Thai Journal of Mathematics
Volume 19 Number 1 (2021)
Pages 1–7

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Coincidence Point Theorem on Hilbert Spaces via Weak Ekeland Variational Principle and Application to Boundary Value Problem

Mehdi Asadi[1,*] **and Erdal Karapınar**[2,3]

[1] *Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran*
e-mail : masadi.azu@gmail.com, masadi@iauz.ac.ir

[2] *Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*
e-mail : erdalkarapinar@tdmu.edu.vn

[3] *Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey*
e-mail : erdalkarapinar@yahoo.com

Abstract In this paper, we give a new coincidence point theorem for two operators on Hilbert spaces for certain operators by using the weak Ekeland variational principle. Our paper extends and improves the results on the topic in the literature. We consider a boundary value problem as an application of our results.

MSC: 34B15; 47G40; 47H10; 58E30
Keywords: fixed point theorem; Hilbert spaces; weak Ekeland variational principle; potential operators; α-positively homogeneous operators

Submission date: 12.02.2018 / Acceptance date: 22.05.2020

1. Introduction and Preliminaries

Riki and Oussaoui in [1] proved a fixed point theorem on Hilbert spaces for potential α-positively homogeneous operators via weak Ekeland variational principle, see also [2, 3]. In this paper, we extend their works by considering two α and β-positively homogeneous operators. We investigate the existence of the coincidence point for these operators and express a boundary value problem to indicate the validity. For the detail of the Ekeland variational principle refer to [4].

Let H be a real Hilbert space endowed with scalar product denoted $(.,.)$.

Definition 1.1. [5] A self-mapping $T : H \to H$ is said to be homogeneous of degree $\alpha > 0$ if there exists an $\alpha > 0$ such that

$$T(tx) = t^{\alpha}T(x), \quad \forall t > 0, x \in H.$$

*Corresponding author.

Published by The Mathematical Association of Thailand.

Let ψ be a Fréchet differentiable functional defined on H and let $\psi'(x)$ denote the differential of ψ at $x \in H$. We denote by $\nabla\psi(x)$ the unique vector of H such that

$$\psi'(x).y = (\nabla\psi(x), y) \quad \forall x, y \in H.$$

Definition 1.2. [5] A self-mapping $T : H \to H$ is said to be a potential or gradient operator if there exists a differentiable functional ψ on H such that $T = \nabla\psi$. In this case, the mapping ψ is called the potential of T.

Let $T : H \to H$ be a potential operator and ψ be its potential, that is, $T = \nabla\psi$. If T is assumed to be continuous, then ψ is of class C^1 and ψ and T are related by the formula

$$\psi(x) = \int_0^1 (T(tx), x)dt.$$

If T is α-positively homogeneous, then

$$\psi(x) = \frac{1}{1+\alpha}(T(x), x).$$

If we let $G(x) = (T(x), x)$, we have that G is of class C^1 and $\nabla G = (\alpha + 1)T$.

Theorem 1.3. [6] *Let E be a complete metric space and let $\varphi : E \to \mathbb{R}$ a functional that is lower semi-continuous, bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in E$ with $\varphi(u_\varepsilon) \leq \inf_E \varphi + \varepsilon$ and whenever $v \in E$ with $v \neq u_\varepsilon$, then*

$$\frac{\varphi(u_\varepsilon) - \varphi(v)}{d(u_\varepsilon, v)} < \varepsilon.$$

For the next section we need the following definition.

Definition 1.4. An operator $S : H \to H$ is called strongly monotone if and only if

$$(Su, u) \geq k\|u\|^2 \quad \forall u \in H,$$

for some $k > 0$.

For the application section, we need to following notes.
$\lambda_1 = \pi^2$ is the first eigenvalue of the linear Dirichlet problem

$$\begin{cases} -u''(t) = \lambda u(t), & 0 < t < 1; \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

We will apply the Poincaré's inequality

$$\|u\|_2 \leq \frac{\|u'\|_2}{\sqrt{\lambda_1}},$$

for every $u \in H_0^1(0, 1)$.

2. Main Results

Theorem 2.1. *Let $T : \overline{U} \to H$ be a compact potential operator, $S : \overline{U} \to H$ a strongly monotone operator, with homogeneous of degrees $\alpha > 0$ and $\beta > 0$, respectively, $ST = TS$, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that*

$$(Tu, u) \leq \frac{\alpha+1}{\beta+1}(Su, u) - C\|Su\| \quad \forall u \in \partial U, \tag{2.1}$$

then T and S have a coincidence point in $\overline{U}$.

