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Spectral Conjugate Gradient Like Method for Signal Reconstruction

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Abstract This paper presents a derivative-free conjugate gradient algorithm for solving the l_1-regularization problem arising in compressive sensing. The search direction of the proposed method is bounded and satisfies the sufficient descent condition. Under some mild assumptions, the global convergence of the proposed algorithm is established. Numerical experiments in recovering sparse signal are performed to illustrate the efficiency of the algorithm compared with existing algorithms.

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1. Introduction

In this paper, we consider the following unconstrained minimization problem for sparse recovery

$$\min_{x} \left\{ \frac{1}{2}\|Ax-b\|_2^2 + \mu\|x\|_1 \right\} \tag{1.1}$$

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where $A \in \mathbb{R}^{m\times n}(m << n)$, $b \in \mathbb{R}^m$, $\|\cdot\|_1$ is the ℓ_1-norm of a vector $x \in \mathbb{R}^n$ usually called *regularizer* and $\mu \in \mathbb{R}^+$ is a regularization parameter that can be interpreted as a trade-off between sparsity and residual error.

There are a lot of solvers for solving the model (1.1). The iterative shrinkage thresholding (IST) and the fast iterative shrinkage thresholding algorithm (FISTA) are among the most common methods for solving (1.1) due to their simplicity and efficiency [1, 2]. Other methods for solving (1.1) includes the fixed-point continuous search method proposed in [3], the Barzilai-Borwein stepsize method [4]. The Gradient descent methods are another type of methods for solving problem (1.1). For instance, Figueiredo [5] developed a gradient method to solve (1.1). Motivated by Figueiredo's method, Xiao et al. [6, 7] then proposed alternative method for solving the model (1.1) using the spectral gradient and the conjugate gradient projection method respectively. Unlike IST and FISTA, the model (1.1) was first transformed into a monotone system of equations. This reformulation procedure can be found in several literature (Ref. [8, 9]). It is noteworthy to state that, with the reformulation of (1.1) into a monotone system of equations, (1.1) is now equivalent to the following convex constrained nonlinear equation

$$F(x) = 0, \; x \in \Omega, \tag{1.2}$$

where $F : \Omega \to \mathbb{R}^n$ is a continuous mapping and $\Omega \subseteq \mathbb{R}^n$ is a convex set. Thus, solving (1.2) is equivalent to solving (1.1). See (Refs. [10–21]) for various algorithms for solving (1.2). Other algorithms for solving the model (1.1) can be found in the following references [22–25].

This paper is structured as follows: In section 2, we introduce the proposed algorithm for solving the model (2). We also give some preliminary concepts. In section 3, we establish the global convergence of the proposed algorithm. In section 4, we illustrate the performance of the proposed algorithm in reconstructing sparse signal. Finally, in the last section, we give the conclusion.

2. Preliminaries and Algorithm

First, we present and describe the proposed method that will be utilized for solving (1.1). To describe the algorithm, we recall the projection map, which is defined as a mapping $\mathrm{P}_\Omega : \mathbb{R}^n \to \Omega$, where Ω is a nonempty closed convex set such that

$$P_\Omega(x) = \arg\min\{\|x-y\| \;|y \in \Omega\}. \tag{2.1}$$

Throughout this article, we denote $\|\cdot\|$, to be the Euclidean norm. A well known characterization of the projection map is its nonexpansive property. That is, for any $x, y \in \mathbb{R}^n$,

$$\|P_\Omega(x) - P_\Omega(y)\| \le \|x-y\|.$$

Consequently,

$$\|P_\Omega(x) - y\| \le \|x-y\|, \; \forall y \in \Omega. \tag{2.2}$$

In what follows, we give a step by step detail of the construction of our method, for convenience, we abbreviated $F(x_k)$ as F_k. Now, by considering the unconstrained problem defined by

