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On the White Noise of the Option on Future

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Abstract In this paper, we studied the the white noise of the option onthe future for the stock price. We obtained the new results which is interesting and we hope that such new results may be useful in the research area of Financial Mathematics.

Keywords: option; stock; Black-Scholes equation

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1. Introduction

At the present the option price on future is the popular one for trading. The equation

$$\frac{\partial}{\partial t}C(F,t)+\frac{1}{2}\sigma^2F^2\frac{\partial^2}{\partial F^2}C(F,t)-rC(F,t)=0 \tag{1.1}$$

with the call payoff

$$C(F_T,T)=\max(F_T-p,0)\equiv(F_T-p)^+ \tag{1.2}$$

can obtain the solution $C(F,t)$ which is the option price on the future, see [[1],pp.118-119] where

$$F=se^{r(T-t)}$$

is the stock price on future at the expiration time T for $0\leq t\leq T$ and p is the strike price. Next consider ξ where $\xi=\frac{d}{dt}B(t)$ and $B(t)$ is the Brownian motion. Actually ξ is

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the parameter that causes the fluctuation of the stock price like the volatility σ. Now we have the formula

$$\xi = \frac{1}{t\sigma} \ln\left(\frac{S}{S_0}\right) - \frac{\mu}{\sigma} + \frac{\sigma}{2} \tag{1.3}$$

such formula can be derived from the stock model $ds = \mu s dt + \sigma s dB$, see [[2],pp.797-804] where s is the stock price, s_0 is the stock price at time $t = 0$, μ is the drift of stock, σ is the volatility of stock and B is Brownian motion.

In this work, we interested in finding the white noise of the option on future. Now form (1.3) with $s = Fe^{-r(T-t)}$, we have

$$\xi = \frac{1}{t\sigma} \ln\left(\frac{F}{s_0} e^{-r(T-t)}\right) - \frac{\mu}{\sigma} + \frac{\sigma}{2} \tag{1.4}$$

and put $C(F,t) = V(\xi,t)$ by using the chian rule then the equation (1.1) is transformed to the equation

$$\frac{\partial}{\partial t} V(\xi,t) - \frac{1}{2}\frac{\sigma}{t}\frac{\partial}{\partial t} V(\xi,t) + \frac{1}{2}\frac{1}{t^2}\frac{\partial^2}{\partial \xi^2} V(\xi,t) - rV(\xi,t) = 0 \tag{1.5}$$

with the call payoff in (1.2)

$$C(F_T,T) = (F_T - p)^+ = \left(s_o \exp(\xi t\sigma + \mu T - \sigma^2\frac{T}{2}) - p\right)^+ = V(\xi,t)$$

from (1.4) with $t = T$. Let $f(\xi) = s_o \exp(\xi t \sigma + \mu T - \sigma^2 \frac{T}{2}) - p$ where f is a continuous function. Then

$$V(\xi,T) = f(\xi) \tag{1.6}$$

is the call payoff of (1.5). Now take the Fourier transform with respect to ξ to the equation (1.5). We obtain the equation

$$\frac{\partial}{\partial t}\widehat{V}(\omega,t) - \frac{1}{2}\frac{\sigma}{t} i\omega \widehat{V}(\omega,t) - \frac{1}{2}\frac{1}{t^2}\omega^2 \widehat{V}(\omega,t) - r\widehat{V}(\omega,t) = 0 \tag{1.7}$$

with the call payoff

$$\widehat{V}(\omega,t) = \widehat{f}(\omega) \tag{1.8}$$

we can solve the solution of (1.7) with the call payoff (1.8). Such solution is i convolution form given by

$$V(\xi,t) = K(\xi,t) * f(\xi) \tag{1.9}$$

where

$$K(\xi,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi \frac{(T-t)}{tT}}} \exp\left(-\frac{\left(\xi - \frac{\sigma}{2}\ln\left(\frac{T}{t}\right)\right)^2}{2\frac{(T-t)}{tT}}\right) \tag{1.10}$$

is the kernel of (1.5). We see that (1.9) is the white noise of the option on future that we need. Now from (1.10) we can show that

$$K(\xi,t) \longrightarrow \delta(\xi) \text{ as } t \longrightarrow T$$

where $\delta(\xi)$ is the Dirac-delta distribution, see [[3], pp.36-37] Thus from (1.9)

$$V(\xi,T) = \delta * f(\xi) = f(\xi)$$

that (1.6) holds.

