Thai Journal of Mathematics
Volume 18 Number 4 (2020)
Pages 1983–1994
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On the Existence and Uniqueness of the Solution of a Probabilistic Functional Equation Approached by the Banach Fixed Point Theorem

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Abstract This paper deals with a particular class of probabilistic functional equations used to observe animals' psychological learning process. Our aims are to find the existence and uniqueness of such functional equations' solution using the Banach fixed point theorem and discuss the Hyers-Ulam and Hyers-Ulam-Rassias type stability of the proposed functional equation.

MSC: 47H09; 47H10
Keywords: probabilistic functional equation; learning process; Banach fixed point theorem; Hyers-Ulam stability; Hyers-Ulam-Rassias stability

Submission date: 20.05.2020 / Acceptance date: 13.12.2020

1. Introduction

A large chunk of psychology is divided into many topics such as excitement, stimulus, learning of the single selective time, time of reaction, etc. Unlike the other extensive group of psychological problems of memory, thinking, and perception, all of the areas mentioned above are having a common theme, which is choice. While studying, sensation choice is lying between stimuli; it lies between responses in the process of learning. In contrast, motivation can be seen in the substitutes of changeable preference evaluations. According to some psychologists' beliefs, these differences are fundamental for understanding behavior, significantly the difference that lies in stimulus and response.

The two-reaction, two-event, path-independent, contingent form of various stochastic models for the learning process is given by the following equation

$$x_{n+1} = \begin{cases} Q_1 x_n & \text{with probability } x_n, \\ Q_2 x_n & \text{with probability } (1 - x_n), \end{cases} \tag{1.1}$$

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Published by The Mathematical Association of Thailand.

where Q_1 and Q_2 act as transition operators, and on trial n, the probabilities of the responses A_1 and A_2 are x_n and $(1 - x_n)$, respectively. Bush and Mosteller [1] defined these transition operators Q_1 and Q_2 as

$$\begin{aligned} Q_1 x_n &= \alpha_1 x_n + (1 - \alpha_1), \\ Q_2 x_n &= \alpha_2 x_n, \end{aligned} \tag{1.2}$$

where $\alpha_1, \alpha_2 \in [0, 1]$. Such type of linear model is known as 'alpha model.' If $g : [0, 1] \to [0, 1]$ is the outcome of the starting probability with boundary conditions $g(0) = 0$ and $g(1) = 1$, then we have the following functional equation

$$g(x) = xg(Q_1(x)) + (1 - x)g(Q_2(x)) \tag{1.3}$$

for all $x \in [0, 1]$.

In 2015, Istrăţescu's result [2] was expanded by Berinde and Khan [3] to prove the existence of a solution of the functional equation (1.3). They defined the transition operators $Q_1, Q_2 : [0, 1] \to [0, 1]$ (satisfying the boundary conditions $Q_1(1) = 1$ and $Q_2(0) = 0$) as Banach contraction mappings with contractive coefficients α and β, respectively, where $0 < \alpha \leq \beta < 1$. Recently, Turab and Sintunavarat [4] used such type of functional equation to observe the behavior of the paradise fish in a two-choice situation.

On the other hand, Epstein [5] proposed the following functional equation, which is very similar to the functional equation (1.3) defined above

$$g(x) = \left(\frac{e^x}{1 + e^x}\right) g(Q_1(x)) + \left(1 - \frac{e^x}{1 + e^x}\right) g(Q_2(x)) \tag{1.4}$$

for all $x \in [0, 1]$, where $Q_1, Q_2 : [0, 1] \to [0, 1]$ are the given transition operators. Such type of this model has many applications in learning theory and mathematical psychology, and named as 'beta model.' With these observations in mind, we first re-write (1.3) and (1.4) into the following more general form

$$g(x) = \phi(x)g(h(x)) + (1 - \phi(x))\, g(k(x)) \tag{1.5}$$

for all $x \in [0, 1]$, where $\phi, h, k : [0, 1] \to [0, 1]$ are the given mappings and $g : [0, 1] \to \mathbb{R}$ is an unknown function.

The objective of this paper is to consider the functional equation (1.5) and investigate the existence and uniqueness of a solution of the proposed functional equation by using the Banach contraction principle or Banach fixed point theorem. After that, we will discuss the Hyers-Ulam and Hyers-Ulam-Rassias stability of the functional equation (1.5). At the end, we will present some examples which show the significance of our results.