$$\min f(x), \; x \in \mathbb{R}^n, \tag{2.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear function whose gradient at x_k is g_k. Amini in [26] proposed a conjugate gradient like algorithm that generates a sequence $\{x_k\}$ by

$$x_{k+1} = x_k + t_k d_k, \tag{2.4}$$

where t_k is the steplength, and d_k is a search direction defined by

$$d_k = \begin{cases} -g_k + \beta_k d_{k-1}, & \text{if } k > 0, \\ -g_k, & \text{if } k = 0, \end{cases} \tag{2.5}$$

and β_k is defined as

$$\beta_k = \tau \frac{\|g_k\|}{\|d_{k-1}\|}, \tag{2.6}$$

where $\tau \in (0,1)$. As shown in [26], the following inequality always holds

$$g_k^T d_k = -(1-\tau)\|g_k\|^2. \tag{2.7}$$

Motivated by the CG-method (2.4)-(2.6), we introduce our method for solving (1.1) by defining d_k as follows

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -N_k F_k + \beta_k d_{k-1}, & \text{if } k > 0, \end{cases} \tag{2.8}$$

where β_k is defined as

$$\beta_k = \tau \frac{\|F_k\|}{\|d_{k-1}\|}, \tag{2.9}$$

with $\tau \in (0,1)$. Note that, for the direction d_k defined by (2.8) with CG-parameter given in (2.9), it is clear that for $k = 0$, the inequality below holds,

$$F_k^T d_k = -(1-\tau)\|F_k\|^2. \tag{2.10}$$

Similarly, for $k \in \mathbb{N}$, we have,

$$F_k^T d_k = -\Big(N_k - \tau \frac{F_k^T d_{k-1}}{\|F_k\|\|d_{k-1}\|}\Big)\|F_k\|^2. \tag{2.11}$$

For (2.11) to satisfy (2.10), we only need that

$$N_k \geq c + \tau \frac{F_k^T d_{k-1}}{\|F_k\|\|d_{k-1}\|} \tag{2.12}$$

where c is a positive constant. Thus, without loss of generality, we choose N_k as

$$N_k = c + \tau \frac{F_k^T d_{k-1}}{\|F_k\|\|d_{k-1}\|}. \tag{2.13}$$

Next, we state our proposed algorithm for solving (1.1). For the direction determined by (2.8), (2.9) and (2.13) we refer to the corresponding algorithm as the SCPM.

Algorithm 2.1.
Input. Set an initial point $x_0 \in \Omega$, the positive constants: $Tol > 0$, $\tau \in (0,1)$, $\rho \in (0,1)$, $\kappa > 0, \sigma > 0$, $c > 0$. Set $k = 0$.

Step 0. If $\|F_k\| \leq Tol$, stop. Otherwise, generate the search direction d_k using (2.8), (2.9) and (2.13).

Step 1. Let $t_k = \max\{\kappa\rho^i | i = 0, 1, 2, \cdots\}$, we set $w_k = x_k + t_k d_k$, to satisfy

$$F(w_k)^T d_k \geq \sigma t_k \|d_k\|^2. \tag{2.14}$$

Step 2. If $w_k \in \Omega$ and $\|F(w_k)\| = 0$, stop. Otherwise, compute the next iterate by

$$x_{k+1} = P_\Omega[x_k - \xi_k F(x_k + t_k d_k)], \tag{2.15}$$

where

$$\xi_k = \frac{F(x_k + t_k d_k)^T (x_k - w_k)}{\|F(w_k)\|^2}.$$

Step 3. Finally we set $k = k + 1$ and return to step 1.

Lemma 2.2. *Let d_k be the search direction generated by SCPM, then d_k is a sufficient descent direction. That is,*

$$F_k^T d_k \leq -c\|F_k\|^2, \ c > 0 \tag{2.16}$$

for all $k \geq 0$.

Remark 2.3. Clearly,

$$\|F_k\|\|d_{k-1}\| > 0.$$

This indicates that the parameters (2.9) − (2.13) are well defined.

