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Hybrid Algorithm for System of Nonlinear Monotone Equations Based on the Convex Combination of Fletcher-Reeves and a New Conjugate Residual Parameters

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Abstract In this paper, based on the projection strategy of Solodov and Svaiter (1998, *Reformulation: Nonsmooth, Piecewise Smooth, Semismooth, and Smoothing Methods* (M. Fukushima & L. Qi eds) Dordrecht: Kluwer, pp. 355-369), we present a hybrid conjugate residual algorithm for nonlinear monotone equations with convex constraints. The parameter is computed as a convex combination of the Fletcher-Reeves (FR) and a new conjugate residual parameters. Furthermore, the convex combination parameter is chosen in such a way that the search direction satisfied the descent property, independent of any line search. The global convergence of the proposed hybrid algorithm was given under some suitable conditions. The proposed approach is shown to be efficient and promising based on the preliminary computational experiments performed on some standard problems.

MSC: 65H10; 90C52; 49M37

Keywords: nonlinear systems of equations; large-scale problems; conjugate residual method; global convergence; computational results

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1. Introduction

The problem of finding solution of the system of nonlinear equation of the form

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is mostly assumed to be continously differentiable, appears in various disciplines such as engineering, physics, chemistry, etc. Thus, it becomes one of the most interesting problems in numerical analysis and optimization.

Recently, a considerable attention is given to the problem (1.1) in which F is monotone that is

$$(F(x) - F(y))^T(x - y) \geq 0, \quad \forall x, y \in \mathbb{R}^n.$$

Based on the monotonicity and Lipschitz continuity assumptions on F, several matrix-free iterative approaches for solving (1.1) have been proposed recently (cf.[1–9, 12, 13, 18–24, 26, 27]). The main motivation on the study of these algorithms is due to the fact that most of them achieved global convergence without differentiability assumption. It is well known that the descent property plays a major role in achieving matrix-free global convergence algorithms, in view of this, [17] proposed a derivative-free spectral conjugate residual projection algorithm for nonlinear monotone equations with convex constraints, in which the spectral parameter is chosen such that the search direction is sufficiently descent at each iteration. For some related works on gradient methods for split feasibility problems in Hilbert space, interested reader may refer to the following references [14, 15, 25], among others.

Our main contribution in this paper is motivated by the success of the algorithms that can generate descent directions for nonlinear monotone equations with convex constraints. We aim at studying a hybrid conjugate residual projection algorithm in which the parameter is a convex combination of FR parameter and a new parameter. Moreover, the convex combination parameter is chosen such that the search direction generated by the proposed algorithm is sufficiently descent at every iteration independent on the line search used. Under the monotonicity and Lipschitz continuity assumptions, we show that the proposed algorithm is globally convergent.

The outstanding part of this paper is organized as follows. In Section 2, we present the motivation and general algorithm of the proposed method. In section 3, we prove the global convergence of the algorithm. In Section 4, we present the numerical experiments, and conclusions in Section 5. Unless otherwise stated, throughout this paper $\|\cdot\|$ stands for the Euclidean norm of vectors in $\mathbb{R}^n$.

2. Motivation and Algorithm

We consider a projection method for finding the solution of the nonlinear system of equation of the form:

$$F(x) = 0, \quad s.t \quad x \in \Omega \tag{2.1}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, is Lipchitz continuous and monotone. The set $\Omega \subseteq \mathbb{R}^n$ is closed and convex. The method can be described as follows. Given the initial iterate $x_0 \in \mathbb{R}^n$, the next iterate is obtained as

$$x_{k+1} = P_\Omega[x_k - \xi_k F(z_k)], \tag{2.2}$$

with

$$\xi_k = \frac{(x_k - z_k)^T F(z_k)}{\|F(z_k)\|^2},$$

$z_k = x_k + \alpha_k d_k$, where $\alpha_k = \rho^{i_k}$ is the step length usually satisfying

$$-F(x_k + \rho^{i_k} d_k)^T d_k \geq \sigma \rho^{i_k} \|F(x_k + \rho^{i_k} d_k)\| \|d_k\|^2, \tag{2.3}$$

