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Fixed Point Theorems Concerning Rational Geraghty Contraction in a b-Metric Space

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Abstract In this paper, we introduce some new type of rational Geraghty contractive mappings in the setup of complete b-metric spaces and investigate the existence of fixed points for such mappings. We also provide an example to illustrate the presented results.

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1. Introduction

Banach contraction principle has been extended and generalized by many researchers by using different forms of contractive conditions in various spaces (see [1–6] and the references therein) . Some of such generalizations are obtained by contraction conditions containing rational expressions. In this direction, in 1973, Geraghty [7] introduced a contraction in which the contraction constant was replaced by a function having some specific properties. Since then, several papers dealt with fixed point theory for rational Geraghty contractive mappings (see, e.g., [8–13]).

Czerwik in [14] introduced the concept of a b-metric space. Since then many mathematicians done several work on involving fixed point for single-valued and multivalued operators in b-metric spaces (see [14–21] and the references therein).

Definition 1.1 ([14])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to \mathbb{R}^+$ is said to be a *b-metric* if for all $x, y, z \in X$ the following conditions are satisfied:

(bM_1) $d(x,y) = 0$ if and only if $x = y$;
(bM_2) $d(x,y) = d(y,x)$;
(bM_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X, d) is called a b-metric space (with constant s).

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Definition 1.2 ([22])**.** Let (X, d) be a b-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is called:

(A) *convergent* if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $lim_{n\to\infty}x_n = x$.
(B) *Cauchy* if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Remark 1.3 ([22])**.** Notice that in a b-metric space (X, d) the following assertions hold:
(A) a convergent sequence has a unique limit;
(B) each convergent sequence is Cauchy;
(C) in general, a b-metric is not continuous;
(D) in general, a b-metric does not induce a topology on X.

Definition 1.4 ([22])**.** The b-metric space (X, d) is *complete* if every Cauchy sequence in X converges in X.

The aim of this paper is to present some fixed point theorems for mappings in complete b-metric spaces satisfying several versions of rational Geraghty-type contractive conditions. Our results extend some existing results in the literature. An example is presented to showing the usefulness of these results and they are indeed more general than some known ones.

2. Main Results

Definition 2.1. Let (X, d) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be *Suzuki-rational Geraghty contraction* if there exist $\delta \in (s, \infty)$, $k \in (0, \infty)$ and the function $\varphi : [0, \infty) \to [0, \frac{1}{\delta})$ such that for all $x, y \in X$ with $x \neq y$

$$\frac{1}{2s}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) \leq \varphi(M_T(x, y))M_T(x, y),$$

where, $M_T(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)d(y,Ty)}{\max\{k,d(x,y)\}}, \frac{d(x,Tx)d(y,Ty)}{\max\{k,d(Tx,Ty)\}}\right\}$.

Theorem 2.2. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ be a Suzuki-rational Geraghty contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof. Chose $x \in X$. Set

$$x_1 = Tx,\ x_2 = Tx_1 = T^2x,\ \cdots, x_n = Tx_{n-1} = T^n x, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, the proof is complete. So we assume that for every $n \in \mathbb{N}$, $0 < d(x_n, Tx_n)$. Therefore

$$\frac{1}{2s}d(x_n, Tx_n) < d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \tag{2.2}$$

So from assumption of theorem, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq \varphi(M_T(x_{n-1}, x_n))M_T(x_{n-1}, x_n) \\ &\leq \frac{1}{\delta}M_T(x_{n-1}, x_n), \end{aligned} \tag{2.3}$$

where,

$$M_T(x_{n-1},x_n) = \max\left\{ \begin{array}{c} d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{\max\{k,d(x_{n-1},x_n)\}}, \\ \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{\max\{k,d(Tx_{n-1},Tx_n)\}} \end{array} \right\}$$
$$\leq \max\left\{ \begin{array}{c} d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)}, \\ \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_n,x_{n+1})} \end{array} \right\}$$
$$= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} \tag{2.4}$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ then from (2.3) and (2.4), we get

$$d(x_n,x_{n+1}) \leq \frac{1}{\delta}d(x_n,x_{n+1}).$$

Since $\frac{1}{\delta} < 1$, we get a contradiction. Therefore

$$\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}.$$

Thus, from (2.3) and (2.4), we have

$$d(x_n,x_{n+1}) \leq \frac{1}{\delta}d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}. \tag{2.5}$$