2. Preliminaries

Following definitions and known results will be needed in the sequel.

Definition 2.1. Let A and B are two nonempty sets and $Z : A \to B$ be a mapping. A point $a \in A$ is called a *fixed point* of Z if and only if $a = Za$.

Definition 2.2. Let (X, d) be a metric space and $Z : X \to X$ be a mapping.

(1) Z is called a *Banach contraction mapping* if there is a nonnegative real number $k < 1$ such that

$$d(Zx, Zy) \leq kd(x, y) \tag{2.1}$$

for all $x, y \in X$.

(2) Z is called a *contractive mapping* if

$$d(Zx, Zy) < d(x, y) \tag{2.2}$$

for all $x, y \in X$ with $x \neq y$.

(3) Z is called a *non-expansive mapping* if

$$d(Zx, Zy) \leq d(x, y) \tag{2.3}$$

for all $x, y \in X$.

Remark 2.3. For a metric space (X, d) and a mapping $Z : X \to X$, the following obserbvations hold:

(1) if Z is a Banach contraction mapping, then Z is a contractive mapping;
(2) if Z is a contractive mapping, then Z is a nonexpansive mapping.

Theorem 2.4 (Banach contraction principle or Banach fixed point theorem, [6]). *Let (X, d) be a complete metric space. A Banach contraction mapping $Z : X \to X$ has precisely one fixed point. Moreover, the Picard iteration $\{x_n\}$ in X which is defined by $x_n = Zx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$, converges to the unique fixed point of Z.*

3. Main Results

Let X be the collection of all continuous real-valued functions $g : [0, 1] \to \mathbb{R}$ such that $g(0) = 0$ and

$$\sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \quad < \quad \infty. \tag{3.1}$$

If $\|\cdot\| : X \to \mathbb{R}$ is defined by

$$\|g\| \quad = \quad \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \tag{3.2}$$

for all $g \in X$, then $(X, \|\cdot\|)$ is a Banach space. Throughout this paper, unless otherwise specified, $\|\cdot\|$ is a norm on X defined by (3.2). Furthermore, we shall be interested with the existence of a solution of the following functional equation

$$g(x) \quad = \quad \phi(x)g(h(x)) + (1 - \phi(x))\, g(k(x)) \tag{3.3}$$

for all $x \in [0, 1]$, where $g : [0, 1] \to \mathbb{R}$ is an unknown function, $h, k : [0, 1] \to [0, 1]$ are given contraction mappings with contractive coefficients α_1 and α_2, respectively. Also, $\phi : [0, 1] \to [0, 1]$ is a given non-expansive mapping. We now turn to our main result in this paper.

Theorem 3.1. *Consider the functional equation (3.3). If $h(0) = 0 = k(0)$ and $\alpha_1 + \alpha_2 < \frac{1}{2}$, then the mapping $Z : X \to X$ defined for each $g \in X$ by*

$$(Zg)(x) \quad = \quad \phi(x)g(h(x)) + (1 - \phi(x))g(k(x)) \tag{3.4}$$

for all $x \in [0, 1]$ is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.

Proof. Let $d : X \times X \to \mathbb{R}$ be a metric induced by $\|\cdot\|$ on X. Then (X, d) is a complete metric space. First, we want to claim that Z is well-defined. For each $g \in X$, we obtain

$$(Zg)(0) = \phi(0)g(h(0)) + (1 - \phi(0))g(k(0)) = 0.$$

Also, Zg is continuous and $\|Zg\| < \infty$ for all $g \in X$. Therefore, Z is a self operator on X and so it is well-defined. Furthermore, it is clear that the solution of the functional equation (3.4) is equivalent to the fixed point of an operator Z. As Z is a linear mapping, so that for $g_1, g_2 \in X$, we have

$$\|Zg_1 - Zg_2\| \quad = \quad \|Z(g_1 - g_2)\|.$$

Thus, to estimate $\|Zg_1 - Zg_2\|$, we let $g_1, g_2 \in X$ and for each distinct points $x, y \in [0, 1]$, we obtain