Proof. Define

$$\varphi(u) \quad := \quad \int_0^1 S(tu,u)dt - \int_0^1 T(tu,u)dt. \tag{2.2}$$

Since T and S are homogeneous of degree α and β, respectively, then, we have

$$= \quad \frac{1}{\beta+1}(Su,u) - \frac{1}{\alpha+1}(Tu,u). \tag{2.3}$$

It is clear that Fréchet φ is differentiable with $\varphi' = S - T$.

We shall divide the rest of the proof in four parts.

Part 1. We shall show that φ is bounded below. Since S is a strongly monotone operator, we have that

$$(Su,u) \geq k\|u\|^2,$$

for some $k > 0$. By using Cauchy-Schwarz inequality, we find that

$$\|Su\|\|u\| \geq (Su,u) \geq k\|u\|^2$$

which implies the following inequality

$$\|Su\| \geq k\|u\| \text{ for some } k > 0.$$

On the other hand, since $\overline{U}$ is a bounded set with $0 \in U$ and since the operator T is compact, there exists $M > 0$ such that $\|T(u)\| \leq M$ for all $u \in \overline{U}$. Again by using Cauchy-Schwarz inequality, we derive that

$$\begin{aligned}
\varphi(u) \quad &:= \quad \int_0^1 (S(tu,u)dt - \int_0^1 (T(tu,u)dt \\
&= \quad \frac{1}{\beta+1}(Su,u) - \frac{1}{\alpha+1}(Tu,u) \\
&\geq \quad \frac{1}{\beta+1}k\|u\|^2 - \frac{1}{\alpha+1}\|Tu\|\|u\| \\
&\geq \quad \frac{1}{\beta+1}k\|u\|^2 - \frac{1}{\alpha+1}M\|u\|.
\end{aligned}$$

Thus, we conclude that φ is bounded below.

Pick $0 < \varepsilon < \frac{kC}{\alpha+1}$. By Theorem 1.3 there exists $u_\varepsilon \in \overline{U}$ with $\varphi(u_\varepsilon) \leq \inf_{\overline{U}} \varphi + \varepsilon$ and whenever $v \in \overline{U}$ with $v \neq u_\varepsilon$, then

$$\varphi(u_\varepsilon) < \varphi(v) + \varepsilon\|u_\varepsilon - v\|.$$

Part 2. In this step, we shall indicate that $u_\varepsilon \notin \partial\overline{U}$.

If $u_\varepsilon \in \partial\overline{U}$, then, for $v = 0$, we have $\varphi(u_\varepsilon) < \varphi(v) + \varepsilon\|u_\varepsilon - v\|$. Due to the fact $\varphi(0) = 0$, we derive that

$$\begin{aligned}
\varphi(u_\varepsilon) \quad &< \quad \varepsilon\|u_\varepsilon\| \\
&\leq \quad \frac{kC}{\alpha+1}\|u_\varepsilon\| \\
&\leq \quad \frac{C}{\alpha+1}\|Su_\varepsilon\|.
\end{aligned}$$

Note also that

$$\varphi(u_\varepsilon) = \frac{1}{\beta+1}(Su_\varepsilon,u_\varepsilon) - \frac{1}{\alpha+1}(Tu_\varepsilon,u_\varepsilon),$$

which yields that

$$\frac{1}{\beta+1}(Su_\varepsilon, u_\varepsilon) - \frac{1}{\alpha+1}(Tu_\varepsilon, u_\varepsilon) < \frac{C}{\alpha+1}\|Su_\varepsilon\|.$$

It contradicts the inequality(2.1) which completes the proof of the claim.

Part 3. Here, we claim that u_ε is an approximate coincidence point of T and S.

Let $h \in H$. Since φ is Frechét differentiable, we have

$$|\langle \varphi(u_\varepsilon), h\rangle| \leq \varepsilon\|h\|.$$

Therefore, we get that

$$\|\varphi'(u_\varepsilon)\| \leq \varepsilon.$$

Thus, we obtain that

$$\|Su_\varepsilon - Tu_\varepsilon\| \leq \varepsilon$$

which means that u_ε is an approximate critical point of φ and then it is an approximate coincidence point of T and S.

