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A Set-Valued Fixed Point Theorem for Nonexpansive Mappings in Partially Ordered Ultrametric and Non-Archimedean Normed Spaces

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Abstract In this paper, a set-valued fixed point theorem for a class of generalized nonexpansive mappings on partially ordered ultrametric spaces and partially ordered non-Archimedean normed spaces is proved.

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1. Introduction and Preliminaries

We begin by recalling definition of an ultrametric space. The classical definition goes back over fifty years [1]. A metric space (X, d) is called an ultrametric space if the metric d satisfies the strong triangle inequality; namely for all $x, y, z \in X$: $d(x, y) \leq max\{d(x, z), d(y, z)\}$. A non-Archimedean normed space [1] $(X, \|.\|)$ is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection [1]. In 1993, Petals proved a fixed point theorem on non-Archimedean normed space using a contractive condition [2]. This result is extended by Kubiaczyk (1996) from single valued to set-valued contractive mapping [3]. Also for nonexpansive set-valued mappings, some fixed point theorems are proved.

In this paper, we investigate the existence of a fixed point for set-valued nonexpansive mappings in partially ordered ultrametric spaces and non-Archimedean normed spaces and we also give more constructive proof for our theorem and obtain a useful conclusion. It would be interesting to study the conclusions that obtained by Xu et al. [4] and Mursaleen et al. [5] in 2016 and compare with our results. Therefore, we can find out the importance of our results and get such results for p-adic fuzzy non-Archimedean numbers and the applications of Schwarz lemma involving the boundary fixed point in non-Archimedean complex analysis.

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2. Main Results

In the following proposition, we prove that if (X, d) is an ultrametric space, then the Hausdorff metric H [6] on $CB(X)$ is also an ultrametric. Where $H(A, B) = \max\Big\{\delta(A, B), \delta(B, A)\Big\}$, $\delta(A, B) = \sup_{a\in A} d(a, B)$ and $d(a, B) = \inf_{b\in B} d(a, b)$.

Proposition 2.1. *Let (X, d) be an ultrametric space. Then the metric H is an ultrametric on $CB(X)$.*

Proof. Let $a \in A$. Then for each $b \in B$, $d(a, B) \leq d(a, b)$, which implies that $d(a, B) \leq \max\{d(a, c), d(c, b)\}$ for all $b \in B, c \in C$. Because $b \in B$ was arbitrary, for each $\varepsilon > 0$ we can choose $b \in B$ such that $d(c, b) \leq d(c, B) + \varepsilon$ and hence

$$\begin{aligned} &d(a, B) \leq \max\{d(a, c), d(c, B) + \varepsilon\} \quad (c \in C),\\ &d(a, B) \leq \max\{d(a, c), \delta(C, B) + \varepsilon\} \qquad (c \in C). \end{aligned}$$

Similarly, because $c \in C$ was arbitrary, we can choose $c \in C$ such that $d(a, c) \leq d(a, C)+\varepsilon$ and hence

$$\begin{aligned} d(a, B) &\leq \max\{d(a, C) + \varepsilon, \delta(C, B) + \varepsilon\},\\ d(a, B) &\leq \max\{\delta(A, C) + \varepsilon, \delta(C, B) + \varepsilon\},\\ d(a, B) &\leq \max\{\delta(A, C), \delta(C, B)\} + \varepsilon. \end{aligned}$$

Because $a \in A$ and $\varepsilon > 0$ were arbitrary, we conclude that $\delta(A, B) \leq \max\{\delta(A, C), \delta(C, B)\}$. Therefore, $H(A, B) \leq \max\{H(A, C), H(C, B)\}$. ∎

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists an ultrametric d in X such that (X, d) is an ultrametric space, and $T : X \to CB(X)$ is a mapping. A closed ball $B(x, r)$ is said to be a partially T-invariant ball if for any $u \in B(x, r)$ comparable to x, there exists $v \in Tu$ such that $d(x, v) \leq r$. Also, the closed ball $B(x, r)$ is called minimal partially T-invariant ball if $B(x, r)$ is partially T-invariant and $d(u, Tu) = r$ for any $u \in B(x, r)$ comparable to x.

Theorem 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is an ultrametric space and a mapping $T : X \to CB(X)$. Suppose also that the following properties are satisfied:*

- **(C1)** *$H(Tx, Ty) \leq d(x, y)$ for every comparable $x, y \in X$.*
- **(C2)** *If $d(x, y) < 1$ for some $x \in X$ and some $y \in Tx$, then $x \preceq y$;*
- **(C3)** *There exist an $x_0 \in X$ and an $x_1 \in Tx_0$ such that $d(x_0, x_1) < 1$;*
- **(C4)** *If $\{x_n\}$ is a non-decreasing sequence in X and $\{B(x_n, r_n)\}$ is a descending sequence of closed balls in X, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_n\}$ has an upper bound $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$,*

and set $X_T = \{x \in X \ : \ \text{there exists } x' \in Tx \text{ such that } d(x, x') < 1\}$. Then for any $x \in X_T$, the ball $B(x, d(x, Tx))$ contains either a fixed point of T or a minimal partially T-invariant closed ball.