$$\begin{aligned}
&\frac{|(Zg_1 - Zg_2)(x) - (Zg_1 - Zg_2)(y)|}{|x-y|} \\
&= \bigg| \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(x)) + (1 - \phi(x))(g_1 - g_2)(k(x)) \\
&\quad -\phi(y)(g_1 - g_2)(h(y)) - (1 - \phi(y))(g_1 - g_2)(k(y))] \bigg| \\
&= \bigg| \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(x)) + (1 - \phi(x))(g_1 - g_2)(k(x)) \\
&\quad -\phi(y)(g_1 - g_2)(h(y)) - (1 - \phi(y))(g_1 - g_2)(k(y)) \\
&\quad +\phi(x)(g_1 - g_2)(h(y)) + (1 - \phi(x))(g_1 - g_2)(k(y)) \\
&\quad -\phi(x)(g_1 - g_2)(h(y)) - (1 - \phi(x))(g_1 - g_2)(k(y))] \bigg| \\
&= \bigg| \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(x)) - \phi(x)(g_1 - g_2)(h(y))] \\
&\quad + \frac{1}{x-y}[(1 - \phi(x))(g_1 - g_2)(k(x)) - (1 - \phi(x))(g_1 - g_2)(k(y))] \\
&\quad + \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(y)) - \phi(y)(g_1 - g_2)(h(y))] \\
&\quad + \frac{1}{x-y}[(1 - \phi(x))(g_1 - g_2)(k(y)) - (1 - \phi(y))(g_1 - g_2)(k(y))] \bigg| \\
&\leq \phi(x)\|g_1 - g_2\| \frac{|h(x) - h(y)|}{|x-y|} + (1 - \phi(x))\|g_1 - g_2\| \frac{|k(x) - k(y)|}{|x-y|} \\
&\quad + \left| \frac{\phi(x) - \phi(y)}{x-y} \right| \|g_1 - g_2\| \frac{|h(y) - h(0)|}{|y-0|} |y - 0| \\
&\quad + \left| \frac{\phi(x) - \phi(y)}{x-y} \right| \|g_1 - g_2\| \frac{|k(y) - k(0)|}{|y-0|} |y - 0| \\
&\leq \alpha_1 \phi(x)\|g_1 - g_2\| + \alpha_2(1 - \phi(x))\|g_1 - g_2\| \\
&\quad + \alpha_1 |y - 0| \|g_1 - g_2\| + \alpha_2 |y - 0| \|g_1 - g_2\| \\
&\leq 2(\alpha_1 + \alpha_2)\|g_1 - g_2\|.
\end{aligned}$$

This gives that

$$d(Zg_1, Zg_2) = \|Zg_1 - Zg_2\| \leq 2(\alpha_1 + \alpha_2)\|g_1 - g_2\| = 2(\alpha_1 + \alpha_2)d(g_1, g_2).$$

It follows from $0 \leq 2(\alpha_1 + \alpha_2) < 1$ that Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$. ∎

From Theorem 3.1, we get the following result related to the existence and uniqueness of a solution of the functional equation (3.3).

Theorem 3.2. *The functional equation (3.3) has a unique solution in X provided that $h(0) = 0 = k(0)$ and $\alpha_1 + \alpha_2 < \frac{1}{2}$. Moreover, the iteration $\{g_n\}$ in X which is defined by*

$$(g_n)(x) = \phi(x)g_{n-1}(h(x)) + (1 - \phi(x))g_{n-1}(k(x))$$

for all $n \in \mathbb{N}$, where $g_0 \in X$, converges to the unique solution of the functional equation (3.3) in the sense of the metric d induced by $\|\cdot\|$.

Proof. By using the Banach contraction principle with Theorem 3.1, we get the conclusion in this theorem. ∎

Remark 3.3. The condition $h(0) = 0 = k(0)$ is sufficient to prove the existence and uniqueness of a solution of the proposed functional equation (3.3) but not necessary. Our next theorems are independent of this condition.

Theorem 3.4. *Consider the functional equation (3.3). Suppose that $\phi(0) = 0 = k(0)$ and there exists an $\alpha_3 \geq 0$ such that*

$$|h(x) - k(x)| \leq \alpha_3 \tag{3.5}$$

for all $x \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Then the mapping $Z : X \to X$ defined for each $g \in X$ by

$$(Zg)(x) = \phi(x)g(h(x)) + (1 - \phi(x))g(k(x)) \tag{3.6}$$

for all $x \in [0, 1]$ is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.