Lemma 2.4. *Let $\{d_k\}$ and $\{x_k\}$ be two sequences generated by SCPM method. Then, there exists a step size t_k satisfying the line search (2.14) for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. For any $i \geq 0$, suppose (2.14) does not hold for the iterate k_0-th, then we have

$$-\langle F(x_{k_0} + \kappa\rho^i d_{k_0}), d_{k_0}\rangle < \sigma\kappa\rho^i \|d_{k_0}\|^2.$$

Thus, by the continuity of F and with $0 < \rho < 1$, it follows that by letting $i \to \infty$, we have

$$-F(x_{k_0})^T d_{k_0} \leq 0,$$

which contradicts (2.16).

□

3. Global Convergence

The convergence analysis of our proposed method is presented in detail in this section. To achieve our goal, two crucial assumptions are made on the mapping F.

Assumption 3.1.

- The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is *Lipschiz* continuous, that is there exists a positive constant L such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{3.1}$$

- Xiao et al. [6] proved that for the problem (1.2), F is monotone. That is,

$$(F(x) - F(y))^T (x - y) \geq 0, \quad \forall x, y \in \mathbb{R}^n. \tag{3.2}$$

In the analysis that follows, $F_k \neq 0$ is assumed for all $k \geq 0$.

Lemma 3.2. *Let the sequences* $\{x_k\}$ *and* $\{w_k\}$ *be generated by the SCPM method under Assumption 3.1, then*

$$t_k \geq \max\left\{\kappa, \frac{\rho c\|F_k\|^2}{(L+\sigma)\|d_k\|^2}\right\}. \tag{3.3}$$

Proof. Let $\bar{\alpha}_k = t_k\rho^{-1}$. Assume $t_k \neq \kappa$, from (2.14), $\bar{\alpha}_k$ does not satisfy (2.14), that is,

$$-F(x_k+\bar{\alpha}_k d_k)^T d_k < \sigma\bar{\alpha}_k\|d_k\|^2.$$

From (2.16) and (3.1), it can be obviously seen that

$$\begin{aligned} c\|F_k\|^2 &\leq -F_k^T d_k \\ &= (F(x_k+\bar{\alpha}_k d_k) - F_k)^T d_k - F(x_k+\bar{\alpha}_k d_k)^T d_k \\ &\leq L\bar{\alpha}_k\|d_k\|^2 + \sigma\bar{\alpha}_k\|d_k\|^2 \\ &\leq \bar{\alpha}_k(L+\sigma)\|d_k\|^2. \end{aligned}$$

This gives the desired inequality (3.3). □

Lemma 3.3. *Suppose that Assumption 3.1 holds. Let* $\{x_k\}$ *and* $\{w_k\}$ *be sequences generated by the SCPM method, then for any solution* $\bar{x}^*$ *contained in the solution set* Ω^* *the inequality*

$$\|x_{k+1}-\bar{x}^*\|^2 \leq \|x_k-\bar{x}^*\|^2 - \sigma^2\|x_k-w_k\|^4. \tag{3.4}$$

holds. In addition, $\{x_k\}$ *is bounded and*

$$\sum_{k=0}^{\infty}\|x_k-w_k\|^4 < +\infty. \tag{3.5}$$

Proof. First, we begin by using the monotonicity of the mapping F. Thus, for any solution $\bar{x}^* \in \Omega^*$,

$$\langle F(w_k), x_k-\bar{x}^*\rangle \geq \langle F(w_k), x_k-w_k\rangle.$$

The above inequality together with (2.14) gives

$$\langle F(x_k+t_k d_k), x_k-w_k\rangle \geq \sigma\alpha_k^2\|d_k\|^2 \geq 0. \tag{3.6}$$