2. Preliminaries

The following definitions and some lemmas are needed.

Definition 2.1. Let f be the integrable function on the set of real number $\mathbb{R}$. Then the Fourier transform of f is defined by

$$\mathcal{F}f(x) = \widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{2.1}$$

and the $\mathcal{F}^{-1}$ inverse Fourier transorm of $\widehat{f}(\omega)$ is also defined by

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}f(x)) = \mathcal{F}^{-1}\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega. \tag{2.2}$$

Definition 2.2. The Dirac-delta or the impulse function is denoted by δ and also defined by

$$\langle \delta(x), \phi(x)\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)dx = \phi(0)$$

where $\phi(x)$ is the testing function of infinitely differentiable with compact support.

Lemma 2.3. *Recall the equation* (1.5) *and the call payoff* (1.6)

$$\frac{\partial}{\partial t}V(\xi,t) - \frac{1}{2}\frac{\sigma}{t}\frac{\partial}{\partial t}V(\xi,t) + \frac{1}{2}\frac{1}{t^2}\frac{\partial^2}{\partial \xi^2}V(\xi,t) - rV(\xi,t) = 0 \tag{2.3}$$

with the call payoff

$$V(\xi,T) = f(\xi) \tag{2.4}$$

then $V(\xi,t) = K(\xi,t) * f(\xi)$ *is the solution of* (2.3) *where*

$$K(\xi,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\dfrac{(T-t)}{tT}}}\exp\left(-\frac{\left(\xi - \dfrac{\sigma}{2}\ln\left(\dfrac{T}{t}\right)\right)^2}{2\dfrac{(T-t)}{tT}}\right)$$

is the kernel or Green function of (2.3).

Proof. Take the Fourier transform with respect to ξ given by (2.1) to the equation (2.3). Then we obtain

$$\frac{\partial}{\partial t}\widehat{V}(\omega,t) - \frac{1}{2}\frac{\sigma}{t}i\omega\widehat{V}(\omega,t) - \frac{1}{2}\frac{1}{t^2}\omega^2\widehat{V}(\omega,t) - r\widehat{V}(\omega,t) = 0. \tag{2.5}$$

With the call payoff $\widehat{V}(\omega,T) = \widehat{f}(\omega)$.

Then we obtain $\widehat{V}(\omega,t) = C(\omega)\exp\left(-\frac{1}{2t}\omega^2 + \frac{1}{2}i\omega\sigma\ln t + rt\right)$ as the solution of (2.4) where $C(\omega)$ is constant. Now we are finding $C(\omega)$. Since

$$\widehat{V}(\omega,T) = C(\omega)\exp\left(-\frac{1}{2T}\omega^2 + \frac{1}{2}i\omega\sigma\ln T + rT\right) = \widehat{f}(\omega).$$

hence

$$C(\omega,t)=\frac{\widehat{f}(\omega)}{\exp\left(-\frac{1}{2t}\omega^2+\frac{1}{2}i\omega\sigma\ln t+rt\right)}.$$

It follows that

$$\widehat{V}(\omega,t)=\exp\left(-\frac{1}{2}(\frac{1}{t}-\frac{1}{T})\omega^2+\frac{1}{2}i\omega\sigma\ln\left(\frac{t}{T}\right)+(t-T)r\right)\widehat{f}(\omega).$$

Since $V(\xi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega\xi}\widehat{V}(\omega,t)d\omega$ from (2.2). Hence

$$V(\xi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega\xi}\exp\left(-\frac{1}{2}(\frac{1}{t}-\frac{1}{T})\omega^2+\frac{1}{2}i\sigma\ln\left(\frac{t}{T}\right)\omega+(t-T)r\right)\widehat{f}(\omega)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\omega\xi}e^{-i\omega y}\exp\left(-\frac{1}{2}(\frac{1}{t}-\frac{1}{T})\omega^2+\frac{1}{2}i\sigma\ln\left(\frac{t}{T}\right)\omega+(t-T)r\right)f(y)dyd\omega$$

where $\widehat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega y}f(y)dy$.

