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An Application of Hypergeometric Distribution Type Series on Certain Analytic Functions

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Abstract In the present paper, we introduce hypergeometric distribution type series and obtain some necessary and sufficient condition for this series belonging to certain classes of analytic univalent functions. We also consider an integral operator related to this series.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in $\mathbb{U}$.

A function f of the form (1.1) is said to be starlike of order α if it satisfies the following condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ \ z \in \mathbb{U},$$

and is said to be convex of order α if it satisfies the following condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ \ z \in \mathbb{U}.$$

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The classes of all starlike and convex functions of order α are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively, studied by Robertson [1].

Further, $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

Let $\mathcal{TS}^*(\alpha) = \mathcal{S}^*(\alpha) \cap \mathcal{T}$ and $\mathcal{TK}(\alpha) = \mathcal{K}(\alpha) \cap \mathcal{T}$. The classes $\mathcal{TS}^*(\alpha)$ and $\mathcal{TK}(\alpha)$ were studied by Silverman [2].

The work of Silverman motivates various researchers to introducing new subclasses of analytic univalent and multivalent functions with negative coefficients. In 1988 Altintas and Owa [3] generalized the classes $\mathcal{TS}^*(\alpha)$ and $\mathcal{TK}(\alpha)$ in to the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$, respectively, which is defined in the following way

A function f of the form (1.2) is said to be in the class $\mathcal{T}(\lambda, \alpha)$ if it satisfy the following condition

$$\Re\left\{\frac{zf'(z)}{\lambda zf'(z) + (1-\lambda)f(z)}\right\} > \alpha, \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$ and for all $z \in \mathbb{U}$.

A function f of the form (1.2) is said to be in the class $\mathcal{C}(\lambda, \alpha)$ if it satisfy the following condition

$$\Re\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)}\right\} > \alpha, \tag{1.4}$$

for some $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$ and for all $z \in \mathbb{U}$.

From (1.3) and (1.4) it is easy to verify that

$$f(z) \in \mathcal{C}(\lambda, \alpha) \Leftrightarrow zf'(z) \in \mathcal{T}(\lambda, \alpha).$$

It is worthy to note that for $\lambda = 0$ the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ reduce to the classes $\mathcal{TS}^*(\alpha)$ and $\mathcal{TK}(\alpha)$, respectively. The certain conditions for hypergeometric functions, generalized Bessel functions for these classes were studied by Mostafa [4] and Porwal and Dixit [5].

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A, B)$ if it satisfy the following condition

$$\left|\frac{f'(z) - 1}{(A-B)\tau - B(f'(z) - 1)}\right| < 1,$$

where $\tau \in \mathbb{C}\backslash\{0\}, -1 \leqslant B < A \leqslant 1, z \in \mathbb{U}$. The class $\mathcal{R}^\tau(A, B)$ was studied by Dixit and Pal [6].

Recently Porwal [7] gives a beautiful application of Poisson distribution series to these classes. Motivating with the above mentioned work, we obtain necessary and sufficient conditions for hypergeometric distribution type series belonging to these classes. First we introduce hypergeometric type distribution as follows, for this purpose we recall the definition of hypergeometric series. The power series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$ and $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$\begin{aligned}(a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)} \\ &= \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)\ldots\ldots\ldots(a+n-1) & \text{if } n \in N = \{1, 2, 3, \ldots\} \end{cases}\end{aligned}$$

is called the hypergeometric series. The series converges absolutely if $|z| < 1$ and diverges if $|z| > 1$ and for $|z| = 1$ the series is absolutely convergent if $\Re(c - a - b) > 0$. It is denoted by $F(a, b; c; z)$.

From the above definition it is easy to see that for $a, b, c > 0$ and $0 < m < 1$ the series

$$F(a, b; c; m) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} m^n$$

is convergent. The above series is also convergent if $a, b, c > 0$, $c > a + b$ and $m = 1$. Now we introduce hypergeometric type distribution whose probability mass function is

$$P(a, b; c; m; n) = \frac{(a)_n (b)_n m^n}{(c)_n n! F(a, b; c; m)}, \; n = 0, 1, 2, \ldots.$$

It is easy to see that

$$P(a, b; c; m; n) \geq 0$$

and

$$\sum_{n=0}^{\infty} P(a, b; c; m; n) = 1.$$

Now we introduce a new series $I(a, b; c; m; z)$ in the following way

$$I(a, b; c; m; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1} m^{n-1}}{(c)_{n-1} (n-1)! F(a, b; c; m)} z^n,$$

where $a, b, c > 0$ and $0 < m < 1$.