We have the following from (2.2) and (3.6),

$$\begin{aligned} \|x_{k+1}-\bar{x}^*\|^2 &= \|P_\Omega(x_k-\xi_k F(x_k+t_k d_k)) - \bar{x}^*\|^2 \leq \|x_k-\xi_k F(x_k+t_k d_k)-\bar{x}^*\|^2 \\ &= \|x_k-\bar{x}^*\|^2 - 2\xi_k\langle F(x_k+t_k d_k), x_k-\bar{x}^*\rangle + \|\xi_k F(x_k+t_k d_k)\|^2 \\ &\leq \|x_k-\bar{x}^*\|^2 - 2\xi_k\langle F(x_k+t_k d_k), x_k-w_k\rangle + \|\xi_k F(x_k+t_k d_k)\|^2 \\ &= \|x_k-\bar{x}^*\|^2 - \frac{\langle F(x_k+t_k d_k), x_k-w_k\rangle^2}{\|F(x_k+t_k d_k)\|^2} \\ &\leq \|x_k-\bar{x}^*\|^2 - \sigma^2\|x_k-w_k\|^4. \end{aligned}$$

Thus, the sequence $\{\|x_k-\bar{x}^*\|\}$ has a nonincreasing and convergent property. Therefore, this makes $\{x_k\}$ to be bounded and therefore the following holds.

$$\sigma^2\sum_{k=0}^{\infty}\|x_k-w_k\|^4 < \|x_0-\bar{x}^*\|^2 < +\infty.$$ □

Remark 3.4. Taking into account of the definition of w_k and also by (3.5), it can be deduced that

$$\lim_{k\to\infty} t_k\|d_k\| = 0. \tag{3.7}$$

Theorem 3.5. *Suppose Assumption 3.1 holds. Let $\{x_k\}$ and $\{w_k\}$ be sequences generated by the SCPM method, then*

$$\liminf_{k\to\infty} \|F_k\| = 0. \tag{3.8}$$

Proof. Suppose (3.8) is not valid, that is, there exist a constant say $r > 0$ such that $r \leq \|F_k\|$, $k \in \mathbb{N}\cup\{0\}$. Then this along with (2.16) implies that

$$\|d_k\| \geq cr, \quad \forall k \geq 0. \tag{3.9}$$

It can be obviously seen from Lemma (3.3) and Remark (3.4), that the sequences $\{x_k\}$ and $\{w_k\}$ are bounded. In addition with the continuity of F, it further implies that $\{\|F_k\|\}$ and $\{\|F(w_k)\|\}$ are bounded by a constant say b. From (2.8) – (2.13), it follows that for all $k \geq 1$,

$$\begin{aligned}\|d_k\| &\leq c\|F_k\| + \tau\frac{\|F_k\|^2}{\|F_k\|\|d_{k-1}\|}\|d_{k-1}\| + \tau\frac{\|F_k\|}{\|d_{k-1}\|}\|d_{k-1}\| \\ &= (c+2\tau)\|F_k\| \leq b(c+2\tau) \triangleq \Gamma.\end{aligned}$$

Note that, by using Remark (2.3) and Cauchy Schwarz inequality, the first inequality is easily obtained. Similarly, from (3.9), the second inequality follows. Now, from (3.3), we have

$$\begin{aligned}t_k\|d_k\| &\geq \max\left\{\kappa, \frac{\rho c\|F_k\|^2}{(L+\sigma)\|d_k\|^2}\right\}\|d_k\| \\ &\geq \max\left\{\kappa cr, \frac{\rho cr^2}{(L+\sigma)\Gamma}\right\} > 0,\end{aligned}$$

which contradicts (3.7). Hence (3.8) is valid. □

4. Numerical Experiment

In this section, we consider recovering a sparse signal of length n from k samples with Gaussian noise. The sample is normally smaller than the actual signal. The quality of recovered signal is assessed by the metric called mean of squared error (MSE) defined as

$$MSE := \frac{1}{n}\|\tilde{x} - x^*\|^2,$$

where $\tilde{x}$ is the actual signal and x^* is the recovered signal. We select the signal size to be $n = 2^{12}$, $k = 2^{10}$. The actual signal $\tilde{x}$ contains 2^6 randomly nonzero elements. A random matrix B is generated using the Matlab command rand(n,k) during the experiment. The observed measurement y is

$$y = B\tilde{x} + b,$$

where b is the Gaussian noise which is normally distributed with mean 0 and variance 10^{-4}.