$$V(\xi,t)=\frac{e^{-r(t-T)}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{(t-T)}{tT}\omega^2+i\left(\frac{\sigma}{2}+\left(\frac{t}{T}\right)+\xi-y\right)\omega\right)f(y)dyd\omega.$$

By using technic of completing the square and compute directly. We obtain

$$V(\xi,t)=\frac{e^{-r(t-T)}}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{(t-T)}{tT}\left(\omega-\frac{itT}{T-t}(\frac{\sigma}{2}\ln\frac{t}{T}+\xi-y)\right)^2\right)d\omega$$

$$\times\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{tT}{T-t}\left(\frac{\sigma}{2}\ln\frac{t}{T}+\xi-y\right)^2\right)f(y)dy.$$

Put $u=\sqrt{\frac{1}{2}\frac{T-t}{tT}}\left(\omega-\frac{itT}{T-t}(\frac{\sigma}{2}\ln\frac{t}{T}+\xi-y)\right)$ then $d\omega=\sqrt{\frac{2tT}{T-t}}du$. Thus

$$V(\xi,t)=\frac{e^{-r(t-T)}}{2\pi}\sqrt{\frac{2tT}{T-t}}\int_{-\infty}^{\infty}e^{-u^2}du$$

$$\times\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{tT}{T-t}(\frac{\sigma}{2}\ln\frac{t}{T}+\xi-y)^2\right)f(y)dy$$

$$=\frac{e^{-r(t-T)}}{2\pi}\sqrt{\frac{2tT}{T-t}}\sqrt{\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{tT}{T-t}(\frac{\sigma}{2}\ln\frac{t}{T}+\xi-y)^2\right)f(y)dy$$

$$=\frac{e^{-r(T-t)}}{\sqrt{2\pi\frac{(T-t)}{tT}}}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(\frac{\sigma^2}{2}\ln\frac{t}{T}+\xi-y\right)^2}{2\frac{T-t}{tT}}\right)f(y)dy$$

(note that $\int_{-\infty}^{\infty}e^{-u^2}du=\sqrt{\pi}$). Thus we have $V(\xi,t)=K(\xi,t)*f(\xi)$ where

$$K(\xi,t)=\frac{e^{-r(T-t)}}{\sqrt{2\pi\frac{(T-t)}{tT}}}\exp\left(-\frac{(\xi-\frac{\sigma}{2}\ln\frac{T}{t})^2}{2\frac{T-t}{tT}}\right)$$

is the kernel. Moreover $K(\xi,t)$ is Normal distribution with mean $= e^{-r(T-t)}\frac{\sigma}{2}\ln\frac{T}{t}$ and variance $= e^{-2(T-t)}\frac{(T-t)}{tT}$ and as $t\to T, K*\xi,t)\to\delta(\xi)$.
Thus $V(\xi,T)=\delta(\xi)*f(\xi)=f(\xi)$ that (2.4) holds. ∎

3. Main Results

The results of Lemma 2.1 leads to the following theorem.

Theorem 3.1. *Given the equation*

$$\frac{\partial}{\partial t}V(\xi,t)-\frac{1}{2}\frac{\sigma}{t}\frac{\partial}{\partial t}V(\xi,t)+\frac{1}{2}\frac{1}{t^2}\frac{\partial^2}{\partial \xi^2}V(\xi,t)-rV(\xi,t)=0 \tag{3.1}$$

which the call payoff

$$V(\xi,T)=f(\xi) \tag{3.2}$$

then $v(\xi,t)=K(\xi,t)*f(\xi)$, *where*

$$K(\xi,t)=\frac{e^{-r(T-t)}}{\sqrt{2\pi\frac{(T-t)}{tT}}}\exp\left(-\frac{(\xi-\frac{\sigma}{2}\ln\frac{T}{t})^2}{2\frac{T-t}{tT}}\right)$$

is the kernel.

Proof. By Lemma 2.3. ∎

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