$\rho \in (0,1)$, i_k is a nonnegative integer and d_k is the direction normally satisfying

$$F(x_k)^T d_k \leq -c\|F(x_k)\|^2, \tag{2.4}$$

where c is a positive constant. The symbol $P_\Omega(\cdot)$ is the projection of a point onto the closed and convex set Ω which is defined by

$$P_\Omega[x] = \text{argmin}\{\|x - y\| : y \in \Omega\},$$

and satisfies

$$\|P_\Omega[x] - P_\Omega[y]\| \leq \|x - y\| \ \forall x, y \in \mathbb{R}^n. \tag{2.5}$$

The direction of the proposed method is defined as,

$$d_k = \begin{cases} -F_k, \text{ if } k = 0 \\ -F_k + \beta_k d_{k-1}, \text{ if } k \geq 1. \end{cases} \tag{2.6}$$

where $F_k = F(x_k)$,

$$\beta_k = (1 - \tau)\beta_k^{FR} + \tau \beta_k^*, \tag{2.7}$$

$$\beta_k^{FR} = \frac{\|F_k\|^2}{\|F_{k-1}\|^2}, \quad \beta_k^* = \frac{\|F_k\|^2}{F_k^T d_{k-1}} \tag{2.8}$$

and the convex combination parameter $\tau \in (0,1)$ is chosen such that the direction d_k satisfies (2.4).

Next, we state the steps of the proposed algorithm.

Algorithm 1.

Hybrid conjugate residual algorithm (HCRA)

Step 0: Given $x_0 \in \Omega$, ρ, σ $\tau \in (0,1), \mu \in (0,2)$ *and* $\epsilon > 0$. *Set* $k = 0$.

Step 1: Compute $\|F_k\|$, *if* $\|F_k\| \leq \epsilon$, *stop.*

Step 2: If $k = 0$, *set* $d_k = -F_k$. *Else compute* $d_k = -F_k + \beta_k d_{k-1}$, *where,*

$$\beta_k = (1 - \tau)\beta_k^{FR} + \tau \beta_k^*,$$
$$\beta_k^{FR} = \frac{\|F_k\|^2}{\|F_{k-1}\|^2}, \quad \beta_k^* = \frac{\|F_k\|^2}{F_k^T d_{k-1}},$$
$$\beta_k = 0 \quad \text{provided} \quad \|F_{k-1}\|^2 - F_k^T d_{k-1} < 0.$$

Step 3: Compute the trial point $z_k = x_k + \alpha_k d_k$, *where* $\alpha_k = \rho^{i_k}$ *with* i_k *being the smallest nonegative integer* i *such that*

$$-F(z_k)^T d_k \geq \sigma \alpha_k \|F(z_k)\| \|d_k\|^2.$$

Step 4: If $z_k \in \Omega$ *and* $F(z_k) \leq \epsilon$, *set* $x_{k+1} = z_k$. *Otherwise, compute the next iterate*

$$x_{k+1} = P_\Omega[x_k - \mu \xi_k F(z_k)],$$

where $\xi_k = \dfrac{(x_k - z_k)^T F(z_k)}{\|F(z_k)\|^2}$.