To illustrate the performance of SCPM in signal recovery, we compare it with three existing algorithms, namely, SGCS, CGD and PCG proposed in [6], [7] and [27], respectively. The parameters chosen for SCPM are: $\kappa = 1, \rho = 0.7, c = 1, \tau = 0.1,\ \sigma = 10^{-4}$. The parameters chosen for SGCS, CGD and PCG are as in their respective papers. In order to be fair in comparing the methods, we initialize from $x_0 = B^T y$ and stop when $Tol := \frac{\|f_k - f_{k-1}\|}{\|f_{k-1}\|} < 10^{-5}$ where $f_k = \frac{1}{2}\|y - Bx\|_2^2 + \eta\|x\|_1$ is the objective function.

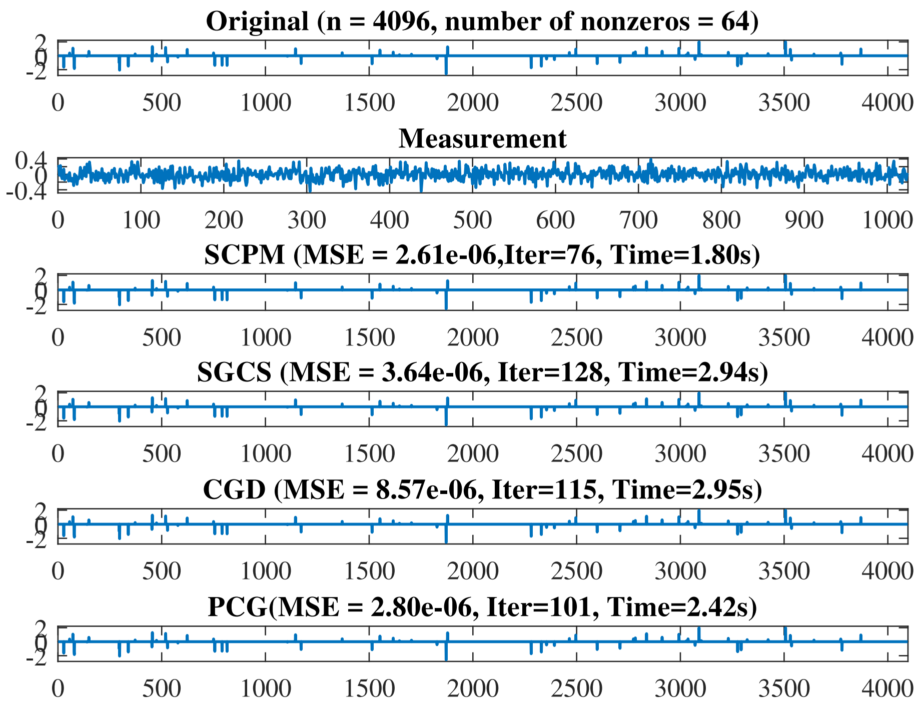

FIGURE 1. Illustration of the sparse signal recovery. From the top to the bottom is the original signal (First plot), the measurement (Second plot), and the reconstructed signals by SCPM (Third plot), SGCS (Fourth plot), CGD (fifth plot) and PCG (sixth plot).

Figure 1 reveal the original signal $\tilde{x}$, the observed measurement y and the recovered signal x^* by all algorithms. Figure 2 show the rate of decrease of MSE and objective function values with respect to number of iterations and CPU time. From the Figure 1 and 2, it can be observed that the proposed algorithm SCPM prove to be more efficient as it recover the measurement with less MSE, number of iterations and CPU time. We repeat the experiment ten times with the original signal $\tilde{x}$ randomly generated. From Table 1, it is not difficult to see that the proposed algorithm was consistently more efficient than SGCS, CGD and PCG.