Therefore $\{d(x_n,x_{n+1})\}_{n=1}^{\infty}$ is a nonnegative decreasing sequence of real numbers, so

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \gamma \geq 0.$$

We prove $\gamma = 0$. Suppose on contrary that $\gamma > 0$. Then, letting $n \to \infty$, from (2.5) we have $\gamma \leq \frac{1}{\delta}\gamma < \gamma$. It is a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(x_n,Tx_n) = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.6}$$

Now, we claim that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \quad d(x_{p(n)},x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1},x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

From the triangle inequality, for all $n \in \mathbb{N}$, we have the following two inequalities:

$$\epsilon \leq d(x_{p(n)},x_{q(n)}) \leq s[d(x_{p(n)},x_{q(n)+1}) + d(x_{q(n)+1},x_{q(n)})].$$

and

$$\epsilon \leq d(x_{p(n)},x_{q(n)}) \leq s[d(x_{p(n)},x_{p(n)-1}) + d(x_{p(n)-1},x_{q(n)})].$$

It follows from (2.6) that

$$\frac{\epsilon}{s} \leq \liminf_{n\to\infty} d(x_{p(n)},x_{q(n)+1}) \;\; and \;\; \frac{\epsilon}{s} \leq \liminf_{n\to\infty} d(x_{p(n)-1},x_{q(n)}). \tag{2.8}$$

From (2.6) and (2.8), there exists $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2s}d(x_{p(n)},Tx_{p(n)}) < \frac{\epsilon}{s} \leq d(x_{p(n)},x_{q(n)+1}), \quad \forall n \geq n_1.$$

Again, From (2.8), there exists $n_2 \in \mathbb{N}$ such that

$$\frac{\epsilon}{s} \leq d(x_{p(n)-1},x_{q(n)}), \quad \forall n \geq n_2.$$

So from assumption of theorem, for all $n \geq n_3 = \max\{n_1, n_2\}$ we have

$$\begin{aligned} d(x_{p(n)}, x_{q(n)+1}) = d(Tx_{p(n)-1}, Tx_{q(n)}) &\leq \varphi(M_T(x_{p(n)-1}, x_{q(n)}))M_T(x_{p(n)-1}, x_{q(n)}) \\ &\leq \frac{1}{\delta} M_T(x_{p(n)-1}, x_{q(n)}). \end{aligned} \tag{2.9}$$

Observe that

$$\begin{aligned} M_T(x_{p(n)-1}, x_{q(n)}) &= \max \left\{ \begin{array}{c} d(x_{p(n)-1}, x_{q(n)}), \\ \frac{d(x_{p(n)-1}, Tx_{p(n)-1}) d(x_{q(n)}, Tx_{q(n)})}{\max\{k, d(x_{p(n)-1}, x_{q(n)})\}}, \\ \frac{d(x_{p(n)-1}, Tx_{p(n)-1}) d(x_{q(n)}, Tx_{q(n)})}{\max\{k, d(Tx_{p(n)-1}, Tx_{q(n)})\}} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} d(x_{p(n)-1}, x_{q(n)}), \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{d(x_{p(n)-1}, x_{q(n)})}, \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{d(x_{p(n)}, x_{q(n)+1})} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} \epsilon, \\ \frac{s}{\epsilon} d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1}), \\ \frac{s}{\epsilon} d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1}) \end{array} \right\}. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10) for all $n \geq n_3$, we have

$$d(x_{p(n)}, x_{q(n)+1}) \leq \frac{1}{\delta} \max \left\{ \begin{array}{c} \epsilon, \frac{s}{\epsilon} d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1}), \\ \frac{s}{\epsilon} d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1}) \end{array} \right\}.$$