Step 5*: Set* $k = k + 1$ *and go to Step* 1*.*

end

The following Lemma gives the descent property of the direction generated by Algorithm 1

Lemma 2.1. *The search direction* $\{d_k\}$ *generated by Algorithm* 1 *(HCGA) is sufficiently descent. That is*

$$F_k^T d_k \leq -\|F_k\|^2 \quad \forall k \geq 0. \tag{2.9}$$

Proof. For $k = 0$,

$$F_0^T d_0 = -F_0^T F_0 = -\|F_0\|^2.$$

Now for $k \geq 1$, we have

$$F_k^T d_k = -F_k^T (F_k - \beta_k d_{k-1}). \tag{2.10}$$

Combining (2.10), (2.8) and (2.7), we have:

$$F_k^T d_k = -\left(1 - \left[\frac{(1-\tau)F_k^T d_{k-1}}{\|F_{k-1}\|^2} + \tau\right]\right)\|F_k\|^2 < 0, \tag{2.11}$$

if and only if

$$1 > \frac{(1-\tau)F_k^T d_{k-1} + \|F_{k-1}\|^2\tau}{\|F_{k-1}\|^2}, \tag{2.12}$$

simplifying (2.12) we have

$$\tau(\|F_{k-1}\|^2 - F_k^T d_{k-1}) < \|F_{k-1}\|^2 - F_k^T d_{k-1}. \tag{2.13}$$

From (2.13)

$$\tau < 1, \text{ provided } \|F_{k-1}\|^2 - F_k^T d_{k-1} > 0, \tag{2.14}$$

and

$$\tau > 1, \text{ provided } \|F_{k-1}\|^2 - F_k^T d_{k-1} < 0. \tag{2.15}$$

Inequality (2.15) cannot hold since $\tau \in (0, 1)$. Therefore, we choose $\beta_k = 0$ in the algorithm whenever $\|F_{k-1}\|^2 - F_k^T d_{k-1} < 0$.

If $\|F_{k-1}\|^2 - F_k^T d_{k-1} > 0$, then $F_k^T d_k < 0 = (-0)\|F_k\|^2$.

Therefore, since $\max\{0, 1\} = 1$, we have $F_k^T d_k \leq -\|F_k\|^2$ for all $k \geq 0$. ∎

The following Lemma shows that the HCGA algorithm is well-defined.

Lemma 2.2. *Let the sequence* $\{d_k\}$ *and* $\{x_k\}$ *be generated by the HCGA algorithm, then there always exists a step-size* $\alpha_k = \rho^{i_k}$ *satisfying the line search* (2.3) *for some positive number* i_k*.*

Proof. Suppose there is an iterate k_0 for which inequality (2.3) does not hold for any non-negative integer i_k, that is,

$$-F(x_{k_0} + \rho^{i_k} d_{k_0})^T d_{k_0} < \sigma\rho^{i_k}\|F(x_{k_0} + \rho^{i_k} d_{k_0})\|\|d_{k_0}\|^2$$

Since F is continuous and $\rho \in (0,1)$, let $i_k \longrightarrow \infty$, it holds that

$$-F_{k_0}^T d_{k_0} \leq 0.$$

This contradicts (2.9). Hence, the proof. ∎

3. Convergence Analysis

In this section, we analyze the convergence of the proposed algorithm (HCGA), before that, we state the following assumptions:

Assumption 3.1. The solution set of (1.1) Ω' is nonempty.

Assumption 3.2. The mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is Lipschiz continuous, i.e., there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{3.1}$$

Assumption 3.3. The mapping F is monotone.

Lemma 3.4. *Suppose that Assumptions (3.1)-(3.3) hold and $\{x_k\}$ and $\{z_k\}$ are sequences generated by Algorithm 1, then $\{x_k\}$ and $\{z_k\}$ are bounded. Furthermore, it holds that either $\{x_k\}$ is finite and the last iteration is the solution, or the sequence $\{x_k\}$ is infinite and $\lim_{k\to+\infty} \|x_k - z_k\| = 0$.*

Proof. If the algorithm terminates at some iteration k, then $\|F(x_k)\| = 0$ or $\|F(z_k)\| = 0$. This imply that either x_k or z_k is a solution of (2.1).