Proof. Let $d : X \times X \to \mathbb{R}$ be a metric induced by $\|\cdot\|$ on X. Then (X, d) is a complete metric space. First, we want to claim that Z is well-defined. For each $g \in X$, we obtain

$$(Zg)(0) = \phi(0)g(h(0)) + (1 - \phi(0))g(k(0)) = 0.$$

Also, Zg is continuous and $\|Zg\| < \infty$ for all $g \in X$. Therefore, Z is a self operator on X and so it is well-defined. Furthermore, it is clear that the solution of the functional equation (3.6) is equivalent to the fixed point of an operator Z. As Z is a linear mapping, so that for $g_1, g_2 \in X$, we have

$$\|Zg_1 - Zg_2\| \quad = \quad \|Z(g_1 - g_2)\|.$$

Thus, to estimate $\|Zg_1 - Zg_2\|$, let $g_1, g_2 \in X$ and for each distinct points $x, y \in [0,1]$, we obtain

$$\begin{aligned}
&\frac{|(Zg_1 - Zg_2)(x) - (Zg_1 - Zg_2)(y)|}{|x-y|} \\
&= \Bigg| \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(x)) + (1-\phi(x))(g_1 - g_2)(k(x)) \\
&\quad -\phi(y)(g_1 - g_2)(h(y)) - (1-\phi(y))(g_1 - g_2)(k(y))] \Bigg| \\
&= \Bigg| \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(x)) + (1-\phi(x))(g_1 - g_2)(k(x)) \\
&\quad -\phi(y)(g_1 - g_2)(h(y)) - (1-\phi(y))(g_1 - g_2)(k(y)) \\
&\quad +\phi(x)(g_1 - g_2)(h(y)) + (1-\phi(x))(g_1 - g_2)(k(y)) \\
&\quad -\phi(x)(g_1 - g_2)(h(y)) - (1-\phi(x))(g_1 - g_2)(k(y))] \Bigg| \\
&= \Bigg| \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(x)) - \phi(x)(g_1 - g_2)(h(y))] \\
&\quad + \frac{1}{x-y}[(1-\phi(x))(g_1 - g_2)(k(x)) - (1-\phi(x))(g_1 - g_2)(k(y))] \\
&\quad + \frac{1}{x-y}[\phi(x)(g_1 - g_2)(h(y)) - \phi(y)(g_1 - g_2)(h(y))] \\
&\quad + \frac{1}{x-y}[(1-\phi(x))(g_1 - g_2)(k(y)) - (1-\phi(y))(g_1 - g_2)(k(y))] \Bigg| \\
&\leq \phi(x)\|g_1 - g_2\| \frac{|h(x)-h(y)|}{|x-y|} + (1-\phi(x))\|g_1 - g_2\| \frac{|k(x)-k(y)|}{|x-y|} \\
&\quad + \left| \frac{\phi(x)-\phi(y)}{x-y} \right| \|g_1 - g_2\| \, |h(y) - k(y)| \\
&\leq \alpha_1 \phi(x)\|g_1 - g_2\| + \alpha_2(1-\phi(x))\|g_1 - g_2\| + \alpha_3\|g_1 - g_2\| \\
&\leq (\alpha_1 + \alpha_2 + \alpha_3)\|g_1 - g_2\|.
\end{aligned}$$

This gives that

$$d(Zg_1, Zg_2) = \|Zg_1 - Zg_2\| \leq (\alpha_1 + \alpha_2 + \alpha_3)\|g_1 - g_2\| = (\alpha_1 + \alpha_2 + \alpha_3)d(g_1, g_2).$$

It follows from $0 \leq \alpha_1 + \alpha_2 + \alpha_3 < 1$ that Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$. ■

From Theorem 3.4, we get the following result related to the existence and uniqueness of a solution of the functional equation (3.3).

Theorem 3.5. *The functional equation (3.3) has a unique solution in X provided that $\phi(0) = 0 = k(0)$ and there exists an $\alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that (3.5) holds. Moreover, the iteration $\{g_n\}$ in X which is defined by*

$$(g_n)(x) = \phi(x)g_{n-1}(h(x)) + (1-\phi(x))g_{n-1}(k(x))$$

for all $n \in \mathbb{N}$, where $g_0 \in X$, converges to the unique solution of the functional equation (3.3) in the sense of the metric d induced by $\|\cdot\|$.

