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Smooth Fuzzy Topology on Crisp Sets

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Abstract In almost all examples available in the literature, while introducing fuzzy concepts, certain properties like intelligence and beauty are considered on objects like students and flowers. The properties are fuzzy in nature whereas the objects are crisp in nature. The motive for introducing fuzzy mathematics is to discuss about the fuzzy properties on crisp objects. But no significant separate theory of fuzzy properties on crisp objects is available in the literature. In this paper we develop a theory, named co-smooth fuzzy theory, exclusively to study fuzzy properties of crisp objects. We investigate co-smooth fuzzy topology, basis, sub-basis, product topology and discuss Hausdorffness in the new context.

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1. Introduction

The concept of fuzzy mathematics is well known. Intelligence, beauty, and temperature are some of many examples available in the literature to introduce the concept fuzziness. These properties are discussed on objects like students, flowers, and so on. It is obvious that the properties are fuzzy in nature and the objects are crisp in nature. So naturally the main aim of fuzzy mathematics is to discuss fuzzy properties defined on crisp objects. But in the literature of fuzzy mathematics, mostly crisp properties on fuzzy objects are studied.

C. Chang [1] introduced the concept of fuzzy topology on a set X as a collection τ of fuzzy sets of X satisfying the well-known conditions for a collection of sets to be a topology, and called each member of τ as a fuzzy open set. As a fuzzy set can be considered as a single crisp object, the objects considered in Chang fuzzy topology are crisp in nature and the properties like open, closed, are also crisp in nature. For example, a fuzzy set s either open or it is not open. Hence, as rightly pointed out by A. P. Šostak [2], there is a lack of fuzziness in Chang's approach.

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In 1985, A. P. Šostak [2] developed a theory by declaring fuzzy topology as a function τ, called degree of openness, from the collection of all fuzzy subsets of X to $[0, 1]$ satisfying some properties. Ramadan [3] gave a similar definition of fuzzy topology on a fuzzy set in Šostak's sense under the name of "smooth fuzzy topological spaces". Many others like Rekha Srivastava, Kalaivani and Roopkumar [4, 5] studied the concept of fuzzy topological spaces in Šostak's sense. Gregori and Vidal [6] gave some sort of fuzziness to open sets of a Chang's fuzzy topology by giving a gradation of openness to each fuzzy set in Šostak sense. Mingsheng Ying [7] cruised the theory of Šostak in the context of crisp sets in the name of fuzzyfying topology.

Höhle [8] developed a theory by defining a function $\mathscr{D}$ from the collection $\mathscr{P}(X)$ of all subsets of a set X to a complete lattice L with interval topology to study fuzzy measurable spaces; he also defined L-stochastic topology as a L-fuzzy subsets of $X \times \mathscr{P}(X)$. Rekha Srivastava [4] studied the concept of base, subbase, product topology and separation axioms in Šostak sense; a drawback in this theory is, the fuzzy product topology obtained from a base is not unique. Moreover a fuzzy topology is not a base in this sense as a base is a collection of fuzzy sets whereas a topology is a function from the collection of all fuzzy sets.

In Section 2, we give the basic definitions and results from the literature which we need to develop our theory; in Section 3, we study co-smooth fuzzy topology and basis, and prove that the co-smooth fuzzy topology generated by a basis is unique and every co-smooth fuzzy topology is a basis for itself as in the crisp theory; we prove some more results in the same section; in Section 4, we extend the theory in the turf of product topology and subbasis. Finally we discuss Hausdorffness in Section 5.

2. Preliminary Definitions and Results

First let us fix the notations. For any set X, by $\mathscr{P}(X)$ we denote the collection of all subsets of X; by $\mathbb{R}$ and $\emptyset$, we denote the set of real numbers and the empty set respectively. For $a, b \in \mathbb{R}$ the minimum of a and b is denoted by $a \wedge b$ and the maximum of a and b is denoted by $a \vee b$. If f and g are real valued function, we write $f \leq g$ to mean $f(x) \leq g(x)$ for all x.

We now give some definitions and results available in the literature. In [4] a base (called a basis by others) is defined as follows.

Definition 2.1 ([4])**.** Let (X, τ) be a fuzzy topological space. Then a family $\mathscr{B} = \{B \in I^X : \tau(B) > 0\}$ is called a base of (X, τ) if and only if for all $U \in I^X$ with $\tau(U) > 0$ and for all fuzzy points $x_\alpha \in U$, there exists a $B \in \mathscr{B}$ such that $x_\alpha \in B \subseteq U$.

In [4] a subbase of (X, τ) is defined as a family $\mathscr{S}$ of fuzzy sets for which the family $\mathscr{B}_{\mathscr{S}}$ of finite intersections of members of $\mathscr{S}$ is a base of (X, τ). According to these definitions a basis or a subbasis is defined only if a fuzzy topology is available on X; but the definitions for a basis and a subbasis which we are going to give shortly do not need any topological structure on X.

Theorem 2.2 ([4])**.** *Let $\mathscr{S} \subseteq I^X$, contain $\tilde{0}$ and $\tilde{1}$. Let τ be any map from $\mathscr{S}$ to $[0, 1]$ such that $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$ and $\tau(U) > 0$, for all $U \in \mathscr{S}$. Then the extension $\tau_{\mathscr{S}} : I^X \to [0, 1]$ given as follows: for each $U \in I^X$,*

$$\tau_{\mathscr{S}}(U) = \begin{cases} \inf\{\tau(U_1), \tau(U_2)\} & \textit{if } U = U_1 \cap U_2 \textit{ where } U_1, U_2 \in \mathscr{S} \\ \sup\{\tau(W_i)\} & \textit{if } U = \cup W_i \textit{ where each } W_i \in \mathscr{B}_{\mathscr{S}} \\ 0 & \textit{otherwise} \end{cases}$$

defines a gradation of openness on X.

The fuzzy topological space (X, τ) described in the above theorem is called a fuzzy topological space generated by $\mathscr{S}$ in [4] and in the same reference it is stated that the fuzzy topology on X generated by $\mathscr{S}$, is not unique. However, the basis and subbasis we are going to define generate unique fuzzy topological spaces.

Definition 2.3 ([9])**.** A smooth fuzzy topological space (μ, τ) is said to be a fuzzy Hausdorff space if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in I^X$ with $\tau(U) > 0$ and $\tau(V) > 0$ such that $U(x) = \mu(x), V(y) = \mu(y)$ and $U \wedge V = 0_X$.

According to this definition a smooth fuzzy topological space is either fuzzy Hausdorff or it is not fuzzy Hausdorff. So the property of being Hausdorff is crisp in nature. In this paper we define Hausdorffness as one which is fuzzy in nature. Of course, there are a few articles are available in the literature which deals Hausdorffness in fuzzy nature; for example in [6], gradation of openness is assigned for the open sets of a Chang fuzzy topological space by means of a map $\sigma : I^X \to [0, 1]$ and a type of Hausdorffness is discussed; but our approach is entirely different.

3. Co-Smooth Fuzzy Topological Spaces

We start the section, with the definition of co-smooth fuzzy topological space.

Definition 3.1. Let X be any set and let $\mathcal{T} : \mathscr{P}(X) \to [0, 1]$ be a mapping satisfying the following conditions.

- i. $\mathcal{T}(X) = 1$
- ii. $\mathcal{T}(\emptyset) = 1$
- iii. $\mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for any two subsets A, B of X
- iv. $\mathcal{T}(\cup A_\lambda) \geq \wedge \mathcal{T}(A_\lambda)$ for any collection $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of X.

then $\mathcal{T}$ is called a co-smooth fuzzy topology on X and the pair $(X, \mathcal{T})$ is called a co-smooth fuzzy topological space. If $A \subseteq X$, then $\mathcal{T}(A)$ is called the degree of openness of the set A in $(X, \mathcal{T})$.

Let $\mathcal{C} : \mathscr{P}(X) \to [0, 1]$ be a mapping defined by $\mathcal{C}(A) = \mathcal{T}(A^c)$ where A^c is the complement of A in X; $\mathcal{C}(A)$ is called the degree of closedness of A.