Suppose that $\|F(x_k)\| \neq 0$ and $\|F(z_k)\| \neq 0$ for all k. Then an infinite sequence is generated. From Step 3 of Algorithm 1

$$\begin{aligned}(x_k - z_k)^T F(z_k) &= -\alpha_k F(z_k)^T d_k \\ &\geq \sigma\alpha_k^2 \|F(z_k)\|\|d_k\|^2 \\ &= \sigma\|F(z_k)\|\|x_k - z_k\|^2 \\ &> 0.\end{aligned} \tag{3.2}$$

Let $\tilde{x} \in \Omega$ such that $F(\tilde{x}) = 0$, then by monotonicity of F, it holds that

$$\begin{aligned}F(z_k)^T(x_k - \tilde{x}) &= F(z_k)^T(x_k - z_k) + F(z_k)^T(z_k - \tilde{x}) \\ &\geq F(z_k)^T(x_k - z_k) + F(\tilde{x})^T(z_k - \tilde{x}) \\ &= F(z_k)^T(x_k - z_k).\end{aligned} \tag{3.3}$$

Also, using (2.5), (3.2), (3.3) and the definition of ξ_k given in Section 2 we have

$$\begin{aligned}
\|x_{k+1}-\tilde{x}\|^2 &= \|P_\Omega[x_k-\mu\xi_k F(z_k)]-P_\Omega(\tilde{x})\|^2\\
&\le \|x_k-\mu\xi_k F(z_k)-\tilde{x}\|^2\\
&= \|x_k-\tilde{x}\|^2-2\mu\xi_k F(z_k)^T(x_k-\tilde{x})+\|\mu\xi_k F(z_k)\|^2\\
&\le \|x_k-\tilde{x}\|^2-\mu(2-\mu)\frac{\big(F(z_k)^T(x_k-z_k)\big)^2}{\|F(z_k)\|^2}\\
&\le \|x_k-\tilde{x}\|^2-\mu(2-\mu)\frac{\sigma^2\|F(z_k)\|^2\|x_k-z_k\|^4}{\|F(z_k)\|^2}\\
&= \|x_k-\tilde{x}\|^2-\mu(2-\mu)\sigma^2\|x_k-z_k\|^4. \qquad (3.4)
\end{aligned}$$

Thus, the sequence $\{\|x_k-\tilde{x}\|\}$ is nonincreasing and convergent, and hence $\{x_k\}$ is bounded. In addition, from (3.4). Since $\mu\in(0,2)$, we have

$$\|x_{k+1}-\tilde{x}\|^2\le\|x_k-\tilde{x}\|^2,$$

which implies

$$\|x_k-\tilde{x}\|^2\le\|x_0-\tilde{x}\|^2,\ \forall k\ge 0.$$

Therefore, by Assumption 3.2 and letting $\kappa := L\|x_0-\tilde{x}\|$, we have

$$\|F(x_k)\|=\|F(x_k)-F(\tilde{x})\|\le L\|x_k-\tilde{x}\|\le\kappa. \qquad (3.5)$$

From (3.2), monotonicity of F and Cauchy-Schwarz inequality,

$$0<\sigma\|F(z_k)\|\|x_k-z_k\|^2\le F(z_k)^T(x_k-z_k)\le\|F(z_k)\|\|x_k-z_k\|,$$

which implies that

$$\sigma\|x_k-z_k\|\le 1. \qquad (3.6)$$

Thus, from (3.6) we have that the sequence $\{z_k\}$ is bounded.
It follows from (3.4) that

$$\sigma^2\sum_{k=0}^{\infty}\|x_k-z_k\|^4\le\sum_{k=0}^{\infty}(\|x_k-\tilde{x}\|^2-\|x_{k+1}-\tilde{x}\|^2)<\infty. \qquad (3.7)$$

which implies

$$\lim_{k\to+\infty}\|x_k-z_k\|=0. \qquad (3.8)$$

∎

Lemma 3.5. *Suppose that $\{d_k\}$ is generated by Algorithm 1 such that $\|F_{k-1}\|^2\le|F_k^T d_{k-1}|$. If there exists a constant γ such that $\|F_k\|\ge\gamma\quad\forall k\ge 0$, then there is a positive constant m such that*