Proof. By using the Banach contraction principle with Theorem 3.4, we get the conclusion in this theorem. ∎

From the above results, we get the following corollaries.

Corollary 3.6. *Consider the functional equation (3.3). Suppose that* $\phi(0) = 0 = k(0)$ *and* $h, k : [0, 1] \to [0, 1]$ *are bounded by constants* $m_1, m_2 \geq 0$*, respectively. For* $0 < \alpha_1 + \alpha_2 + m_1 + m_2 < 1$*, the mapping* $Z : X \to X$ *defined for each* $g \in X$ *by*

$$(Zg)(x) = \phi(x)g(h(x)) + (1 - \phi(x))g(k(x)) \tag{3.7}$$

for all$x \in [0, 1]$ *is a Banach contraction mapping with the metric* d *induced by* $\|\cdot\|$.

Corollary 3.7. *The functional equation (3.3) has a unique solution provided that* $\phi(0) = 0 = k(0)$ *and* $h, k : [0, 1] \to [0, 1]$ *are bounded by constants* $m_1, m_2 \geq 0$*, respectively, such that* $0 < \alpha_1 + \alpha_2 + m_1 + m_2 < 1$*. Moreover, the iteration* $\{g_n\}$ *in* X *which is defined by*

$$(g_n)(x) = \phi(x)g_{n-1}(h(x)) + (1 - \phi(x))g_{n-1}(k(x))$$

for all $n \in \mathbb{N}$ *, where* $g_0 \in X$*, converges to the unique solution of the functional equation (3.3) in the sense of the metric* d *induced by* $\|\cdot\|$.

Remark 3.8. Our proposed functional equation (3.3) generalizes many functional equations in the existing literature.

4. Stability Analysis of the Proposed Functional Equation

We start this section from the following question of Ulam [7] regarding the stability of group homomorphisms:

Let $d(\cdot, \cdot)$ be a metric and we denote $(Z_1, \diamond), (Z_2, \cdot, d)$ be a group and a metric group, respectively. For given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $g_1 : Z_1 \to Z_2$ satisfies $d(g_1(x \diamond y), g_1(x).g_1(y)) < \delta$ for all $x, y \in Z_1$, then there exists a homomorphism $g_2 : Z_1 \to Z_2$ with $d(g_1(x), g_2(x)) < \epsilon$ for all $x \in Z_1$?

Hyers [8] was the first who partially answered the question of Ulam for Banach spaces. Certainly, he also demonstrated that for Banach spaces X and Y, every solution of the inequality

$$\|g(x + y) - g(x) - g(y)\| \quad \leq \quad \epsilon \tag{4.1}$$

for all $x, y \in X$, where $g : X \to Y$ is an unknown function and $\epsilon > 0$, can be approximated by an additive function. After that, Rassias [9] tried to weak the condition for the bound of the norm of the Cauchy difference in (4.1) as follows

$$\|g(x + y) - g(x) - g(y)\| \quad \leq \quad \epsilon(\|x\|^p + \|y\|^p) \tag{4.2}$$

for all $x, y \in X$, where $g : X \to Y$ is an unknown function, $\epsilon > 0$ and $0 \leq p < 1$. The work of Rassias in [9] (known as, Hyers-Ulam-Rassias stability theorem) has the great impact on the theory of stability analysis and has many applications (see [10, 11]).

On the other hand, the stability of solutions has the great importance in the theory of mathematical modeling. For instance, in physical problems, slight deviations from the mathematical model caused by unavoidable errors in measurement do not have a correspondingly slight effect on the solution, the mathematical equations describing the problem will not accurately predict the future outcome. Therefore, it is very important to discuss the stability of a solution of the proposed functional equation (3.3) here.

Now, we prove the following result related to the Hyers-Ulam-Rassias stability of a solution of the functional equation (3.3).