Proof. Let $z \in X$, put $r = d(z, Tz)$ and pick $u \in B(z, r)$ comparable to z. Then

$$d(z, Tu) \leq \{d(z, Tz), H(Tu, Tz)\} \leq \max\{d(z, Tz), d(u, z)\} = d(z, Tz).$$

Thus, there exists $v \in Tu$ such that $d(z,v) \leq d(z,Tz)$. Therefore, every ball in X of the form $B\big(z,d(z,dz)\big)$ is partially T-invariant. Now, let $x_0 \in X_T$ and put $x_1 = x_0$, $r_1 = d(x_1,Tx_1)$ and $\lambda_1 = \inf\{d(x,Tx) : x \in B(x_1,r_1) \cap Tx_1,\ x_1 \preceq x\}$. If $x \in B(x_1,r_1)$ and $x_1 \preceq x$, then

$$\begin{aligned} d(x,Tx) &\leq \max\{d(x,Tx_1),H(Tx_1,Tx)\} \\ &\leq \max\{d(x,Tx_1),d(x_1,x)\} \\ &\leq \max\{d(x,x_1),d(x_1,Tx_1),d(x_1,x)\} \leq d(x_1,Tx_1). \end{aligned}$$

Hence $\lambda_1 \leq r_1$. Suppose $\{\epsilon_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. If $r_1 = \lambda_1$, then the proof is completed because in this case either $r_1 = \lambda_1 = 0$ therefore x_1 is a fixed point of T in $B(x_1,r_1)$ or $B(x_1,r_1)$ is minimal partially T-invariant. Otherwise, if $x \in B(x_1,r_1) \cap Tx_1$ and $x_1 \preceq x$, then

$$\begin{aligned} d(x,Tx) &\leq \max\{d(x,Tx_1),H(Tx,Tx_1)\} \\ &\leq H(Tx,Tx_1) < d(x,x_1) \leq d(x_1,Tx_1). \end{aligned}$$

Hence $\lambda_1 < r_1 < 1$. Choose an $x_2 \in B(x_1,r_1)$ such that $x_1 \preceq x_2$, $x_2 \in Tx_1$ and $r_2 = d(x_2,Tx_2) < \min\{\lambda_1 + \varepsilon_1, r_1\}$. Let $\lambda_2 = \inf\big\{d(x,Tx) \mid x \in B(x_2,r_2) : x_2 \preceq x, x \in Tx_2\big\}$. Choose $x_2 \in B(x_1,r_1)$ such that there exists a path in $\tilde{G}$ between x_1 and x_2 and $r_2 = d(x_2,Tx_2) < \min\{r_1,\lambda_1+\epsilon_1\}$. With the same argemen, if $r_2 = \lambda_2$, then $B(x_2,r_2)$ is minimal partially T-invariant. Otherwise, we have $\lambda_2 < r_2$, and choose an $x_3 \in B(x_2,r_2)$ such that $x_2 \preceq x_3$, $x_3 \in Tx_3$ and $r_3 = d(x_3,Tx_3) < \min\{r_2,\lambda_2+\varepsilon_2\}$. Having defined $x_n \in X$, let

$$\lambda_n = \inf\big\{d(x,Tx) : x \in B(x_n,r_n) \cap Tx_n, x_n \preceq x\big\}.$$

Then we have $\lambda_n \leq r_n$, and choose an $x_{n+1} \in B(x_n,r_n)$ such that $x_n \preceq x_{n+1}$, $x_{n+1} \in Tx_{n+1}$ and $r_{n+1} = d(x_{n+1},Tx_{n+1}) < \min\{r_n,\lambda_n+\varepsilon_n\}$. The sequence $\{x_n\}$ is non-decreasing and $\{B(x_n,r_n)\}$ is a descending sequence of non-trivial closed balls. Thus by assumption, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in \bigcap_{k=1}^{\infty} B(x_{n_k},r_{n_k})$ such that $x_{n_k} \preceq z$ for all $k \geq 1$. Since $\{r_n\}$ is non-increasing, it follows that $r = \lim_{n\to\infty} r_n$ exists. Also, $\{\lambda_n\}$ is non-decreasing and bounded above and so $\lambda := \lim_{n\to\infty} \lambda_n$ also exists, too. Hence $d(z,Tz) \leq \max\big\{d(z,x_{n_k}),d(x_{n_k},Tz)\big\} \leq r_{n_k}$, for all $n \geq 1$. Moreover, $\lambda_{n_k} \leq d(z,Tz) \leq r \leq r_{n_{k+1}} \leq \lambda_{n_k} + \varepsilon_{n_k}$ for all $k \geq 1$. Letting $k \to \infty$, we see that $d(z,Tz) = \lambda = r$. Set $a = \inf\big\{d(x,Tx) : x \in B\big(z,d(z,Tz)\big) \cap Tz, z \preceq x\big\}$. Since $z \in B(x_n,r_n)$ and $x_n \preceq z$ for all $n \geq 1$, it follows that that $d(x,Tx) \leq d(z,Tz) \leq r_n$ for all $x \in B(z,d(z,Tz))$. Hence $a \leq r_n$ for all $n \geq 1$. Moreover, since every closed ball in X is partially T-invariant, we have $\lambda_n \leq a$ for all $n \geq 1$. Thus,

$$a = \inf\big\{d(x,Tx) : x \in B\big(z,d(z,Tz)\big) \cap Tz, z \preceq x\big\} = r = d(z,Tz).$$

If $r = 0$, then z is a fixed point of T in $B(x,d(x,Tx))$, if not, then the closed ball $B(z,d(z,Tz))$ is minimal partially T-invariant. Therefore the proof is completed. ∎

Corollary 2.4. *Theorem 2.3 remains valid if the partially ordered ultrametric space (X,d) is replaced by a partially ordered non-Archimedean normed space over a non-Archimedean valued field $\mathbb{K}$.*

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