Letting $n \to \infty$ in the inequality above and using (2.6) we obtain

$$\limsup_{n \to \infty} d(x_{p(n)}, x_{q(n)+1}) \leq \frac{\epsilon}{\delta}. \tag{2.11}$$

Thus from (2.8) and (2.11), we get $\frac{\epsilon}{s} \leq \frac{\epsilon}{\delta}$, a contradiction since $\delta > s$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. So by completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore,

$$\lim_{n \to \infty} d(x_n, x^*) = 0. \tag{2.12}$$

Now, we will show that for every $n \in \mathbb{N}$,

$$\frac{1}{2s} d(x_n, Tx_n) < d(x_n, x^*) \text{ or } \frac{1}{2s} d(Tx_n, T^2x_n) < d(Tx_n, x^*), \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Arguing by contradiction, we assume that there exist $m \in \mathbb{N}$ such that

$$\frac{1}{2s} d(x_m, Tx_m) \geq d(x_m, x^*) \text{ and } \frac{1}{2s} d(Tx_m, T^2x_m) \geq d(Tx_m, x^*). \tag{2.14}$$

From (2.5), we have

$$d(Tx_m, T^2x_m) < d(x_m, Tx_m) \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\begin{aligned} d(x_m, Tx_m) &\le sd(x_m, x^*) + sd(x^*, Tx_m) \\ &\le \frac{1}{2}d(x_m, Tx_m) + \frac{1}{2}d(Tx_m, T^2x_m) \\ &< \frac{1}{2}d(x_m, Tx_m) + \frac{1}{2}d(x_m, Tx_m) \\ &= d(x_m, Tx_m) \end{aligned}$$

This is a contradiction. Hence (2.13) holds. So from (2.13) and assumption of theorem, for every $n \in \mathbb{N}$, either

$$\begin{aligned} d(Tx_n, Tx^*) &\le \varphi(M_T(x_n, x^*)M_T(x_n, x^*) \\ &\le \frac{1}{\delta}\max\left\{d(x_n, x^*), \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{\max\{k, d(x_n, x^*)\}}, \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{\max\{k, d(Tx_n, Tx^*)\}}\right\} \\ &\le \frac{1}{\delta}\max\left\{d(x_n, x^*), \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{k}, \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{k}\right\} \end{aligned}$$

or

$$\begin{aligned} d(T^2x_n, Tx^*) &\le \varphi(M_T(Tx_n, x^*))M_T(Tx_n, x^*) \\ &\le \frac{1}{\delta}\max\left\{\begin{array}{c} d(Tx_n, x^*), \frac{d(Tx_n, T^2x_n)d(x^*, Tx^*)}{\max\{k, d(Tx_n, x^*)\}}, \\ \frac{d(Tx_n, T^2x_n)d(x^*, Tx^*)}{\max\{k, d(T^2x_n, Tx^*)\}} \end{array}\right\} \\ &\le \frac{1}{\delta}\max\left\{\begin{array}{c} d(x_{n+1}, x^*), \frac{d(x_{n+1}, Tx_{n+1})d(x^*, Tx^*)}{k}, \\ \frac{d(x_{n+1}, Tx_{n+1})d(x^*, Tx^*)}{k} \end{array}\right\} \end{aligned}$$

holds. In the first case from (2.6) and (2.12), we conclude that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = 0. \tag{2.16}$$

Since

$$\begin{aligned} d(x^*, Tx^*) &\le s[d(x^*, Tx_n) + d(Tx_n, Tx^*)] \\ &= sd(x^*, x_{n+1}) + sd(Tx_n, Tx^*). \end{aligned}$$