Theorem 4.1. *Under the assumption of Theorem 3.1, the equation $Zg = g$, where $Z : X \to X$ is defined for each $g \in X$ by*

$$(Zg)(x) = \phi(x)g(h(x)) + (1 - \phi(x))g(k(x)) \tag{4.3}$$

for all $x \in [0, 1]$, has Hyers-Ulam-Rassias stability; that is, there is a function $\varphi : X \to [0, \infty)$ such that for each $g \in X$ with

$$d(Zg, g) \leq \varphi(g),$$

there exists a unique $\bar{g} \in X$ such that $Z\bar{g} = \bar{g}$ and

$$d(g, \bar{g}) \leq \mu\varphi(g)$$

for some $\mu > 0$.

Proof. Let $g \in X$ such that $d(Zg, g) \leq \varphi(g)$. By Theorem 3.1, there is a unique solution $\bar{g} \in X$ of the functional equation (3.3) on X, that is $Z\bar{g} = \bar{g}$. Thus, we have

$$\begin{aligned} d(g, \bar{g}) &\leq d(g, Zg) + d(Zg, \bar{g}) \\ &\leq \varphi(g) + d(Zg, Z\bar{g}) \\ &\leq \varphi(g) + 2(\alpha_1 + \alpha_2)d(g, \bar{g}) \end{aligned}$$

and so

$$d(g, \bar{g}) \leq \mu\varphi(g),$$

where $\mu := \dfrac{1}{1 - 2(\alpha_1 + \alpha_2)} > 0.$ ∎

From the above theorem, we can propose the following result related to the Hyers-Ulam stability.

Corollary 4.2. *Under the assumption of Theorem 3.1, the equation $Zg = g$, where $Z : X \to X$ is defined for each $g \in X$ by*

$$(Zg)(x) = \phi(x)g(h(x)) + (1 - \phi(x))g(k(x)) \tag{4.4}$$

for all $x \in [0, 1]$ has Hyers-Ulam stability, that is, there is an $\epsilon > 0$ such that for each $g \in X$ with

$$d(Zg, g) \leq \epsilon,$$

there exists a unique $\bar{g} \in X$ such that $Z\bar{g} = \bar{g}$ and

$$d(g, \bar{g}) \leq \mu\epsilon$$

for some $\mu > 0$.

5. Some Illustrative Examples

To support our results, we present the following examples.

Example 5.1. Consider the following functional equation

$$g(x) = xg\left(\frac{x}{12}\right) + (1-x)\,g\left(\frac{x}{11}\right) \tag{5.1}$$

for all $x \in [0,1]$, where $g : [0,1] \to \mathbb{R}$ is an unknown function. We set the mappings $\phi, h, k : [0,1] \to [0,1]$ by

$$\phi(x) = x,\ \ h(x) = \frac{x}{12} \text{ and } k(x) = \frac{x}{11}$$

for all $x \in [0,1]$. It can be seen that ϕ is a non-expansive mapping and h, k are contraction mappings with contractive coefficients $\alpha_1 := \frac{1}{12}$ and $\alpha_2 := \frac{1}{11}$, respectively. Clearly, $\alpha_1 + \alpha_2 < \frac{1}{2}$. Also, we have

$$h(0) = 0 = k(0).$$

Therefore, we can apply Theorem 3.2 for claiming the solution of the functional equation (5.1).

If we take an initial approximation $g_0(x) = x$, then the following iterates converge to the unique solution of the functional equation (5.1):

$$\begin{aligned}
g_1(x) &= \frac{-x^2 + 12x}{132}, \\
g_2(x) &= \frac{23x^3 - 1728x^2 + 19008x}{2299968}, \\
&\vdots \\
g_n(x) &= xg_{n-1}\left(\frac{x}{12}\right) + (1-x)g_{n-1}\left(\frac{x}{11}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$. Now, let

$$\mu := \frac{1}{1 - 2(\alpha_1 + \alpha_2)} = \frac{66}{43} > 0.$$

If a function $g \in X$ satisfies the inequality

$$d(Zg, g) \le \varphi(g), \text{ for all } g \in X,$$

then Theorem 4.1 implies that there exists a unique solution $\bar{g} \in X$ of the functional equation (5.1) such that