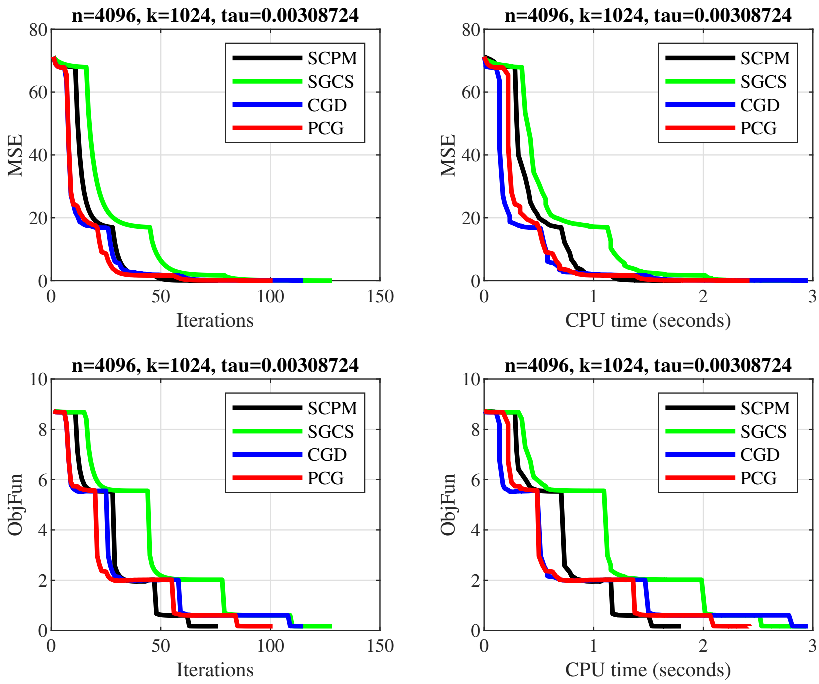

FIGURE 2. The x-axes represent the number of iterations (top left and bottom left) and the CPU time in seconds (top right and bottom right). The y-axes represent the MSE (top left and top right) and the function values (bottom left and right).

TABLE 1. Results for the signal reconstruction by the various algorithms

	SCG			SGCS			CGD			PCG		
	ITER	TIME	MSE	ITER	TIME	MSE	ITER	TIME	MSE	ITER	TIME	MSE
	79	2.09	2.93E-06	127	3.63	3.49E-06	129	3.48	2.92E-06	111	3.06	2.81E-06
	74	2.27	3.15E-06	131	4.09	3.63E-06	117	3.7	3.03E-06	103	4	1.08E-05
	72	1.98	2.98E-06	117	3.14	3.89E-06	101	2.72	3.27E-06	104	2.77	3.19E-06
	75	1.94	2.86E-06	132	3.23	3.33E-06	104	2.64	2.45E-05	114	2.92	2.64E-06
	81	2.45	3.01E-06	131	4.17	4.04E-06	151	4.69	3.45E-06	113	3.44	3.23E-06
	73	1.88	2.78E-06	127	3.27	3.20E-06	112	2.94	3.56E-06	116	3.05	2.60E-06
	72	1.98	2.07E-06	127	3.27	2.58E-06	125	3.52	2.15E-06	91	2.59	3.47E-06
	79	2.14	4.53E-06	133	3.73	5.52E-06	138	3.88	4.37E-06	119	3.11	4.19E-06
	74	1.94	1.87E-06	126	3.48	2.25E-06	123	3.67	3.47E-06	100	2.7	3.21E-06
	90	2.81	3.66E-06	136	4.38	4.91E-06	120	3.81	1.37E-05	108	3.28	8.17E-06
Average	76.9	2.148	2.98E-06	128.7	3.639	3.68E-06	122	3.505	6.44E-06	107.9	3.092	4.43E-06

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