$$\|d_k\|\le m\|F_k\|\quad\forall k\ge 0. \qquad (3.9)$$

Proof. If $k=0$, then

$$\|d_k\|=\|F_k\|. \qquad (3.10)$$

If $k \geq 1$, then

$$\begin{aligned}
\|d_k\| &= \| - F_k + \beta_k d_{k-1}\| \\
&\leq \|F_k\| + |\beta_k| \|d_{k-1}\| \\
&= \|F_k\| + \frac{(1-\tau)\|F_k\|^2}{\|F_{k-1}\|^2}\|d_{k-1}\| + \frac{\tau\|F_k\|^2}{|F_k^T d_{k-1}|}\|d_{k-1}\| \\
&= \|F_k\| + \frac{\|F_k\|^2}{\|F_{k-1}\|^2}\|d_{k-1}\| - \frac{\tau\|F_k\|^2}{\|F_{k-1}\|^2}\|d_{k-1}\| + \frac{\tau\|F_k\|^2}{|F_k^T d_{k-1}|}\|d_{k-1}\| \\
&\leq \|F_k\| + \frac{\|F_k\|^2}{\|F_{k-1}\|^2}\|d_{k-1}\| - \frac{\tau\|F_k\|^2}{|F_k^T d_{k-1}|}\|d_{k-1}\| + \frac{\tau\|F_k\|^2}{|F_k^T d_{k-1}|}\|d_{k-1}\| \\
&= \left(1 + \frac{\|F_k\|\|d_{k-1}\|}{\|F_{k-1}\|^2}\right)\|F_k\|.
\end{aligned}$$

Where the first inequality is obtained using triangular inequality, the second inequality is due to the hypothesis that $\|F_{k-1}\|^2 \leq |F_k^T d_{k-1}|$.

From (3.5) and the hypothesis of the Lemma, we have $\gamma \leq \|F_k\| \leq \kappa$, for all $k \geq 0$. Now since $\|d_0\| = \|F_0\|$, for $k = 1$,

$$\begin{aligned}
\|d_1\| &\leq \left(1 + \frac{\|F_1\|\|d_0\|}{\|F_0\|^2}\right)\|F_1\| \\
&\leq \left(1 + \frac{\kappa}{\gamma}\right)\|F_1\|.
\end{aligned}$$

Also, for $k = 2$,

$$\begin{aligned}
\|d_2\| &\leq \left(1 + \frac{\|F_2\|\|d_1\|}{\|F_1\|^2}\right)\|F_2\| \\
&\leq \left(1 + \frac{\kappa}{\gamma} + \left(\frac{\kappa}{\gamma}\right)^2\right)\|F_2\|.
\end{aligned}$$

Thus,

$$\|d_k\| \leq \left(1 + \frac{\kappa}{\gamma} + \left(\frac{\kappa}{\gamma}\right)^2 + \ldots + \left(\frac{\kappa}{\gamma}\right)^k\right)\|F_k\|.$$

Letting $\ell = \dfrac{\kappa}{\gamma} > 1$ and using the sum of the geometric series formula, we have

$$\begin{aligned}
\|d_k\| &\leq \left(\frac{\ell^k - 1}{\ell - 1}\right)\|F_k\| \\
&\leq l^k\|F_k\|. \qquad (3.11)
\end{aligned}$$

Taking $m := \max\left\{\ell, \ell^2, \ldots, \ell^k\right\}$ inequality (3.11) gives (3.9).

■

Theorem 3.6. *Let Assumptions (3.1)-(3.3) hold and let the sequence* $\{x_k\}$ *be generated by Algorithm 1. Then*

$$\liminf_{k\to\infty} \|F(x_k)\| = 0. \tag{3.12}$$

Furthermore, the sequence $\{x_k\}$ *converges.*

Proof. We consider two cases.