Part 4. In the last part, we shall prove the existence of a coincidence point.

For $n \in \mathbb{N}$, there exists a sequence $\{u_n\}$ in $\overline{U}$ such that $\|\varphi'(u_n)\| \leq \frac{1}{n}$.

Now there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ such that $Tu_{n_k} \to u_0$ as $k \to \infty$, for some $u_0 \in \overline{U}$. Other hand

$$\|Su_{n_k} - u_0\| \leq \|Su_{n_k} - Tu_{n_k}\| + \|Tu_{n_k} - u_0\| \to 0,$$

as $k \to \infty$. So $TSu_{n_k} \to Tu_0$ and $STu_{n_k} \to Su_0$ as $k \to \infty$, and by commuting of T and S we get $Tu_0 = Su_0$. ■

We improve Theorem 2.1 [1] as follows.

Corollary 2.2. *Let $T : \overline{U} \to H$ be a compact potential operator, with α-positively homogeneous, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that*

$$(Tu, u) \leq \frac{\alpha+1}{2}\|u\|^2 - C\|u\| \quad \forall u \in \partial U. \tag{2.4}$$

then T has a fixed point in $\overline{U}$.

Proof. Put $Su := u$. So S is homogeneous of degree 1. Thus $\beta + 1 = 2$. ■

Corollary 2.3. *Let $T : \overline{U} \to H$ be a compact potential operator, $S : \overline{U} \to H$ a strongly monotone operator with constant $k > 0$, with homogeneous of degrees $\alpha > 0$ and $\beta > 0$, respectively, $ST = TS$, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that*

$$\|Tu\| \leq k\frac{\alpha+1}{\beta+1}\|u\| - C\|Su\|/\|u\| \quad \forall u \in \partial U. \tag{2.5}$$

then T and S have a coincidence point in $\overline{U}$.

Corollary 2.4. *Let T and S be compact potential operator with homogeneous of degrees $\alpha > 0$ and $\beta > 0$, respectively, $ST = TS$, satisfying*

$$T(\partial B(0, R)) \subseteq B\left(0, k\frac{\alpha+1}{\beta+1}R - Ck\right) \quad \text{and} \quad S(\partial B(0, R)) \subseteq B(0, kR), \tag{2.6}$$

for some $R > 0$ and $C > 0$ such that $C < \frac{\alpha+1}{\beta+1}R$. Then T and S has a coincidence point in $\overline{B(0,R)}$.

Proof. It is enough we put $U := B(0,R)$ for some $R > 0$. ■

Corollaries 2.2 and 2.3 [1] are also direct consequent of our main Theorem 2.1.

3. An Application

We consider the Dirichlet boundary value problem

$$\begin{cases} -u''(t) = p(t)f(u(t)), & 0 < t < 1; \\ -u''(t) = q(t)g(u(t)), & 0 < t < 1; \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are two continuous functions, $p, q \in L^2(0,1)$ such that $p(t)f(u(t)) - q(t)g(u(t)) \neq 0$. Let A, T and S be the operators defined on the standard Sobolev space $H_0^1(0,1)$ endowed with inner product $(u,v) = \int_0^1 u'(t)v'(t)dt$ and the norm $\|u\|^2 = \int_0^1 {u'}^2(t)dt$ by

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)(p(s)f(u(s)) - q(s)g(u(s)))ds, \\ Tu(t) &= \int_0^1 G(t,s)p(s)f(u(s))ds, \\ Su(t) &= \int_0^1 G(t,s)q(s)g(u(s))ds, \end{aligned}$$

where

$$G(t,s) = \begin{cases} t(1-s), & t \leq s; \\ s(1-t), & s \leq t. \end{cases} \tag{3.2}$$

Then, $A = S - T$ satisfies the problem

$$\begin{cases} -(Au)''(t) = q(t)g(u(t)) - p(t)f(u(t)), & 0 < t < 1; \\ Au(0) = Au(1) = 0, \end{cases}$$

Operator A satisfies also the following property

$$(Au, v) = \int_0^1 (q(s)g(u(s)) - p(s)f(u(s)))v(s)ds \quad \forall u, v \in H_0^1(0,1).$$

