q(z_n-w_n-x^\star)\rangle+c_q\|w_n\|^q\\ &\leq (1-\gamma_n)\|x_n-x^\star\|^q+\gamma_n L^q\|x_n-x^\star\|^q+q\|w_n\|\|z_n-w_n-x^\star\|^{q-1}\\ &\quad +c_q\|w_n\|^q. \qquad (2.5)\end{aligned}$$

Hence, form (2.4) and (2.5), we have

$$\begin{aligned}\|y_n-x^\star\|^q &\leq (1-\beta_n)\|x_n-x^\star\|^q+\beta_n L^q\{(1-\gamma_n)\|x_n-x^\star\|^q+\gamma_n L^q\|x_n-x^\star\|^q\\ &\quad +q\|w_n\|\|z_n-w_n-x^\star\|^{q-1}+c_q\|w_n\|^q\}+q\|v_n\|\|y_n-v_n-x^\star\|^{q-1}+c_q\|v_n\|^q\\ &= \{1-\beta_n+\beta_n L^q(1-\gamma_n)+\beta_n\gamma_n L^{2q}\}\|x_n-x^\star\|^q\\ &\quad +q\beta_n L^q\|w_n\|\|z_n-x^\star-w_n\|^{q-1}+\beta_n L^q c_q\|w_n\|^q\\ &\quad +q\|v_n\|\|y_n-v_n-x^\star\|^{q-1}+c_q\|v_n\|^q, \qquad (2.6)\end{aligned}$$

Combine (2.2)–(2.6), we derive that

$$\|x_{n+1}-x^\star\|^q \leq (1+\delta_n)\|x_n-x^\star\|^q+\theta_n$$

where

$$\delta_n = -\alpha_n+\alpha_n(1-\beta_n)L^q+\alpha_n\beta_n(1-\gamma_n)L^{2q}+\alpha_n\beta_n\gamma_n L^{3q},$$

and

$$\begin{aligned}\theta_n &= q\alpha_n\beta_n L^{2q}\|w_n\|\|z_n-x^\star-w_n\|^{q-1}+\alpha_n\beta_n L^{2q}c_q\|w_n\|^q+q\alpha_n L^q\|v_n\|\|y_n-v_n-x^\star\|^{q-1}\\ &\quad +\alpha_n L^q c_q\|v_n\|^q+q\|u_n\|\|x_{n+1}-u_n-x^\star\|^{q-1}+c_q\|u_n\|^q.\end{aligned}$$

Lemma 2.2. Let E be a real $q-$uniformly smooth Banach space with $q>1$ and K be a nonempty convex subset of E with $K+K \subseteq K$, and $T : K \to 2^K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Furthermore, if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then there is a constant $M > 0$ (e.g. $M = e^{\sum_{n=1}^{\infty} \delta_n}$) such that

$$\|x_{n+m} - x^\star\|^q \leq M\|x_n - x^\star\|^q + M(\sum_{k=n}^{n+m-1} \theta_n), \tag{2.7}$$

$\forall m,\ n \in N$, $\forall x^\star \in F(T)$. In particular,

$$\|x_{n+1} - x^\star\|^q \leq M\|x_1 - x^\star\|^q + M\sum_{k=1}^{n} \theta_k, \tag{2.8}$$

$\forall n \in N$, $\forall x^\star \in F(T)$.

Proof. Since $\sum_{n=1}^{\infty} \alpha_n < \infty$, by Lemma 2.1, $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\forall n \in N$,

$$\begin{aligned} \|x_{n+1} - x^\star\|^q &\leq (1+\delta_n)\|x_n - x^\star\|^q + \theta_n \\ &\leq e^{\delta_n}\|x_n - x^\star\|^q + \theta_n. \end{aligned}$$

Hence, by induction, we have

$$\begin{aligned} \|x_{n+1} - x^\star\|^q &\leq e^{\delta_n}\|x_n - x^\star\|^q + \theta_n \\ &\leq e^{\delta_n}[e^{\delta_{n-1}}\|x_{n-1} - x^\star\|^q + \theta_{n-1}] + \theta_n \\ &\leq \cdots \\ &\leq e^{\sum_{k=1}^{n} \delta_k}\|x_1 - x^\star\|^q + e^{\sum_{k=1}^{n} \delta_k}(\textstyle\sum_{k=1}^{n} \theta_k) \\ &\leq M\|x_1 - x^\star\|^q + M(\textstyle\sum_{k=1}^{n} \theta_k), \end{aligned}$$

for all $n \in N$, and

$$\|x_{n+m} - x^\star\|^q \leq M\|x_n - x^\star\|^q + M\sum_{k=n}^{n+m-1} \theta_k$$

for all $m,\ n \in N$.