`Case 1:` If

$$\liminf_{k\to\infty} \|d_k\| = 0,$$

then by (2.9) and Cauchy-Schwarz inequality, we have

$$\liminf_{k\to\infty} \|F(x_k)\| = 0.$$

`Case 2:` If

$$\liminf_{k\to\infty} \|d_k\| > 0,$$

then by (3.9), we have

$$\liminf_{k\to\infty} \|F(x_k)\| > 0.$$

Using (3.8), we obtain

$$\lim_{k\to\infty} \alpha_k = 0.$$

Also from (2.3), we have that

$$-(F(x_k + \rho^{i_k-1} d_k))^T d_k < \sigma \rho^{i_k-1} \|F(x_k + \ \rho^{i_k-1} d_k)\| \|d_k\|^2. \tag{3.13}$$

Since $\{x_k\}, \{d_k\}$ are bounded, we can select a sub-sequence of ($\{x_k\}$ and $\{d_k\}$) such that by allowing $k \to \infty$, $i_k \to \infty$ in (3.13), we have

$$F(\bar{x})^T \bar{d} \geq 0. \tag{3.14}$$

On the other hand from (2.9), we have

$$F(\bar{x})^T \bar{d} < 0,$$

which contradicts (3.14). Consequently, $\liminf_{k\to\infty} \|F(x_k)\| > 0$ is not possible.

As F is continuous, then the sequence $\{x_k\}$ has some point of accumulation $\bar{x}$ such that $F(\bar{x}) = 0$, that is $\bar{x} \in \Omega^{'}$. Since the sequence $\{\|x_k - \bar{x}\|\}$ converges, it must hold that $\{x_k\}$ converges to $\bar{x}$ as $\bar{x}$ is an accumulation point. ■

4. Numerical Experiments

This section reports numerical experiments conducted using the proposed algorithm HCRA and Algorithm 2.1 proposed in [29]. The experiments compare the performance of HCRA and Algorithm 2.1.

The parameters considered for each algorithm are as follows:

HCRA: $\sigma = 10^{-4}$, $\tau = 0.1$, $\rho = 0.6$, $\mu = 1.8$.

Algorithm 2.1: All parameters are chosen as in [29].
All algorithms were coded in MATLAB using a windows 10 operating system of 2.4GHz Intel(R) Core(TM) i7-7100U CPU with 8GB RAM. The experiments were carried out on eight benchmark test problems using six initial points $x_1 = (0.1, 0.1, ..., 0.1)^T$, $x_2 = (0.2, 0.2, ..., 0.2)^T$, $x_3 = (0.5, 0.5, ..., 0.5)^T$, $x_4 = (1.2, 1.2, ..., 1.2)^T$, $x_5 = (1.5, 1.5, ..., 1.5)^T$, $x_6 = (2, 2, ..., 2)^T$ and five dimensions $n = 1000,\ 5000,\ 10000,\ 50000,\ 100000$.

The stopping condition considered is

$$\|F(x_k)\| \leq 10^{-6}$$

The algorithm is also terminated if the iteration exceeds 1000. The list of the benchmark test problems considered are given below where F is taken as

$$F(x) = (f_1(x), f_2(x), ..., f_n(x))^T \text{ and } x = (x_1, x_2, ..., x_n)^T.$$

Problem 1 [16] Exponential Function.

$$f_1(x) = e^{x_1} - 1,$$
$$f_i(x) = e^{x_i} + x_i - 1, \text{ for } i = 2, 3, \ldots, n,$$
$$\text{and } \Omega = R^n_+.$$

Problem 2 [16] Modified Logarithmic Function.