$$Z\bar{g} = \bar{g} \ \text{ and } \ d(g, \bar{g}) \le \mu\varphi(g).$$

Example 5.2. Consider the following functional equation

$$g(x) = xg\left(\frac{x}{7}\right) + (1-x)g\left(\frac{x}{8}\right) \tag{5.2}$$

for all $x \in [0,1]$, where $g : [0,1] \to \mathbb{R}$ is an unknown function. We set the mappings $\phi, h, k : [0,1] \to [0,1]$ by

$$\phi(x) = x,\ \ h(x) = \frac{x}{7} \text{ and } k(x) = \frac{x}{8}$$

for all $x \in [0,1]$. It can be seen that, ϕ is a non-expansive mapping and h, k are contraction mappings with contractive coefficients $\alpha_1 := \frac{1}{7}$ and $\alpha_2 := \frac{1}{8}$, respectively, and $\phi(0) = 0 = k(0)$. Also, we have

$$|h(x) - k(x)| = \left|\frac{x}{7} - \frac{x}{8}\right| \leq \frac{1}{56} := \alpha_3$$

for all $x \in [0,1]$ and

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{2}{7} < 1.$$

Therefore, we can apply Theorem 3.5 for claiming the solution of the functional equation (5.2).

If we take an initial approximation $g_0(x) = x$, then the following iterates converge to the unique solution of (5.2):

$$\begin{aligned} g_1(x) &= \frac{x^2 + 7x}{56}, \\ g_2(x) &= \frac{15x^3 + 441x^2 + 2744x}{175616}, \\ &\vdots \\ g_n(x) &= x g_{n-1}\left(\frac{x}{7}\right) + (1-x) g_{n-1}\left(\frac{x}{8}\right) \end{aligned}$$

for all $n \in \mathbb{N}$.

Example 5.3. Consider the following functional equation

$$g(x) = x g\left(\frac{(1-a)x}{4}\right) + (1-x) g\left(\frac{bx}{2}\right) \tag{5.3}$$

for all $x \in [0,1]$, where $0 < a \leq b < \frac{1}{2}$. We set the mappings $\phi, h, k : [0,1] \to [0,1]$ by

$$\phi(x) = x, \;\; h(x) = \frac{(1-a)x}{4} \text{ and } k(x) = \frac{bx}{2}.$$

It can be seen that, ϕ is a non-expansive mapping and h, k are contraction mappings with contractive coefficients $\alpha_1 := \frac{1-a}{4}$ and $\alpha_2 := \frac{b}{2}$, respectively. Also, we have

$$\phi(0) = 0 = k(0)$$

and

$$|h(x)| \leq \frac{1-a}{4} =: m_1 \quad \text{and} \quad |k(x)| \leq \frac{b}{2} =: m_2$$

for all $x \in [0,1]$. It is easy to see that

$$\alpha_1 + \alpha_2 + m_1 + m_2 = \frac{1-a+2b}{2}.$$

If $0 < \dfrac{1-a+2b}{2} < 1$, then we can apply Corollary 3.7 for claiming the solution of the functional equation (5.3).

If we take an initial approximation $g_0(x) = x$, then the following iterates converge to the unique solution of (5.3):

$$\begin{aligned} g_1(x) &= \frac{(1-a)x^2 + 2bx(1-x)}{4}, \\ g_2(x) &= \frac{x(x^2(1-a)^3+2bx(1-a)(4-x(1-a)))}{64} + \frac{\left(b^2x^2(1-a)+4b^2x\left(1-\frac{4bx}{2}\right)\right)(1-x)}{16}, \\ &\vdots \\ g_n(x) &= xg_{n-1}\left(\frac{(1-a)x}{4}\right) + (1-x)g_{n-1}\left(\frac{bx}{2}\right) \end{aligned}$$

for all $n \in \mathbb{N}$.

6. Conclusion

In this paper, we proposed a new class of functional equation (1.5) which connects two different categories of functional equations arising in the psychological learning theory. The proposed functional equation (1.5) has the great importance, especially in a two-choice situation, i.e., such type of functional equation describes the relationship between the predator animals and their two choices of prey with their corresponding probabilities. In [2–4], the authors used the boundary conditions in the proof of their main results, but in their comparison, in Theorem 3.4, we do not use such types of conditions to find the existence and uniqueness results of the functional equation (1.5), which shows that our result covers more problems than the previous ones existing in the particular literature.

Acknowledgement

This work was supported by Thammasat University Research Unit in Fixed Points and Optimization.

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