The series $I(a, b; c; m; z)$ is absolutely convergent for $|z| < 1$ because

$$\begin{aligned}&|z| + \sum_{n=2}^{\infty} \left| \frac{(a)_{n-1} (b)_{n-1} m^{n-1}}{(c)_{n-1} (n-1)! F(a, b; c; m)} \right| |z^n| \\ &= |z| + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1} m^{n-1}}{(c)_{n-1} (n-1)! F(a, b; c; m)} |z^n| \\ &\leq |z| + \sum_{n=2}^{\infty} |z^n|, \text{ since} \frac{(a)_{n-1} (b)_{n-1} m^{n-1}}{(c)_{n-1} (n-1)! F(a, b; c; m)} \leq 1 \\ &= \frac{|z|}{1 - |z|}.\end{aligned}$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Now, we consider a linear operator $K(a,b;c;m;z):\mathcal{A}\to\mathcal{A}$ defined by

$$\begin{aligned}K(a,b;c;m;z) &= I(a,b;c;m;z) * f(z)\\ &= z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}a_n z^n.\end{aligned}$$

In the present paper, we obtain necessary and sufficient condition for the function

$$\begin{aligned}I_1(z) &= 2z - I(a,b;c;m;z)\\ &= z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}z^n,\end{aligned}$$

belonging to the classes $\mathcal{T}(\lambda,\alpha)$ and $\mathcal{C}(\lambda,\alpha)$ and connections for the function $I_2f(z) = 2z-K(a,b;c;m;z)$ between the classes $\mathcal{R}^{\tau}(A,B)$ and $\mathcal{C}(\lambda,\alpha)$. Finally, we give a condition for an integral operator $G(a,b;c;m;z)$ belonging to the class $\mathcal{C}(\lambda,\alpha)$.

2. Main Results

To establish our main results, we shall require the following lemmas due to Dixit and Pal [6], Altintas and Owa [3].

Lemma 2.1. ([6]) *If $f\in\mathcal{R}^{\tau}(A,B)$ is of the form (1.1) then*

$$|a_n|\le\frac{(A-B)|\tau|}{n},\quad (n\in\mathbb{N}\setminus\{1\}). \tag{2.1}$$

The bounds given in (2.1) is sharp.

Lemma 2.2. ([3]) *A function $f(z)$ defined by (1.2) is in the class $\mathcal{T}(\lambda,\alpha)$, if and only if*

$$\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]\,|a_n|\le 1-\alpha.$$

Lemma 2.3. ([3]) *A function $f(z)$ defined by (1.2) is in the class $\mathcal{C}(\lambda,\alpha)$, if and only if*

$$\sum_{n=2}^{\infty}n\,[n-\lambda\alpha n-\alpha+\lambda\alpha]\,|a_n|\le 1-\alpha.$$

Theorem 2.4. *If $a,b,c>0$ and $0<m<1$ then $I_1(z)$ is in $\mathcal{T}(\lambda,\alpha)$, if and only if*

$$(1-\alpha\lambda)\frac{ab}{c}mF(a+1,b+1;c+1;m)\le 1-\alpha. \tag{2.2}$$

Proof. Since

$$I_1(z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}z^n,$$

according to Lemma 2.2, we must show that

$$\sum_{n=2}^{\infty}[n(1-\lambda\alpha)-\alpha(1-\lambda)]\,\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}\le 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1-\alpha\lambda)-\alpha(1-\lambda)]\,\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}\\
&=\frac{1}{F(a,b;c;m)}\left[\sum_{n=2}^{\infty}[(n-1)(1-\alpha\lambda)+(1-\alpha)]\,\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!}\right]\\
&=\frac{1}{F(a,b;c;m)}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-2)!}\right.\\
&\quad\left.+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!}\right]\\
&=\frac{1}{F(a,b;c;m)}\left[(1-\alpha\lambda)\frac{ab}{c}m\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}m^{n-2}}{(c+1)_{n-2}(n-2)!}\right.\\
&\quad\left.+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!}\right]\\
&=\frac{1}{F(a,b;c;m)}\left[(1-\alpha\lambda)\frac{ab}{c}mF(a+1,b+1;c+1;m)+(1-\alpha)\left(F(a,b;c;m)-1\right)\right].
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.2) holds.
Thus the proof of Theorem 2.4 is established. ∎

Theorem 2.5. *If $a,b,c>0$ and $0<m<1$, then $I_1(z)$ is in $\mathcal{C}(\lambda,\alpha)$, if and only if*

$$\begin{aligned}
&(1-\alpha\lambda)\frac{a(a+1)b(b+1)}{c(c+1)}m^2F(a+2,b+2;c+2;m)\\
&+(3-2\alpha\lambda-\alpha)\frac{ab}{c}mF(a+1,b+1;c+1;m)\leq 1-\alpha. \qquad (2.3)
\end{aligned}$$

Proof. Since

$$I_1(z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}z^n,$$

according to Lemma 2.3, we must show that

$$\sum_{n=2}^{\infty}n\,[n(1-\lambda\alpha)-\alpha(1-\lambda)]\,\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}\leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right] \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)} \\
&=\sum_{n=2}^{\infty}\{(1-\alpha\lambda)(n-1)(n-2)+(3-2\alpha\lambda-\alpha)(n-1)+(1-\alpha)\} \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)} \\
&=\frac{1}{F(a,b;c;m)}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-3)!}\right. \\
&\left.+(3-2\alpha\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!}\right] \\
&=\frac{1}{F(a,b;c;m)}\left[(1-\alpha\lambda)\frac{a(a+1)b(b+1)}{c(c+1)}m^2F(a+2,b+2;c+2;m)\right. \\
&+(3-2\alpha\lambda-\alpha)\frac{ab}{c}mF(a+1,b+1;c+1;m) \\
&\left.+(1-\alpha)\left(F(a,b;c;m)-1\right)\right]
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.3) holds.