$$f_i(x) = \ln(x_i + 1) - \frac{x_i}{n}, \text{ for } i = 1, 2, \ldots, n,$$
$$\text{and } \Omega = \{x \in R^n : \sum_{i=1}^{n} x_i \leq n, x_i > -1,\ i = 1, 2, \ldots, n\}.$$

Problem 3 [16] Strictly Convex Function II.

$$f_i(x) = e^{x_i} - 1, \text{ for } i = 1, 2, \ldots, n,$$
$$\text{and } \Omega = R^n_+.$$

Problem 4 [30] Nonsmooth Function I.

$$f_i(x) = 2x_i - \sin|x_i|, \text{ for } i = 1, 2, \ldots, n,$$
$$\text{and } \Omega = R^n_+.$$

Problem 5 [10] Tridiagonal Exponential Function.

$$\begin{aligned}
f_1(x) &= x_1 - e^{\cos(h(x_1+x_2))}, \\
f_i(x) &= x_i - e^{\cos(h(x_{i-1}+x_i+x_{i+1}))}, \text{ for } i = 2, \ldots, n-1, \\
f_n(x) &= x_n - e^{\cos(h(x_{n-1}+x_n))}, \\
h &= \frac{1}{n+1} \text{ and } \Omega = R^n_+.
\end{aligned}$$

Problem 6 [28] Nonsmooth Function II.

$$\begin{aligned}
&f_i(x) = x_i - \sin|x_i - 1|, \text{ for } i = 1, 2, \ldots, n, \\
&\text{and } \Omega = \{x \in R^n : \sum_{i=1}^{n} x_i \leq n, \ x_i \geq -1, \ i = 1, 2, \ldots, n\}.
\end{aligned}$$

Problem 7 [16]

$$\begin{aligned}
f_1(x) &= 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2) \\
f_i(x) &= 3x_i^3 + 2x_{i+1} - 5 + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) \\
&\quad + 4x_i - x_{i-1}e^{x_{i-1}-x_i} - 3, \\
&\text{for } i = 2, 3, ..., n-1 \\
f_n(x) &= x_{n-1}e^{x_{n-1}-x_n} - 4x_n - 3, \\
&\text{and } \Omega = R^n_+.
\end{aligned}$$

Problem 8 Pursuit-Evasion Problem.

$$\begin{aligned}
&f_i(x) = \sqrt{8}x_1 - 1, \text{ for } i = 1, 2, \ldots, n, \\
&\text{and } \Omega = R^n_+.
\end{aligned}$$

Problem 9

$$\begin{aligned}
&f_i(x) = e^{x_i^2} + 3\sin(x_i)\cos(x_i) - 1, \text{ for } i = 1, 2, \ldots, n, \\
&\text{and } \Omega = R^n_+.
\end{aligned}$$

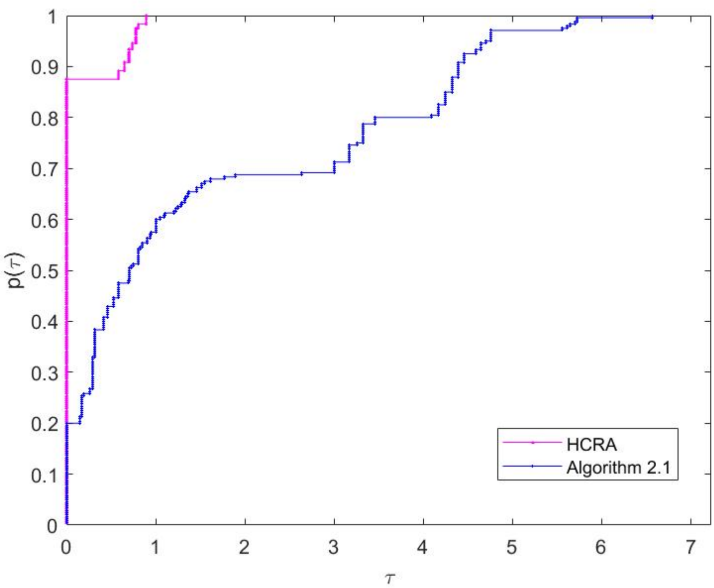

FIGURE 1. Performance based on the number of iterations.