Theorem 2.1. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$. In additional, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x_n, F(T))$ is the distance of x_n to set $F(T)$, i.e., $d(x_n, F(T)) = \inf_{u^\star \in F(T)} \|x_n - u^\star\|$ if and only if the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. If the sequence $\{x_n\}$ converges strongly to a fixed point of T, say, $y^\star \in F(T)$, we can easy to deduce that the sequence $\{x_n\}$ is bounded. Note that

$$d(x_n, F(T)) = \inf_{u^\star \in F(T)} \|x_n - u^\star\| \leq \|x_n - y^\star\| \to 0 \text{ as } n \to \infty.$$

Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Suppose that the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Since the sequence $\{x_n\}$ is bounded and both series $\sum_{n=1}^{\infty} \|u_n\|$, $\sum_{n=1}^{\infty} \|v_n\|$ and $\sum_{n=1}^{\infty} \|w_n\|$ are finite, from (2.8), there is a $\tilde{M} > 0$ such that $\|x_n - x^\star\| < \tilde{M}$, $\|x_{n+1} - u_n - x^\star\| < \tilde{M}$, $\|z_n - w_n - x^\star\| < \tilde{M}$, $\|y_n - v_n - x^\star\| < \tilde{M}$, $\|u_n\| < \tilde{M}$, $\|v_n\| < \tilde{M}$ and $\|w_n\| < \tilde{M}$. Then

$$\begin{array}{rcl} \sum_{n=1}^{\infty} \theta_n & \leq & \sum_{n=1}^{\infty} q\alpha_n L^{2q} \tilde{M}^q + \sum_{n=1}^{\infty} \alpha_n L^{2q} c_q \tilde{M}^q + \sum_{n=1}^{\infty} q\alpha_n L^q \tilde{M}^q \\ & + & \sum_{n=1}^{\infty} \alpha_n L^q c_q \tilde{M}^q + q\sum_{n=1}^{\infty} \|u_n\| \tilde{M}^{q-1} + c_q \sum_{n=1}^{\infty} \|u_n\|^q \\ & < & \infty. \end{array}$$

Hence, the sequence $\{\|x_{n+1} - x^\star\|^q\}$ is bounded, so is $\{\|x_{n+1} - x^\star\|\}$. Also, from (2.1) and Lemma 1.1, we know that $\lim_{n\to\infty} \|x_n - x^\star\|$ exists. Furthermore, from (2.1), we have

$$(d(x_{n+1}, F(T)))^q \leq (d(x_n, F(T)))^q + \delta_n \tilde{M}^q + \theta_n.$$

By Lemma 1.1, we have $\lim_{n\to\infty} (d(x_n, F(T)))^q$ exists and so is $\lim_{n\to\infty} d(x_n, F(T))$. Since $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. By using the similar argument in [2, Theorem 2.1], we have the sequence $\{x_n\}$ converges strongly to a fixed point of T.

As the mention in (1.6), a strictly pseudocontractive mapping is also a Lipschitzian mapping, we can get the following corollary.

Corollary 2.1. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a strictly pseudocontractive mapping and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. In additional, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x_n, F(T))$ is the distance of x_n to set $F(T)$, i.e., $d(x_n, F(T)) = \inf_{u^\star \in F(T)} \|x_n - u^\star\|$ if and only if the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The conclusion of Corollary 2.1 follows immediately from Theorem 2.1 and the fact that a strictly pseudocontractive mapping is also a Lipschitzian mapping.

Theorem 2.2. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a Lipschitzian strongly pseudocontraction mapping with Lipschitzian constant L and strongly accrective constant $k \in (0, 1)$ with respect to $I - T$. The set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. In additional, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x_n, F(T))$ is the distance of x_n to set $F(T)$, i.e., $d(x_n, F(T)) = \inf_{u^\star \in F(T)} \|x_n - u^\star\|$ if and only if the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Since all conditions of Theorem 2.1 holds, from Theorem 2.1, we have the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to a fixed point, say $x^\star$, of T. Actually, the fixed point $x^\star$ is unique. Indeed, if there is another fixed point $\bar{x}$, we have

$$\bar{x} \in T\bar{x}, \text{ and } x^\star \in Tx^\star.$$

If we choose that $0 \in (I - T)\bar{x}$ and $0 \in (I - T)x^\star$, from the strongly pseudocontraction of T, there is a $j_q(\bar{x} - x^\star) \in J_q(\bar{x} - x^\star)$, such that

$$0 = \langle 0 - 0, j_q(\bar{x} - x^\star)\rangle \geq k\|\bar{x} - x^\star\|^q.$$

This implies that $\bar{x} = x^\star$ and we complete prove the Theorem 2.2.

We can deduce the similar conclusion of Theorem 2.2 for a Lipschitzian strongly accrective mapping as following.