Let (X, τ) be a (crisp) topological space and $\mathcal{T} : \mathscr{P}(X) \to [0, 1]$ be map defined by

$$\mathcal{T}(A) = \begin{cases} 1 & \text{if } A \in \tau \\ 0 & \text{otherwise} \end{cases}$$

then, $\mathcal{T}$ is a co-smooth fuzzy topology on X. In this way we can view crisp topologies as co-smooth fuzzy topologies. If $(X, \mathcal{T})$ is a co-smooth fuzzy topology in which $\mathcal{T}$ assumes only the values 1 and 0, then the collection of all sets A for which $\mathcal{T}(A) = 1$ forms a topology on X. More generally, if $(X, \mathcal{T})$ is a co-smooth fuzzy topology then for any α, the collection of all sets A for which $\mathcal{T}(A) \geq \alpha$ forms a (crisp) topology on X.

Definition 3.2. If $\mathcal{T}$ and $\mathcal{T}'$ are two co-smooth fuzzy topologies on a given set X and if $\mathcal{T}' \geq \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ or equivalently $\mathcal{T}$ is coarser than $\mathcal{T}'$. Strict finer and strict coarser can be defined accordingly. We say that $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T}' \geq \mathcal{T}$ or $\mathcal{T} \geq \mathcal{T}'$.

We now give the definition of basis for a co-smooth fuzzy topology.

Definition 3.3. Let X be any set. Define a function $\mathcal{B} : \mathscr{P}(X) \to [0, 1]$ satisfying the following conditions:

i. Given $x \in X$ and $\epsilon > 0$ there exists $A \subseteq X$ such that $x \in A$ and $\mathcal{B}(A) \geq 1 - \epsilon$.
ii. If $x \in A \cap B$ and $\epsilon > 0$ be given then there exists $C \subseteq X$ such that $x \in C \subseteq A \cap B$ and $\mathcal{B}(C) \geq (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon$.

then $\mathcal{B}$ is called a basis for a co-smooth fuzzy topology on X.

In the sequel we prove that every basis generates a unique co-smooth fuzzy topology.

Example 3.4. Let $X = \mathbb{R}$, Define $\mathcal{B} : \mathscr{P}(X) \to [0, 1]$ as

$$\mathcal{B}(B) = \begin{cases} 1 & \text{if } A = X \\ \frac{1}{1+(b-a)} & \text{if } A = (a, b) \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{B}$ is a basis for the usual topology on X in the sense as mentioned below Definition 3.1.

Example 3.5. Let $X = (0, 1)$, Define $\mathcal{B} : \mathscr{P}(X) \to [0, 1]$ as

$$\mathcal{B}(A) = \begin{cases} 1 & \text{if } A = X \\ q & \text{if } A = (q, 1) \text{ where } q \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{B}$ is a basis for a topology on X.

The topology generated by the above basis is different from the usual topology. We note that $\mathcal{B}((a, 1)) = 0$ and $\mathcal{T}((a, 1)) = a$ if a is irrational.

Definition 3.6. A collection $\{A_\lambda\}_{\lambda \in \Lambda}$ of nonempty subsets of a set A is called an inner cover for A if $A = \bigcup\limits_{\lambda \in \Lambda} A_\lambda$.

Definition 3.7. Let $\mathcal{B} : \mathscr{P}(X) \to [0, 1]$ be a basis for a co-smooth fuzzy topology on a set X. Define $\mathcal{T} : \mathscr{P}(X) \to [0, 1]$ as follows. Define $\mathcal{T}(\emptyset) = 1$; let $\emptyset \subsetneq A \subseteq X$ and let $\{C_\Lambda\}_{\Lambda \in \Gamma}$ be the collection of all possible inner covers $\{A_\lambda\}_{\lambda \in \Lambda}$ of A; let

$$\mathcal{T}(A) = \sup_{\Lambda \in \Gamma} \left\{ \inf_{A_\lambda \in C_\Lambda} \{\mathcal{B}(A_\lambda)\} \right\}.$$

Then $\mathcal{T}$ is a co-smooth fuzzy topology which is called the co-smooth fuzzy topology generated by $\mathcal{B}$.

The following theorem proves that the definition is well defined.

Theorem 3.8. *Let $\mathcal{B}$ be a basis for a co-smooth fuzzy topology on a set X. If $\mathcal{T}$ is as defined in Definition 3.7, then $\mathcal{T}$ is a co-smooth fuzzy topology on X.*

Proof. Since $\mathcal{B}$ takes values in $[0,1]$, $\mathcal{T}$ is well defined. Now we prove that $\mathcal{T}(X) = 1$. For each $x \in X$ and $\epsilon > 0$ let $A_{x,\epsilon} \subseteq X$ be such that $x \in A_{x,\epsilon}$ and $\mathcal{B}(A_{x,\epsilon}) \geq 1 - \epsilon$. The collection $\{A_{x,\epsilon}\}_{x\in X}$ is then an inner cover for X and

$$\inf_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\} \geq 1 - \epsilon.$$

Thus for each $\epsilon > 0$ there exists an inner cover $\{A_{x,\epsilon}\}$ such that $\inf_{x}\{\mathcal{B}(A_{x,\epsilon})\} \geq 1 - \epsilon$. This implies that

$$\mathcal{T}(X) \geq \sup_{\epsilon}\{\inf_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\}\} \geq 1$$

and hence $\mathcal{T}(X) = 1$. By the definition of $\mathcal{T}$, it follows that $\mathcal{T}(\emptyset) = 1$.

Now we prove that $\mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for any two subsets A and B of X. If $A \cap B = \emptyset$, then there is nothing to prove. If $A \cap B \neq \emptyset$, then let $C = A \cap B$. For $\epsilon > 0$ let $\{A_\lambda\}_{\lambda\in\Lambda_1}$ and $\{B_\gamma\}_{\gamma\in\Lambda_2}$ be inner covers such that

$$\inf_{\lambda\in\Lambda_1}\{\mathcal{B}(A_\lambda)\} \geq \mathcal{T}(A) - \frac{\epsilon}{2} \text{ and } \inf_{\gamma\in\Lambda_2}\{\mathcal{B}(B_\gamma)\} \geq \mathcal{T}(A) - \frac{\epsilon}{2}.$$

Let $C_{\lambda,\gamma} = A_\lambda \cap B_\gamma$ for $\lambda \in \Lambda_1$ and $\gamma \in \Lambda_2$. Let Λ denote the set containing the pairs (λ,γ) for which $C_{\lambda,\gamma} \neq \emptyset$. Since $A \cap B \neq \emptyset$, we have $\Lambda \neq \emptyset$. For $(\lambda,\gamma) \in \Lambda$, and for $x \in C_{\lambda,\gamma}$, let $D_{\lambda,\gamma,x}$ be such that $D_{\lambda,\gamma,x} \subseteq C_{\lambda,\gamma}$ and

$$\mathcal{B}(D_{\lambda,\gamma,x}) \geq (\mathcal{B}(A_\lambda) \wedge \mathcal{B}(B_\gamma)) - \frac{\epsilon}{2}.$$