We know, if u be a solution of the integral equation

$$u(t) = \int_0^1 G(t,s)(p(s)f(u(s)) - q(s)g(u(s)))ds,$$

then u is a solution of the problem (3.1) and conversely. Let φ be the functional defined on $H_0^1(0,1)$ by

$$\varphi(u) = (Su, u) - (Tu, u). \tag{3.3}$$

Definition 3.1. $u \in H_0^1(0,1)$ is called a weak solution of (3.1) if

$$\int_0^1 [u'(t)v'(t) - (q(t)g(u)(t) - p(t)f(u(t)))v(t)]dt = 0, \quad \forall v \in H_0^1(0,1).$$

Lemma 3.2. [1] *The operator $A : H_0^1(0,1) \to H_0^1(0,1)$ is compact.*

Theorem 3.3. *Let that the following conditions hold:*

i. Let f, g be homogenous of degrees α and β respectively.
ii. There exists $R > 0$ such that

$$M := \max_{|t|<R} |f(t)| \leq \frac{\pi(\alpha+1)R}{2\|p\|_2}, \quad N := \max_{|t|<R} |g(t)| \leq \frac{\pi(\beta+1)R}{2\|q\|_2}.$$

Then the Dirichlet boundary value problem (3.1) has a solution $u \in C^2(0,1)$.

By Lemma 3.2 operator A is compact. Integrating by parts, we get

$$\begin{aligned} \varphi'(u)v &= \int_0^1 [u'(t)v'(t) - (q(t)g(u)(t) - p(t)f(u(t)))v(t)]dt \\ &= -\int_0^1 [u''(t)v(t)dt - (q(t)g(u)(t) - p(t)f(u(t)))v(t)]dt \\ &= -\int_0^1 [(Au)''v(t)]dt \\ &= \int_0^1 [(Au)'v'(t)]dt \\ &= (A(u), v), \end{aligned}$$

for all $u, v \in H_0^1(0,1)$. Therefore $\varphi' = S - T$.

Now we prove that the operator A satisfies the conditions of Corollary (2.4). We also shall show that S is a strongly monotone operator for some $k > 0$.

Take any $C > 0$ such that

$$Ck \leq k\frac{\alpha+1}{\beta+1}R - \frac{1}{\pi}(\|q\|_2 N + \|p\|_2 M)$$

By using the Cauchy-Schwarz inequality and Poincaré's inequality

$$\begin{aligned} \|Au\| &= \sup_{\|v\|\leq 1} |(Au, v)| \\ &= \sup_{\|v\|\leq 1} |\int_0^1 (q(s)g(u(s)) - p(s)f(u(s)))v(s)ds| \\ &\leq \sup_{\|v\|\leq 1} (\int_0^1 (q(s)g(u(s)) - p(s)f(u(s)))^2 ds)^{\frac{1}{2}} (\int_0^1 (v(s))^2 ds)^{\frac{1}{2}} \\ &\leq \sup_{\|v\|\leq 1} \frac{(\|q\|_2 N + \|p\|_2 M)}{\sqrt{\lambda_1}} \\ &\leq k\frac{\alpha+1}{\beta+1}R - Ck. \end{aligned}$$

Also we can show that

$$\|Su\| \leq \frac{\|q\|_2 N}{\sqrt{\lambda_1}}.$$

Now if we take $u \in B(0, \frac{\|q\|_2 N}{\sqrt{\lambda_1}}) \cap B(0, R)$, then S is a strongly monotone operator for some $k > 0$.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] M.B. Riki, T.M. Oussaoui, A fixed point theorem on Hilbert spaces for potential α-positively homogeneous operators via weak Ekeland variational principle, Arab Journal of Mathematical Sciences https://doi.org/10.1016/j.ajmsc.2016.05.001.

[2] H. Aydi, C. Vetro, E. Karapinar, On Ekeland's variational principle in partial metric spaces, Appl. Math. Inf. Sci. 9 (1) (2015) 257–262.

[3] E. Karapinar, S. Romaguera, On the weak form of Ekeland's variational principle in quasi-metric spaces, Topology and Its Applications 184 (2015) 54–60.

[4] C. Mongkolkeha, T. Tanaka, P. Kumam, On some generalizations of Ekelands variational principle and fixed point theorems in quasi-partial metric spaces, J. Nonlinear and Conv. Anal. 18 (11) (2017) 2083–2099.

[5] J. Chabrowski, Variational Methods for Potential Operator Equations With Applications to Nonlinear Elliptic Equations, Walter de Gruyter, Berlin, New York, 1997.

[6] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974) 324–353.