This completes the proof of Theorem 2.5. ∎

Theorem 2.6. *Let $a,b,c>0$ and $0<m<1$. If $f\in\mathcal{R}^{\tau}(A,B)$ and the inequality*

$$(A-B)|\tau|\left[(1-\alpha\lambda)\frac{ab}{c}mF(a+1,b+1;c+1;m)+(1-\alpha)\left(F(a,b;c;m)-1\right)\right]$$

$$\leq F(a,b;c;m)(1-\alpha), \tag{2.4}$$

is satisfied then $I_2f(z)\in\mathcal{C}(\lambda,\alpha)$.

Proof. By Lemma 2.3, it suffices to show that

$$P_1=\sum_{n=2}^{\infty} n\left[n-\lambda\alpha n-\alpha+\lambda\alpha\right]\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}|a_n|\leq 1-\alpha.$$

Since $f\in\mathcal{R}^{\tau}(A,B)$ then by Lemma 2.1 we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Hence

$$\begin{aligned}
P_1&\leq(A-B)|\tau|\sum_{n=2}^{\infty}\left[n(1-\alpha\lambda)-\alpha(1-\lambda)\right]\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)} \\
&=(A-B)|\tau|\sum_{n=2}^{\infty}\left[(n-1)(1-\alpha\lambda)+(1-\alpha)\right]\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)} \\
&=\frac{(A-B)|\tau|}{F(a,b;c;m)}\left[(1-\alpha\lambda)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!}\right] \\
&=\frac{(A-B)|\tau|}{F(a,b;c;m)}\left[(1-\alpha\lambda)\frac{ab}{c}mF(a+1,b+1;c+1;m)+(1-\alpha)\left(F(a,b;c;m)-1\right)\right]
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.4) holds. ■

3. An Integral Operator

In the following theorem, we obtain analogues results in connection with a particular integral operator $G(a,b;c;m;z)$ as follows

$$G(a,b;c;m;z) = \int_0^z \left(\frac{I_1(t)}{t}\right) dt. \tag{3.1}$$

Theorem 3.1. *If $a,b,c>0$ and $0<m<1$ then $G(a,b;c;m;z)$ defined by (3.1) is in $\mathcal{C}(\lambda,\alpha)$, if and only if (2.2) holds.*

Proof. From the representation of (3.1), we have

$$G(a,b;c;m;z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)} \frac{z^n}{n}$$

by Lemma 2.3, we need only to show that

$$\sum_{n=2}^{\infty} n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right] \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)n} \le 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right] \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)n}\\
&=\sum_{n=2}^{\infty} \left[n(1-\lambda\alpha)-\alpha(1-\lambda)\right] \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}\\
&=\sum_{n=2}^{\infty} \left[(n-1)(1-\lambda\alpha)+(1-\alpha)\right] \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}\\
&=(1-\alpha\lambda)\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-2)!F(a,b;c;m)} + (1-\alpha)\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a,b;c;m)}\\
&=\frac{1}{F(a,b;c;m)}\left[(1-\alpha\lambda)\frac{ab}{c}mF(a+1,b+1;c+1;m)+(1-\alpha)(F(a,b;c;m)-1)\right]
\end{aligned}$$

which is bounded above by $1-\alpha$, if and only if (2.2) holds. ■

Remark 3.2. The results of Theorem 2.4-3.1 also holds for $a,b,c>0$, $c>a+b$ and $m=1$.

References

[1] M.S. Robertson, On the theory of univalent functions, Ann. Math. 37 (1936) 374–408.

[2] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975) 109–116.

[3] O. Altintas, S. Owa, On subclasses of univalent functions with negative coefficients, Pusan Kyŏngnam Math. J. 4 (1988) 41–56.

[4] A.O. Mostafa, A study on starlike and convex properties for hypergeometric functions, J. Inequal. Pure Appl. Math. 10 (3) (2009) Article No. 87.

[5] S. Porwal, K.K. Dixit, An application of generalized Bessel functions on certain analytic functions, Acta Universitatis Matthiae Belii, Series Mathematics 21 (2013) 55–61.

[6] K.K. Dixit, S.K. Pal, On a class of univalent functions related to complex order, Indian J. Pure Appl. Math. 26 (9) (1995) 889–896.

[7] S. Porwal, An application of a Poisson distribution series on certain analytic functions, J. Complex Anal. 2014 (2014) Article ID 984135.