Then the collection $\{D_{\lambda,\gamma,x}/(\lambda,\gamma) \in \Lambda,\ x \in C_{\lambda,\gamma}\}$ is an inner cover for C. Indeed, if $x \in C$ then $x \in A_\lambda \cap B_\gamma$ for some $(\lambda,\gamma) \in \Lambda$ and hence $x \in D_{\lambda,\gamma,x}$. Now,

$$\begin{aligned}
\inf_{\substack{x\in C_{\lambda,\gamma}\\(\lambda,\gamma)\in\Lambda}}\{\mathcal{B}(D_{\lambda,\gamma,x})\} &\geq \inf_{(\lambda,\gamma)\in\Lambda}\{\mathcal{B}(A_\lambda) \wedge \mathcal{B}(B_\gamma)\} - \frac{\epsilon}{2}\\
&\geq \left\{\inf_{(\lambda,\gamma)\in\Lambda}\{\mathcal{B}(A_\lambda)\} \wedge \inf_{(\lambda,\gamma)\in\Lambda}\{\mathcal{B}(B_\gamma)\}\right\} - \frac{\epsilon}{2}\\
&\geq \left\{\inf_{\lambda\in\Lambda_1}\{\mathcal{B}(A_\lambda)\} \wedge \inf_{\gamma\in\Lambda_2}\{\mathcal{B}(B_\gamma)\}\right\} - \frac{\epsilon}{2}\\
&\geq \left(\mathcal{T}(A) - \frac{\epsilon}{2}\right) \wedge \left(\mathcal{T}(B) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2}\\
&= (\mathcal{T}(A) \wedge \mathcal{T}(B)) - \frac{\epsilon}{2} - \frac{\epsilon}{2}\\
&= (\mathcal{T}(A) \wedge \mathcal{T}(B)) - \epsilon.
\end{aligned}$$

But,

$$\begin{aligned}
\mathcal{T}(A \cap B) &= \mathcal{T}(C)\\
&\geq \inf_{\substack{x\in C_{\lambda,\gamma}\\(\lambda,\gamma)\in\Lambda}}\{\mathcal{B}(D_{\lambda,\gamma,x})\}\\
&\geq (\mathcal{T}(A) \wedge \mathcal{T}(B)) - \epsilon.
\end{aligned}$$

This is true for every $\epsilon > 0$ and hence $\mathcal{T}(A \cap B) \geq (\mathcal{T}(A) \wedge \mathcal{T}(B))$ for any two subsets A, B of X.

Now we prove that $\mathcal{T}(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \mathcal{T}(A_\lambda)$ for any collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of X. For each $\epsilon > 0$ and for each A_λ, let $\{A_{\lambda,\gamma}\}_{\gamma\in\Gamma_\lambda}$, where Γ_λ is an indexing set depending upon A_λ, be an inner cover for A_λ such that

$$\inf_{\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\} \geq \mathcal{T}(A_\lambda) - \epsilon.$$

Since $\{A_{\lambda,\gamma}\}_{\gamma\in\Gamma_\lambda}$ is an inner cover for A_λ, we have $\{A_{\lambda,\gamma}\}_{\lambda,\gamma}$ is an inner cover for $\underset{\lambda\in\Lambda}{\cup} A_\lambda$. Now,

$$\begin{aligned}
\mathcal{T}(\underset{\lambda\in\Lambda}{\cup} A_\lambda) &\geq \inf_{\lambda,\gamma}\{\mathcal{B}(A_{\lambda,\gamma})\} \\
&= \inf_{\lambda\in\Lambda}\left\{\inf_{\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\}\right\} \\
&\geq \inf_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda) - \epsilon\} \\
&= \inf_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\} - \epsilon.
\end{aligned}$$

Since this is true for every $\epsilon > 0$, we get $\mathcal{T}(\underset{\lambda\in\Lambda}{\cup} A_\lambda) \geq \inf_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\}$. Thus $\mathcal{T}(\underset{\lambda\in\Lambda}{\cup} A_\lambda) \geq \underset{\lambda\in\Lambda}{\wedge}\{\mathcal{T}(A_\lambda)\}$ for any collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of X. Hence $\mathcal{T}$ is a co-smooth fuzzy topology on X. ■

Theorem 3.9. *If $(X, \mathcal{T})$ is a co-smooth fuzzy topological space, then $\mathcal{T}$ is a basis and the co-smooth fuzzy topology generated by $\mathcal{T}$ is itself.*

Proof. We take $\mathcal{B} = \mathcal{T}$ and prove that $\mathcal{B}$ is a basis for the co-smooth fuzzy topology $\mathcal{T}$. For any $x \in X$ and $\epsilon > 0$, taking $A = X$, we have $x \in A$ and $\mathcal{B}(A) = \mathcal{T}(A) = \mathcal{T}(X) \geq 1 - \epsilon$. Let $x \in A \cap B$ and $\epsilon > 0$. Taking $C = A \cap B$, we have $x \in C$ and

$$\mathcal{B}(C) = \mathcal{T}(C) = \mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B) \geq (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon.$$

Thus $\mathcal{B}$ is a basis. Now we prove that the co-smooth fuzzy topology generated by $\mathcal{B}$ is $\mathcal{T}$. Let $\mathcal{T}'$ be the co-smooth fuzzy topology generated by $\mathcal{B}$. Let $E \subseteq X$. Since $\mathcal{T}'$ is the co-smooth fuzzy topology generated by $\mathcal{B}$, we have

$$\mathcal{T}'(E) = \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\}$$

where $\{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $\{E_\lambda\}_{\lambda\in\Lambda}$ of E. Since E itself is an inner cover for E, we have

$$\mathcal{T}'(E) \geq \mathcal{B}(E) = \mathcal{T}(E).$$

Therefore $\mathcal{T}' \geq \mathcal{T}$. But by our assumption, $\mathcal{T}$ is a co-smooth fuzzy topology on X and hence for any collection $\{E_\lambda\}_{\lambda\in\Lambda}$ of subsets of X, we have,

$$\mathcal{T}(\underset{\lambda\in\Lambda}{\cup} E_\lambda) \geq \inf_{\lambda\in\Lambda}\{\mathcal{T}(E_\lambda)\}.$$

Thus for any inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ of E, we have

$$\mathcal{T}(\underset{E_\lambda\in C_\Lambda}{\cup} E_\lambda) \geq \inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}.$$

This implies that,

$$\mathcal{T}(E) \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\right\} = \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\} = \mathcal{T}'(E).$$

Therefore $\mathcal{T} \geq \mathcal{T}'$ and hence $\mathcal{T} = \mathcal{T}'$. ■

Theorem 3.10. *Let $(X, \mathcal{T})$ be a co-smooth fuzzy topological space. Let $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ be a function satisfying*

i. $\mathcal{T}(A) \geq \mathcal{B}(A)$ *for all* $A \subseteq X$
ii. if $A \subseteq X$, $x \in A$ *and* $\epsilon > 0$, *then there exists* $B \subseteq X$ *such that* $x \in B \subseteq A$ *and* $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Then $\mathcal{B}$ *is a basis for the co-smooth fuzzy topology* $\mathcal{T}$ *on* X.

Proof. First we prove that $\mathcal{B}$ is a basis for some co-smooth fuzzy topology on X. Taking $A = X$ in (ii.) we obtain that, for each $x \in X$ and $\epsilon > 0$ there exists $B \subseteq X$ such that

$$\mathcal{B}(B) \geq \mathcal{T}(X) - \epsilon = 1 - \epsilon.$$

Thus (i.) of Definition 3.3 holds. If $x \in A \cap B$ and $\epsilon > 0$, then by (ii.), there exists $C \subseteq X$ such that $x \in C \subseteq A \cap B$ and

$$\mathcal{B}(C) \geq \mathcal{T}(A \cap B) - \epsilon.$$

This implies that,

$$\mathcal{B}(C) \geq \mathcal{T}(A \cap B) - \epsilon \geq (\mathcal{T}(A) \wedge \mathcal{T}(B)) - \epsilon \geq (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon.$$

Thus (ii.) of Definition 3.3 holds and hence $\mathcal{B}$ is a basis.