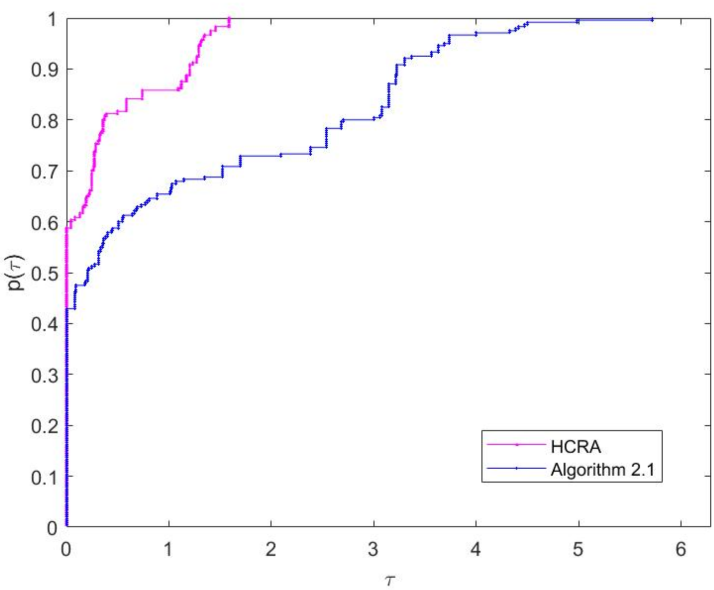

FIGURE 2. Performance based function evaluation.

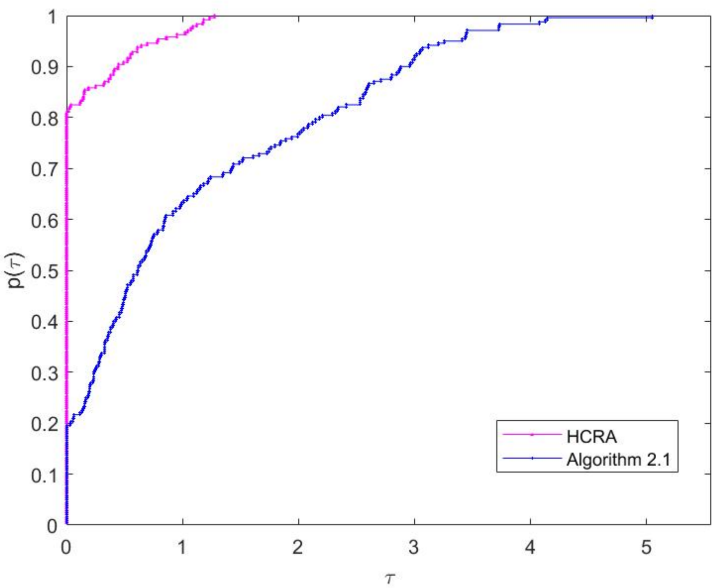

Figure 3. Performance based on CPU time.

To show clearly the performance of each algorithm, we apply the performance profile of Dolan and Morè [11]. The metrics considered are; number of iterations, CPU time (in seconds) and number of function evaluations. Figures 1-3 show that the proposed algorithm HCRA is the best solver as it stays longer on the $y-$axis.

5. Final Remarks

In this paper, we propose a hybrid conjugate residual algorithm for solving nonlinear monotone equations with convex constraints. The algorithm generate a finite sequence of iterates that converges to the solution of nonlinear systems of equations in which the function satisfy the monotonocity and Lipschitz continuity assumptions. Numerical tests on some problems show that the proposed algorithm is robust and efficient compared with the recent algorithm proposed by Li et al. (Journal of Computational and Applied Mathematics, 375:112781, 2020).

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