Theorem 2.3. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a Lipschitzian strongly accrective mapping with Lipschitzian constant L and strongly accrective constant $k \in (0, 1)$. The set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. In additional, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique fixed point of T.

Proof. The uniqueness of the fixed point for the mapping T can be derived from the strongly accrection.

We note that if we take $u_n = 0,\ v_n = 0$ and $w_n = 0\ \forall n \geq 1$ in Theorem 2.1 and 2.2, then we can obtain the corresponding new results on the strong convergence of the three-step iterative process:

$$\begin{cases} z_n \in (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n,\ n \geq 1, \end{cases} \tag{2.9}$$

Corollary 2.2. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E, and $T : K \to 2^K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$. In additional, $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (2.9). Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T. In addition, if T is also a strongly pseudocontractive mapping, then the sequence $\{x_n\}$ converges strongly to an unique fixed point of T.

Proof. It follows from Lemma 2.1 that

$$\|x_{n+1}-x^\star\|^q \leq (1+\delta_n)\|x_n-x^\star\|^q \leq e^{\sum_{j=1}^{n}\delta_j}\|x_1-x^\star\|^q \leq e^{\sum_{j=1}^{\infty}\delta_j}\|x_1-x^\star\|^q < \infty.$$

This shows that $\{x_n\}$ is bounded. The conclusion of the corollary follows from Theorem 2.1 and Theorem 2.2.

From Theorem 2.3, we have the sufficient and necessary condition for three-step iterative approximation of solutions to inclusion $f \in Tx$ in K as follows.

Theorem 2.4. Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$, and $T : K \to 2^K$ be a strongly pseudocontractive mapping such that $I - T : K \to 2^K$ is Lipchitzian with Lipschitzian constant L and strongly accrective constant $k \in (0,1)$ with respect to T. For any given $f \in K$, define $S : K \to 2^K$ by

$$Sx = f - Tx + x,\ \forall x \in K.$$

Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$. In additional, $\sum_{n=1}^{\infty}\alpha_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$ and $\sum_{n=1}^{\infty}\|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process

$$\begin{cases} z_n \in (1-\gamma_n)x_n + \gamma_n S x_n + w_n, \\ y_n \in (1-\beta_n)x_n + \beta_n S z_n + v_n, \\ x_{n+1} \in (1-\alpha_n)x_n + \alpha_n S y_n + u_n,\ n \geq 1. \end{cases} \tag{2.10}$$

with errors. If the set $F(S)$ of fixed points of S is nonempty, then the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(S)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique solution of the inclusion $f \in Tx$ in K.

Proof. Since $T : K \to 2^K$ is a strongly pseudocontractive mapping with strongly accrective constant $k \in (0,1)$ such that $I - T : K \to 2^K$ is Lipchitzian with Lipschitzian constant L, S is a Lipchitzian strongly accrective mapping with constant $k \in (0,1)$ and with Lipschitzian constant L. From Theorem 2.3, the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(S)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique fixed point, say $\hat{x}$, of S. For this fixed point $\hat{x}$ of S, we have $\hat{x} \in S\hat{x} = f - T\hat{x} + \hat{x}$, that is, $f \in T\hat{x}$. Hence the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(S)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique solution $\hat{x}$ of the inclusion $f \in Tx$ in K.

As the mention in (1.8), the new iterative results of Theorem 2.4 include many new particular iterative process for approximating the solution of the inclusion $f \in Tx$ as follows:

(i) Three-step iterative process whenever (1.8) with $u_n = v_n = w_n = 0$, $\forall n \geq 1$,

(ii) Ishikawa iterative process with errors whenever (1.8) with $\gamma_n = 0$ and $w_n = 0$ $\forall n \geq 1$, we refer to [14]. In addition, T is a single-valued mapping, we refer to [15], and $v_n = 0$ $\forall n \geq 1$, we refer to [16],

(iii) Ishikawa iterative process whenever (1.8) with $\gamma_n = 0$ and $u_n = v_n = w_n = 0$ $\forall n \geq 1$,

(iv) Mann iterative process with errors whenever (1.8) with $\beta_n = \gamma_n = 0$ and $w_n = v_n = 0$, $\forall n \geq 1$,

(v) Mann iterative process whenever (1.8) with $\beta_n = \gamma_n = 0$ and $u_n = v_n = w_n = 0$, $\forall n \geq 1$.

Furthermore, as we mention in Corollary 2.2, we can omit the conditions "$K + K \subseteq K$" and "the sequence $\{x_n\}$ is bounded" from Theorem 2.4 whenever one of the cases (i), (iii) and (v) hold.

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Yen-Cherng Lin
General Education Center,
China Medical University
Taichung 404, Taiwan, ROC
E-mail: yclin@mail.cmu.edu.tw