Now we prove that the co-smooth fuzzy topology generated by $\mathcal{B}$ is $\mathcal{T}$. Let $\mathcal{T}'$ be the co-smooth fuzzy topology generated by $\mathcal{B}$. Let $E \subseteq X$ and let $\{C_\Lambda\}_{\Lambda \in \Gamma}$ be the collection of all possible inner covers $\{E_\lambda\}_{\lambda \in \Lambda}$ of E. Now let $C_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$ be an inner cover for E. Then for each E_λ and for each $x \in E_\lambda$, by (ii.), there exists $E_{\lambda,x} \subseteq X$ such that

$$x \in E_{\lambda,x} \subseteq E_\lambda$$

and

$$\mathcal{B}(E_{\lambda,x}) \geq \mathcal{T}(E_\lambda) - \epsilon.$$

Then the collection $\{E_{\lambda,x}\}_{x \in E_\lambda}$ is an inner cover for E_λ and therefore the collection $\{E_{\lambda,x}\}_{\lambda \in \Lambda, x \in E_\lambda}$ is an inner cover for E. Thus for any given inner cover $C_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$ for E, there exists an inner cover $\{E_{\lambda,x}\}_{\lambda \in \Lambda, x \in E_\lambda}$ for E such that

$$\mathcal{B}(E_{\lambda,x}) \geq \mathcal{T}(E_\lambda) - \epsilon,$$

for all $\lambda \in \Lambda$, $x \in E_\lambda$. This implies that

$$\inf_{\substack{E_\lambda \in C_\Lambda \\ x \in E_\lambda}} \{\mathcal{B}(E_{\lambda,x})\} \geq \inf_{E_\lambda \in C_\Lambda} \{\mathcal{T}(E_\lambda) - \epsilon\} \geq \inf_{E_\lambda \in C_\Lambda} \{\mathcal{T}(E_\lambda)\} - \epsilon.$$

Since this is true for every inner cover $C_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$, we have

$$\sup_{\Lambda \in \Gamma} \left\{ \inf_{\substack{E_\lambda \in C_\Lambda \\ x \in E_\lambda}} \{\mathcal{B}(E_{\lambda,x})\} \right\} \geq \sup_{\Lambda \in \Gamma} \left\{ \inf_{E_\lambda \in C_\Lambda} \{\mathcal{T}(E_\lambda)\} \right\} - \epsilon.$$

But by definition of $\mathcal{T}'$ we have $\mathcal{T}'(E) \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in E_\lambda}}\{\mathcal{B}(E_{\lambda,x})\}\right\}$. This implies that,

$$\begin{aligned}\mathcal{T}'(E) &\geq \sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in E_\lambda}}\{\mathcal{B}(E_{\lambda,x})\}\right\}\\ &\geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\right\} - \epsilon\\ &= \mathcal{T}(E) - \epsilon.\end{aligned}$$

Since this is true for every $\epsilon > 0$, we have $\mathcal{T}'(E) \geq \mathcal{T}(E)$ and therefore $\mathcal{T}' \geq \mathcal{T}$. Let $E \subseteq X$ and $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ be an inner cover for E. Then by (ii.), we have, $\mathcal{T}(E_\lambda) \geq \mathcal{B}(E_\lambda)$ for all $\lambda \in \Lambda$. This implies,

$$\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\} \geq \inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}.$$

Since this is true for every inner cover $\{E_\lambda\}_{\lambda\in\Lambda}$, we have

$$\sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\right\} \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\}$$

Thus $\mathcal{T}(E) \geq \mathcal{T}'(E)$ for all subsets E of X, and hence $\mathcal{T} = \mathcal{T}'$. ∎

Theorem 3.11. *Let $(X, \mathcal{T})$ be a co-smooth fuzzy topological space. Let $\mathcal{B}$ be a basis for $\mathcal{T}$, then*

i. $\mathcal{T}(A) \geq \mathcal{B}(A)$ for all $A \subseteq X$
ii. if $A \subseteq X$, $x \in A$ and $\epsilon > 0$, then there exists $B \subseteq X$ such that $x \in B \subseteq A$ and $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Proof. Let A be a subset of X. Since $\mathcal{T}$ is a topology and $\mathcal{B}$ is a basis for $\mathcal{T}$ we have,

$$\mathcal{T}(A) = \sup_{\Lambda\in\Gamma}\left\{\inf_{A_\lambda\in C_\Lambda}\{\mathcal{B}(A_\lambda)\}\right\}$$

where $\{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $\{A_\lambda\}_{\lambda\in\Lambda}$ of A. Since A itself is an inner cover for A, we have $\mathcal{T}(A) \geq \mathcal{B}(A)$ for all $A \subseteq X$. Thus (i.) follows.

To prove (ii.), let $A \subseteq X$, $x \in A$ and $\epsilon > 0$. Then there exists an inner cover $\{A_\lambda\}_{\lambda\in\Lambda}$ such that $\inf_{\lambda\in\Lambda}\{\mathcal{B}(A_\lambda)\} \geq \mathcal{T}(A) - \epsilon$. Since $\{A_\lambda\}_{\lambda\in\Lambda}$ is an inner cover for A, there exists $A_{\lambda_0} \in \{A_\lambda\}_{\lambda\in\Lambda}$ such that $x \in A_{\lambda_0}$. Clearly $A_{\lambda_0} \subseteq A$ and since $A_{\lambda_0} \in \{A_\lambda\}_{\lambda\in\Lambda}$, we have $\mathcal{B}(A_{\lambda_0}) \geq \mathcal{T}(A) - \epsilon$. Thus, (ii.) follows. ∎

Theorem 3.10 and Theorem 3.11 together give a necessary and sufficient condition for a function $\mathcal{B} : \mathscr{P}(X) \to [0, 1]$ to be a basis for a co-smooth fuzzy topology.

Theorem 3.12. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the co-smooth fuzzy topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then the following conditions are equivalent.*

i. $\mathcal{T}'$ is finer than $\mathcal{T}$.
ii. if $B \subseteq X$, $x \in B$ and $\epsilon > 0$, then there exists $B' \subseteq X$ such that $x \in B' \subseteq B$ and $\mathcal{B}'(B') \geq \mathcal{B}(B) - \epsilon$.

Proof. Let $\mathcal{T}'$ be finer than $\mathcal{T}$. Let $B \subseteq X$, $x \in B$ and $\epsilon > 0$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Theorem 3.11, there exists $B' \subseteq X$ such that $x \in B'$, $B' \subseteq B$ and $\mathcal{B}'(B') \geq \mathcal{T}'(B) - \epsilon$. By our assumption $\mathcal{T}'(B) \geq \mathcal{T}(B)$, and hence we have

$$\mathcal{B}'(B') \geq \mathcal{T}'(B) - \epsilon \geq \mathcal{T}(B) - \epsilon \geq \mathcal{B}(B) - \epsilon.$$

Conversely assume that (ii.) holds. Let E be a subset of X and $\epsilon > 0$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, there exists an inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ such that $\inf\limits_{E_\lambda \in C_\Lambda} \{\mathcal{B}(E_\lambda)\} \geq \mathcal{T}(E) - \epsilon$. Then by (ii.), for each E_λ and for each $x \in E_\lambda$, there exists $E_{\lambda,x} \subseteq X$ such that $x \in E_{\lambda,x} \subseteq E_\lambda$ and

$$\mathcal{B}'(E_{\lambda,x}) \geq \mathcal{B}(E_\lambda) - \epsilon.$$

Then the collection $\{E_{\lambda,x}\}_{x\in E_\lambda}$ is an inner cover for E_λ and therefore the collection $\{E_{\lambda,x}\}_{\lambda\in\Lambda, x\in E_\lambda}$ is an inner cover for E. Thus for any given inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ of E, there exists an inner cover $\{E_{\lambda,x}\}_{\lambda\in\Lambda, x\in E_\lambda}$ of E such that

$$\mathcal{B}'(E_{\lambda,x}) \geq \mathcal{B}(E_\lambda) - \epsilon,$$

for all $\lambda \in \Lambda$, $x \in E_\lambda$. This implies that, $\inf\limits_{x\in E_\lambda} \{\mathcal{B}'(E_{\lambda,x})\} \geq \mathcal{B}(E_\lambda) - \epsilon$ and hence

$$\inf_{\substack{E_\lambda \in C_\Lambda \\ x\in E_\lambda}} \{\mathcal{B}'(E_{\lambda,x})\} \geq \inf_{E_\lambda \in C_\Lambda} \{\mathcal{B}(E_\lambda) - \epsilon\} \geq \inf_{E_\lambda \in C_\Lambda} \{\mathcal{B}(E_\lambda)\} - \epsilon.$$

Since this is true for every inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$, we have

$$\sup_{\Lambda\in\Gamma} \left\{ \inf_{\substack{E_\lambda \in C_\Lambda \\ x\in E_\lambda}} \{\mathcal{B}'(E_{\lambda,x})\} \right\} \geq \sup_{\Lambda\in\Gamma} \left\{ \inf_{E_\lambda \in C_\Lambda} \{\mathcal{B}(E_\lambda)\} \right\} - \epsilon.$$

But by definition of $\mathcal{T}'$ we have

$$\begin{aligned} \mathcal{T}'(E) &\geq \sup_{\Lambda\in\Gamma} \left\{ \inf_{\substack{E_\lambda \in C_\Lambda \\ x\in E_\lambda}} \{\mathcal{B}'(E_{\lambda,x})\} \right\} \\ &\geq \sup_{\Lambda\in\Gamma} \left\{ \inf_{E_\lambda \in C_\Lambda} \{\mathcal{B}(E_\lambda)\} \right\} - \epsilon \\ &= \mathcal{T}(E) - \epsilon. \end{aligned}$$

Since this is true for every $\epsilon > 0$ we have, $\mathcal{T}'(E) \geq \mathcal{T}(E)$, and hence $\mathcal{T}' \geq \mathcal{T}$. ∎

Theorem 3.13. *If $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ is a family of co-smooth fuzzy topologies on X, then $\inf\limits_{\lambda\in\Lambda}\{\mathcal{T}_\lambda\}$ is also a co-smooth fuzzy topology on X.*

Proof. Let $\mathcal{T} = \inf\limits_{\lambda\in\Lambda}\{\mathcal{T}_\lambda\}$. Clearly we have $\mathcal{T}(X) = 1$ and $\mathcal{T}(\emptyset) = 1$. Let A and B be subsets of X. Then

$$\begin{aligned} \mathcal{T}(A\cap B) &= \inf_{\lambda\in\Lambda}\{\mathcal{T}_\lambda(A\cap B)\} \\ &\geq \inf_{\lambda\in\Lambda}\{\mathcal{T}_\lambda(A) \wedge \mathcal{T}_\lambda(B)\} \\ &= \inf_{\lambda\in\Lambda}\{\mathcal{T}_\lambda(A)\} \wedge \inf_{\lambda\in\Lambda}\{\mathcal{T}_\lambda(B)\} \\ &= \mathcal{T}(A) \wedge \mathcal{T}(B). \end{aligned}$$

Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be any collection of subsets of X. Then

$$\begin{aligned}\mathcal{T}(\bigcup_{\gamma\in\Gamma} A_\gamma) &= \inf_{\lambda\in\Lambda}\{\mathcal{T}_\lambda(\bigcup_{\gamma\in\Gamma} A_\gamma)\}\\ &\geq \inf_{\lambda\in\Lambda}\{\bigwedge_{\gamma\in\Gamma}\{\mathcal{T}_\lambda(A_\gamma)\}\}\\ &= \bigwedge_{\gamma\in\Gamma}\{\inf_{\lambda\in\Lambda}\{\mathcal{T}_\lambda(A_\gamma)\}\}\\ &= \bigwedge_{\gamma\in\Gamma}\mathcal{T}(A_\gamma).\end{aligned}$$

Thus $\mathcal{T}$ is a co-smooth fuzzy topology on X. ∎

As in the case of crisp topology, if $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ is a family of co-smooth fuzzy topologies on X, then $\sup_\lambda\{\mathcal{T}_\lambda\}$ need not be a co-smooth fuzzy topology on X.

Example 3.14. The collection $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ of functions $\mathscr{P}(\mathbb{R})$ defined as

$$\mathcal{T}_n(A) = \begin{cases} 1 & \text{if } A \in \{\mathbb{R}, \emptyset, \{1\}, \{1,2\}, \dots, \{1,2,\dots,n\}\} \\ 0 & \text{otherwise} \end{cases}$$

are co-smooth fuzzy topologies on $\mathbb{R}$ whereas $\mathcal{T} = \sup\{\mathcal{T}_n\}$ is not. Indeed, if $A_n = \{1, 2, \dots, n\}$, then $\mathcal{T}(\bigcup_{n=1}^{\infty} A_n) \not\geq \wedge\mathcal{T}(A_n)$.

4. Product of Co-Smooth Fuzzy Topologies

Theorem 4.1. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two co-smooth fuzzy topological spaces and let $X = X_1 \times X_2$. Let $\mathcal{B}$ be a map from $\mathscr{P}(X)$ to $[0,1]$ defined as*

$$\mathcal{B}(E) = \begin{cases} \inf\{\mathcal{T}_1(E_1), \mathcal{T}_2(E_2)\} & \text{if } E = E_1 \times E_2 \\ 0 & \text{otherwise} \end{cases}$$

then $\mathcal{B}$ is a basis for a co-smooth fuzzy topology on X.

Proof. Since $\mathcal{B}(X) = \inf\{\mathcal{T}_1(X_1), \mathcal{T}_2(X_2)\} = 1$, (i.) of Definition 3.3 follows. Let $(x, y) \in A \cap B$ and $\epsilon > 0$. Suppose any one of these sets, say A, cannot be written as $A_1 \times A_2$ with $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$, then $\mathcal{B}(A) = 0$ and hence (ii.) of Definition 3.3 follows in this case. Otherwise let $A = A_1 \times A_2$ and $B = B_1 \times B_2$, where A_1, B_1 are subsets of X_1 and A_2, B_2 are subsets of X_2. Let $C = A \cap B$. Clearly $(x, y) \in C$. Now,

$$\begin{aligned}\mathcal{B}(C) &= \mathcal{B}(A\cap B)\\ &= \mathcal{B}((A_1\times A_2)\cap(B_1\times B_2))\\ &= \mathcal{B}((A_1\cap B_1)\times(A_2\cap B_2))\\ &= \inf\{\mathcal{T}_1((A_1\cap B_1), \mathcal{T}_2(A_2\cap B_2))\\ &\geq \inf\{\mathcal{T}_1(A_1)\wedge\mathcal{T}_1(B_1), \mathcal{T}_2(A_2)\wedge\mathcal{T}_2(B_2)\}\\ &= \inf\{\mathcal{T}_1(A_1), \mathcal{T}_2(A_2), \mathcal{T}_1(B_1), \mathcal{T}_2(B_2)\}\\ &= \inf\{\mathcal{T}_1(A_1)\wedge\mathcal{T}_2(A_2), \mathcal{T}_1(B_1)\wedge\mathcal{T}_2(B_2)\}\\ &= \inf\{\mathcal{T}_1(A_1), \mathcal{T}_2(A_2)\}\wedge\inf\{\mathcal{T}_1(B_1), \mathcal{T}_2(B_2)\}\\ &= \mathcal{B}(A)\wedge\mathcal{B}(B)\\ &\geq \mathcal{B}(A)\wedge\mathcal{B}(B)-\epsilon\end{aligned}$$

and hence (ii.) of Definition 3.3 follows in this case also and hence $\mathcal{B}$ is a basis. ∎

Definition 4.2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two co-smooth fuzzy topological spaces. The function $\mathcal{B} : \mathscr{P}(X \times Y) \to [0, 1]$ defined as

$$\mathcal{B}(E) = \begin{cases} \inf\{\mathcal{T}(E_1), \mathcal{T}'(E_2)\} & \text{if } \; E = E_1 \times E_2 \\ 0 & \text{otherwise} \end{cases}$$

then $\mathcal{B}$ is a basis for a co-smooth fuzzy topology on $X \times Y$ and the co-smooth fuzzy topology generated by $\mathcal{B}$ is called the co-smooth fuzzy product topology of $\mathcal{T}$ and $\mathcal{T}'$ on $X \times Y$.

Theorem 4.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two co-smooth fuzzy topological spaces. Let $\mathcal{B}_X, \mathcal{B}_Y$ be bases for the co-smooth fuzzy topologies $\mathcal{T}_X, \mathcal{T}_Y$. Define a function*

$$\mathcal{B}_{X \times Y} : \mathscr{P}(X \times Y) \to [0, 1]$$

as

$$\mathcal{B}_{X \times Y}(E) = \begin{cases} \inf\{\mathcal{B}_X(A), \mathcal{B}_Y(B)\} & \text{if } \; E = A \times B \\ 0 & \text{otherwise} \end{cases}$$

then $\mathcal{B}_{X \times Y}$ is a basis for the co-smooth fuzzy product topology.

Proof. Let $(x, y) \in X \times Y$ and $\epsilon > 0$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for $\mathcal{T}_X$ and $\mathcal{T}_Y$, there exists $A \subseteq X$ and $B \subseteq Y$ such that $x \in A$, $y \in B$ with

$$\mathcal{B}_X(A) \geq 1 - \epsilon \text{ and } \mathcal{B}_Y(B) \geq 1 - \epsilon.$$

Taking $E = A \times B$, we have $(x, y) \in E$ and

$$\mathcal{B}_{X \times Y}(E) = \mathcal{B}_{X \times Y}(A \times B) = \inf\{\mathcal{B}_X(A), \mathcal{B}_Y(B)\} \geq 1 - \epsilon.$$

Thus (i.) of Definition 3.3 follows. Let $(x, y) \in E_1 \cap E_2$ and $\epsilon > 0$. Suppose any one of these sets, say E_1, cannot be written as $A \times B$ with $A \subseteq X$ and $B \subseteq Y$, then $\mathcal{B}_{X \times Y}(E_1) = 0$. Now taking $E_3 = E_1 \cap E_2$, then clearly $(x, y) \in E_3$ and $\mathcal{B}_{X \times Y}(E_1) \wedge \mathcal{B}_{X \times Y}(E_2) = 0$. Hence it follows that

$$\mathcal{B}_{X \times Y}(E_3) \geq (\mathcal{B}_{X \times Y}(E_1) \wedge \mathcal{B}_{X \times Y}(E_2)) - \epsilon.$$

Now suppose that $E_1 = A_1 \times B_1$ and $E_2 = A_2 \times B_2$ where A_1, A_2 are subsets of X and B_1, B_2 are subsets of Y. Let $(x, y) \in E_1 \cap E_2$ and $\epsilon > 0$. Then,

$$(x, y) \in (A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

and hence $x \in A_1 \cap A_2$ and $y \in B_1 \cap B_2$. Thus there exists $A_3 \subseteq X$ and $B_3 \subseteq Y$ such that $x \in A_3 \subseteq A_1 \cap A_2$ and $y \in B_3 \subseteq B_1 \cap B_2$, with

$$\mathcal{B}_X(A_3) \geq (\mathcal{B}_X(A_1) \wedge \mathcal{B}_X(A_2)) - \epsilon \text{ and } \mathcal{B}_Y(B_3) \geq (\mathcal{B}_Y(B_1) \wedge \mathcal{B}_Y(B_2)) - \epsilon.$$

Let $E_3 = A_3 \times B_3$, then clearly $(x, y) \in E_3 \subseteq E_1 \cap E_2$. Now

$$\begin{aligned}
\mathcal{B}_{X\times Y}(E_3) &= \mathcal{B}_{X\times Y}(A_3 \times B_3)\\
&= \inf\{\mathcal{B}_X(A_3), \mathcal{B}_Y(B_3)\}\\
&\geq \inf\{(\mathcal{B}_X(A_1) \wedge \mathcal{B}_X(A_2)) - \epsilon, (\mathcal{B}_Y(B_1) \wedge \mathcal{B}_Y(B_2)) - \epsilon\}\\
&= \inf\{(\mathcal{B}_X(A_1) \wedge \mathcal{B}_X(A_2)), (\mathcal{B}_Y(B_1) \wedge \mathcal{B}_Y(B_2))\} - \epsilon\\
&= \inf\{(\mathcal{B}_X(A_1) \wedge \mathcal{B}_Y(B_1)), (\mathcal{B}_X(A_2) \wedge \mathcal{B}_Y(B_2))\} - \epsilon\\
&= \inf\{\mathcal{B}_{X\times Y}(A_1 \times B_1), \mathcal{B}_{X\times Y}(A_2 \times B_2)\} - \epsilon\\
&= \inf\{\mathcal{B}_{X\times Y}(E_1), \mathcal{B}_{X\times Y}(E_2)\} - \epsilon\\
&= (\mathcal{B}_{X\times Y}(E_1) \wedge \mathcal{B}_{X\times Y}(E_2)) - \epsilon.
\end{aligned}$$

Thus (ii.) of Definition 3.3 follows and hence $\mathcal{B}_{X\times Y}$ is a basis for a co-smooth fuzzy topology on $X\times Y$. Let $\mathcal{T}$ be the co-smooth fuzzy topology generated by $\mathcal{B}_{X\times Y}$. let $\mathcal{T}_{X\times Y}$ be the co-smooth fuzzy product topology on $X \times Y$ and $\mathcal{B}^p_{X\times Y}$ be the basis for $\mathcal{T}_{X\times Y}$ as described in Definition 4.2. Now we prove that $\mathcal{T}_{X\times Y} = \mathcal{T}$. Let $E \subseteq X \times Y$; then

$$\mathcal{T}_{X\times Y}(E) = \sup_{\Lambda\in\Gamma}\left\{\inf_{\lambda\in\Lambda}\left\{\mathcal{B}^p_{X\times Y}(E_\lambda)\right\}\right\}$$

where $\mathfrak{C} = \{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ of E. We divide the collection $\mathfrak{C}$ into two subcollections $\mathfrak{C}'$ and $\mathfrak{C}''$ where $\mathfrak{C}'$ is the collection all possible inner covers $\{E_\lambda\}_{\lambda\in\Lambda}$ of E so that for all $\lambda \in \Lambda$, E_λ is of the form $A_\lambda \times B_\lambda$, and $\mathfrak{C}''$ is the complement of $\mathfrak{C}'$ in $\mathfrak{C}$. If an inner cover $\{E_\lambda\}_{\lambda\in\Lambda}$ is in $\mathfrak{C}''$, then at least one E_λ is not of the form $A_\lambda \times B_\lambda$; Therefore

$$\inf_{\lambda\in\Lambda}\left\{\mathcal{B}^p_{X\times Y}(E_\lambda)\right\} = 0 = \inf_{\lambda}\left\{\mathcal{B}_{X\times Y}(E_\lambda)\right\}$$

and hence

$$\begin{aligned}
\mathcal{T}_{X\times Y}(E) &= \sup_{\mathfrak{C}}\{\inf_{\lambda}\{\mathcal{B}^p_{X\times Y}(E_\lambda)\}\}\\
&= \sup_{\mathfrak{C}'}\{\inf_{\lambda}\{\mathcal{B}^p_{X\times Y}(E_\lambda)\}\}\\
&= \sup_{\mathfrak{C}'}\{\inf_{\lambda}\{\mathcal{B}^p_{X\times Y}(A_\lambda \times B_\lambda)\}\}\\
&= \sup_{\mathfrak{C}'}\{\inf_{\lambda}\{\mathcal{T}_X(A_\lambda) \wedge \mathcal{T}_Y(B_\lambda)\}\}\\
&\geq \sup_{\mathfrak{C}'}\{\inf_{\lambda}\{\mathcal{B}_X(A_\lambda) \wedge \mathcal{B}_Y(B_\lambda)\}\}\\
&= \sup_{\mathfrak{C}}\{\inf_{\lambda}\{\mathcal{B}_{X\times Y}(E_\lambda)\}\}\\
&= \mathcal{T}(E).
\end{aligned}$$

This implies that $\mathcal{T}_{X\times Y} \geq \mathcal{T}$. To prove the reverse inequality, let $E \subseteq X\times Y$ and $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}''$ be as above. If an inner cover $\{E_\lambda\}_{\lambda\in\Lambda}$ is in $\mathfrak{C}''$, then we have nothing to prove. So let $\{E_\lambda\}_{\lambda\in\Lambda} \in \mathfrak{C}'$. Then for all λ, let $E_\lambda = A_\lambda \times B_\lambda$. Now since, $\mathcal{B}_X, \mathcal{B}_Y$ are bases for $\mathcal{T}_X, \mathcal{T}_Y$, by Theorem 3.11, for any $\epsilon > 0$, $\lambda \in \Lambda$, $x \in A_\lambda$, $y \in B_\lambda$, there exist sets, $A_{\lambda,x}$ and $B_{\lambda,y}$ such that,

$$B_X(A_{\lambda,x}) + \epsilon \geq \mathcal{T}_X(A_\lambda)$$
$$\mathcal{B}_Y(B_{\lambda,y}) + \epsilon \geq \mathcal{T}_Y(B_\lambda).$$

Clearly $\{A_{\lambda,x}\}_{x\in A_\lambda}$ and $\{B_{\lambda,y}\}_{y\in B_\lambda}$ are inner covers for A_λ and B_λ respectively. This implies that,

$$A_\lambda \times B_\lambda = \bigcup_{x\in A_\lambda, y\in B_\lambda} (A_{\lambda,x} \times B_{\lambda,y}).$$

This implies, the collection $\{A_{\lambda,x} \times B_{\lambda,y}\}_{\lambda\in\Lambda, x\in A_\lambda, y\in B_\lambda}$ is an inner cover for E. Now since this is true for all $\{E_\lambda\}_{\lambda\in\Lambda} \in \mathfrak{C}'$, we have,

$$\begin{aligned}
\mathcal{T}_{X\times Y}(E) &= \sup_{\mathfrak{C}}\{\inf_{\lambda}\{\mathcal{T}_X(A_\lambda) \wedge \mathcal{T}_Y(B_\lambda)\}\} \\
&= \sup_{\mathfrak{C}'}\{\inf_{\lambda}\{\mathcal{T}_X(A_\lambda) \wedge \mathcal{T}_Y(B_\lambda)\}\} \\
&\leq \sup_{\mathfrak{C}'}\{\inf_{\lambda,x,y}\{\mathcal{B}_X(A_{\lambda,x}) \wedge \mathcal{B}_Y(B_{\lambda,y})\}\} + \epsilon \\
&= \sup_{\mathfrak{C}'}\{\inf_{\lambda,x,y}\{\mathcal{B}_{X\times Y}(A_{\lambda,x} \times B_{\lambda,y})\}\} + \epsilon \\
&\leq \mathcal{T}(E) + \epsilon.
\end{aligned}$$

Since this is true for every $\epsilon > 0$ we have, $\mathcal{T}_{X\times Y} \leq \mathcal{T}$ and hence both the co-smooth fuzzy topologies are the same. ■

Definition 4.4. Let X be any set and define a function $\mathcal{S} : \mathscr{P}(X) \to [0,1]$ satisfying the following condition: Given $x \in X$ and $\epsilon > 0$ there exists $A \subseteq X$ such that $x \in A$ and $\mathcal{S}(A) \geq 1 - \epsilon$. Then $\mathcal{S}$ is called a subbasis for a co-smooth fuzzy topology on X.

Theorem 4.5. *Let $\mathcal{S} : \mathscr{P}(X) \to [0,1]$ be a subbasis of a co-smooth fuzzy topology on a set X. Define a function $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ as follows:*

$$\mathcal{B}(A) = \sup_{D\in\mathfrak{D}} \left\{ \inf_{i\in I_D} \{\mathcal{S}(A_i)\} \right\}$$

where $\mathfrak{D}$ is the collection of all possible finite intersections $D = \{A_i\}_{i\in I_D}$ of subsets of X for some finite indexing set I_D such that $A = \bigcap_{i\in I_D} A_i$, then $\mathcal{B}$ is a basis for a co-smooth fuzzy topology on X.

Proof. Since every set A is the intersection of members of the collection consisting of A alone, and $0 \leq \mathcal{S}(E) \leq 1$, $\mathcal{B}$ is well defined. As $\mathcal{B}$ clearly satisfies (i.) of Definition 3.3, we prove (ii.) of Definition 3.3 only. Let $x \in A \cap B$ and $\epsilon > 0$. By definition of $\mathcal{B}$ there exists a collections $\{A_i\}_{i=1,2,\ldots,n}$ and $\{B_j\}_{j=1,2,\ldots,m}$ of subsets of X in $\mathfrak{D}$ such that

$$\inf_i\{\mathcal{B}(A_i)\} \geq \mathcal{B}(A) - \epsilon \text{ and } \inf_j\{\mathcal{B}(B_j)\} \geq \mathcal{B}(B) - \epsilon.$$

Then

$$A \cap B = (\bigcap_{i=1}^{n} A_i) \cap (\bigcap_{j=1}^{m} B_j).$$

Now define a collection of sets C_k, for $k = 1, 2, 3, \ldots, n+m$, as

$$C_k = \begin{cases} A_k & \text{if } k \leq n \\ B_{k-n} & \text{if } k > n \end{cases}$$

Let $C = \bigcap_{k=1}^{n+m} C_k$, then clearly $x \in C$. By definition of $\mathcal{B}$, we have

$$\mathcal{B}(C) = \sup_{D\in\mathfrak{D}} \left\{ \inf_{i\in I_D} \{\mathcal{S}(E_i)\} \right\}$$

where $\mathfrak{D}$ is the collection of all possible finite intersections $D = \{E_i\}_{i\in I_D}$ of subsets of X for some finite indexing set I_D such that $C = \bigcap_{i\in I_D} E_i$. Now,

$$\begin{aligned}
\mathcal{B}(C) &= \sup_{D\in\mathfrak{D}} \left\{ \inf_{i\in I_D} \{\mathcal{S}(E_i)\} \right\} \\
&\geq \inf\{\mathcal{S}(C_k)\} \\
&= \inf_i\{\mathcal{S}(A_i)\} \wedge \inf_j\{\mathcal{S}(B_j)\} \\
&\geq (\mathcal{B}(A) - \epsilon) \wedge (\mathcal{B}(B) - \epsilon) \\
&= (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon.
\end{aligned}$$

Thus $\mathcal{B}(C) \geq (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon$ and hence, $\mathcal{B}$ is a basis. ∎

Theorem 4.6. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be any two co-smooth fuzzy topological spaces. Define $\mathcal{S} : \mathscr{P}(X \times Y) \to [0, 1]$ as*

$$\mathcal{S}(E) = \begin{cases} \mathcal{T}_X(A) & \text{if } E = A \times Y \\ \mathcal{T}_Y(B) & \text{if } E = X \times B \\ 0 & \text{otherwise} \end{cases}.$$

Then $\mathcal{S}$ is a subbasis for the co-smooth fuzzy product topology on $X \times Y$.

Proof. As $\mathcal{S}(X \times Y) = 1$, taking $A = X \times Y$, it clearly follows that $\mathcal{S}$ is a subbasis for a co-smooth fuzzy topology on $X \times Y$. Let $\mathcal{B}'$ be the basis generated by $\mathcal{S}$. Then for any set $E \subseteq X \times Y$, we have

$$\mathcal{B}'(E) = \sup_{D\in\mathfrak{D}} \left\{ \inf_{i\in I_D} \{\mathcal{S}(E_i)\} \right\}$$

where $\mathfrak{D}$ is the collection of all possible finite intersections $D = \{E_i\}_{i\in I_D}$ of subsets of $X \times Y$ for some finite indexing set I_D such that $E = \bigcap_{i\in I_D} E_i$. Let $\mathcal{B}$ be the basis for the co-smooth fuzzy product topology on $X \times Y$ as defined in Definition 4.2. Now we prove that $\mathcal{B} = \mathcal{B}'$. Let $E \subseteq X \times Y$. Suppose E is not of form $A \times B$ for any $A \subseteq X$ and $B \subseteq Y$. Then by Definition 4.2 we have, $\mathcal{B}(E) = 0$. Now let us compute $\mathcal{B}'(E)$. Let $E = E_1 \cap E_2 \cap \cdots \cap E_n$ be a representation of E as a finite intersection of subsets of $X \times Y$. We claim that E_j is neither of the form $(A_j \times Y)$ nor of the form $(X \times B_j)$ for at least one j. Suppose $E_j = (A_j \times Y)$ or $E_j = (X \times B_j)$ for all j. Without loss of generality, let $E_j = (A_j \times Y)$ for $j = 1, 2, \ldots, m$ and $E_j = (X \times B_j)$ for $j = m + 1, m + 2, \ldots, n$, then

$$\begin{aligned}
E &= E_1 \cap E_2 \cap \cdots \cap E_n \\
&= \{(A_1 \times Y) \cap \cdots \cap (A_m \times Y)\} \cap \{(X \times B_{m+1}) \cap \cdots \cap (X \times B_n)\} \\
&= \{(A_1 \cap A_2 \cap \cdots \cap A_m) \times Y\} \cap \{X \times (B_{m+1} \cap B_{m+2} \cap \cdots \cap B_n)\}
\end{aligned}$$

Let $A = A_1 \cap A_2 \cap \cdots \cap A_m$ and $B = B_{m+1} \cap B_{m+2} \cap \cdots \cap B_n$, then $E = (A \times Y) \cap (X \times B) = A \times B$ which is a contradiction to our assumption that E is not of the form $A \times B$. This proves the claim and hence $\inf\{\mathcal{S}(E_i)\} = 0$. Since this is true for any representation of E as a finite intersection, by the definition of $\mathcal{B}'$ we have, $\mathcal{B}'(E) = 0$. Thus $\mathcal{B} = \mathcal{B}'$ in this case. Now let E be of the form $A \times B$ for some $A \subseteq X$, $B \subseteq Y$ and let $E = E_1 \cap E_2 \cap \cdots \cap E_n$ be a representation of E as a finite intersection of subsets of $X \times Y$. If, for at least one j, E_j is neither of the form $(A_j \times Y)$ nor of the form $(X \times B_j)$, then $\inf\{\mathcal{S}(E_i)\} = 0$. This implies $\mathcal{B}(E) \geq \inf\{\mathcal{S}(E_i)\}$. If $E_j = (A_j \times Y)$ for $j = 1, 2, \ldots, m$ and $E_j = (X \times B_j)$ for $j = m+1, m+2, \ldots, n$, then

$$\begin{aligned} E &= E_1 \cap E_2 \cap \cdots \cap E_n \\ &= \{(A_1 \cap A_2 \cap \cdots \cap A_m) \times Y\} \cap \{X \times (B_{m+1} \cap B_{m+2} \cap \cdots \cap B_n)\} \end{aligned}$$

Since $E = A \times B$, we have,

$$A = A_1 \cap A_2 \cap \cdots \cap A_m \text{ and } B = B_{m+1} \cap B_{m+2} \cap \cdots \cap B_n.$$

Now,

$$\begin{aligned} \mathcal{B}(E) &= \inf\{\mathcal{T}_X(A), \mathcal{T}_Y(B)\} \\ &\geq \inf\{\bigwedge_{j=1}^{m} \mathcal{T}_X(A_j), \bigwedge_{j=m+1}^{n} \mathcal{T}_Y(B_j)\} \\ &= \inf\{\mathcal{T}_X(A_1), \ldots, \mathcal{T}_X(A_m), \mathcal{T}_Y(B_{m+1}), \ldots, \mathcal{T}_Y(B_n)\} \\ &= \inf\{\mathcal{S}(E_1), \ldots, \mathcal{S}(E_m), \mathcal{S}(E_{m+1}), \ldots, \mathcal{S}(E_n)\} \\ &= \inf\{\mathcal{S}(E_i)\}. \end{aligned}$$

Since this is true for any representation of E as a finite intersection of subsets of $X \times Y$, we have,

$$\mathcal{B}(E) \geq \mathcal{B}'(E).$$

But, since $E = A \times B = (A \times Y) \cap (X \times B)$ we have,

$$\mathcal{B}'(E) \geq \inf\{\mathcal{S}(A \times Y), \mathcal{S}(X \times B)\} = \inf\{\mathcal{T}_X(A), \mathcal{T}_Y(B)\} = \mathcal{B}(E).$$

Thus $\mathcal{B}'(E) \geq \mathcal{B}(E)$ and hence $\mathcal{B} = \mathcal{B}'$ in this case also. Thus $\mathcal{S}$ is a subbasis for the product topology on $X \times Y$. ■

5. Hausdorffness on Co-Smooth Fuzzy Topological Spaces

As stated in Section 2, "Being Hausdorff" is a property hold by topological spaces. In this section we give a fuzzy version of Hausdorff spaces.

Definition 5.1. A subset A of a co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to be α-open if $\mathcal{T}(A) > \alpha$ and is said to be α-closed if the set $X - A$ is α-open.

If A is α-closed, then $\mathcal{T}(X - A) > \alpha$ and hence $\mathcal{C}(A) > \alpha$ where $\mathcal{C}(A)$ is the degree of closedness of A as defined in Definition 3.1, and conversely. So A is α-closed if and only if $\mathcal{C}(A) > \alpha$. If U is an α-open set containing x, then we say that U is an α-neighbourhood of x.

Definition 5.2. A co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to be α-Hausdorff if for each pair x, y of distinct points of X, there exist disjoint α-open sets U and V containing x and y respectively. Hausdorffness of a topological space is defined as the supremum of all α such that $(X, \mathcal{T})$ is α-Hausdorff.

If $(X, \mathcal{T})$ is a (crisp) topological space, then it can be viewed as a co-smooth fuzzy topological space as mentioned below Definition 3.1. If $(X, \mathcal{T})$ is Hausdorff, in the crisp sense, then it is α-Hausdorff for all α and hence its Hausdorffness is 1.

Theorem 5.3. *In a α-Hausdorff space $\mathcal{C}(A) \geq \alpha$ for any finite set A.*

Proof. Let A be a finite set in the α-Hausdorff space $(X, \mathcal{T})$. It suffices to show that $\mathcal{C}(\{x\}) \geq \alpha$ for all $x \in X$. Let $x \in X$ and y be any point of X different from x; then there exist disjoint α-neighbourhoods U_x and V_y of x and y respectively. Let $V = \bigcup_{y \neq x} V_y$. Then $V = X - \{x\}$ and $\mathcal{T}(V) \geq \alpha$. Hence $\mathcal{C}(\{x\}) \geq \alpha$. ∎

Definition 5.4. A co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to satisfy the α-T_1 axiom, if $\mathcal{C}(\{x\}) > \alpha$ for all $\{x\} \in \mathscr{P}(X)$.

Definition 5.5. A sequence (x_n) of points of a co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to converge to a point $x \in X$ at level α if for every α-neighbourhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

Theorem 5.6. *If $(X, \mathcal{T})$ is a α-Hausdorff space, then a sequence of points of X converges to at most one point of X at level α.*

Since the proof follows as in the classical theory we omit the proof.

Theorem 5.7. *Let $(X, \mathcal{T}_X)$ be a α-Hausdorff space and $(Y, \mathcal{T}_Y)$ be a β-Hausdorff space and let $\gamma = \min\{\alpha, \beta\}$. Then $X \times Y$ is a γ-Hausdorff space.*

Proof. Let $(a, b), (c, d)$ be two distinct points in $X \times Y$. If $a \neq c$, then there exist disjoint α-open sets U_a and U_c containing a and c. The sets $U_a \times Y$ and $U_c \times Y$ are disjoint γ-open sets containing (a, b) and (c, d). If $b \neq d$, then a similar argument holds. ∎

Corollary 5.8. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are two α-Hausdorff spaces, then the product space $X \times Y$ is also α-Hausdorff.*

Conclusion

One cannot talk about the beauty, which is fuzzy in nature, of an object without knowing the existence of the object. As any object has a sure base, the concept of fuzziness should be build on crisp objects. The theory developed here is a good model to study fuzzy properties of crisp objects. With this theory one may study many more concepts in topology.

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