It follows from (2.6) and (2.16) that $d(x^*, Tx^*) = 0$, therefore $x^* = Tx^*$. In the second case from (2.6) and (2.12), we conclude that

$$\lim_{n\to\infty} d(T^2x_n, Tx^*) = 0. \tag{2.17}$$

Since

$$\begin{aligned} d(x^*, Tx^*) &\le s[d(x^*, T^2x_n) + d(T^2x_n, Tx^*)] \\ &= sd(x^*, x_{n+2}) + sd(T^2x_n, Tx^*) \\ &\le s^2d(x^*, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + sd(T^2x_n, Tx^*) \end{aligned}$$

It follows from (2.6), (2.12) and (2.17) that $d(x^*, Tx^*) = 0$, therefore $x^* = Tx^*$. Now, we show that T has a unique fixed point. For this, we assume that y^* is another fixed point of T in X such that $d(x^*, y^*) > 0$. Therefore $\frac{1}{2}d(x^*, Tx^*) < d(x^*, y^*)$, so from assumption of theorem

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\le \varphi(M_T(x^*, y^*))M_T(x^*, y^*) \\ &\le \frac{1}{\delta} \max\left\{d(x^*, y^*), \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{\max\{k, d(x^*, y^*)\}}, \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{\max\{k, d(Tx^*, Ty^*)\}}\right\} \\ &= \frac{1}{\delta}d(x^*, y^*) \\ &< d(x^*, y^*), \end{aligned}$$

which is a contradiction. Hence, we conclude that x^* is a unique fixed point of T. ■

Corollary 2.3. *Let (X, d) be a complete b-metric space with constant $s \ge 1$ and T be a self mapping on X. Suppose that there exist $\delta \in (s, \infty)$, $k \in (0, \infty)$ and the function $\varphi : [0, \infty) \to [0, \frac{1}{\delta})$ such that for all $x, y \in X$ with $x \ne y$*

$$\frac{1}{2s}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty)) \le \varphi(M(x, y))M(x, y),$$

where, $M(x, y) = \max\left\{d(x, y), \frac{d(x,Tx)d(y,Ty)}{k+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{k+d(Tx,Ty)}\right\}$. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof. Since $M(x, y) \le M_T(x, y)$, so from Theorem 2.2 the proof is complete. ■

Definition 2.4. Let (X, d) be a b-metric space with constant $s \ge 1$. A mapping $T : X \to X$ is said to be a *generalized rational Geraghty contraction of type A* if there exist $\delta \in (s, \infty)$, $k \in (0, \infty)$ and the function $\varphi : [0, \infty) \to [0, \frac{1}{\delta})$ such that

$$d(Tx, Ty) \le \begin{cases} \varphi(M_{A,T}(x, y))M_{A,T}(x, y), & if \quad \max\{d(x, Ty), d(Tx, y)\} \ne 0, \\ 0, & if \quad \max\{d(x, Ty), d(Tx, y)\} = 0, \end{cases} \tag{2.18}$$

where,

$$M_{A,T}(x, y) = \max\left\{\begin{array}{l} d(x, y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{k, s[d(x,Tx)+d(y,Ty)]\}}, \\ \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(Tx,y)\}} \end{array}\right\}.$$

Theorem 2.5. *Let (X, d) be a complete b-metric space with constant $s \ge 1$ and $T : X \to X$ be a generalized rational Geraghty contraction of type A. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof. Let $x_0 = x \in X$. Put $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If, there exists $n \in \mathbb{N}$ such that $x_n = x_{n-1}$, then x_{n-1} is a fixed point of T. This completes the proof. Therefore, we suppose $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$. We shall divide the proof into two cases.

Cases 1. Assume that

$$\max\{d(x_m, Tx_n), d(Tx_m, x_n)\} \ne 0, \quad \forall m \in \mathbb{N}, \forall n \in \mathbb{N}_0. \tag{2.19}$$

Then from (2.18), we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\le \varphi(M_{A,T}(x_{n-1},x_n))M_{A,T}(x_{n-1},x_n) \\ &< \frac{1}{\delta}\max\left\{ \begin{array}{c} d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)+d(x_n,Tx_n)d(x_n,Tx_{n-1})}{\max\{k,s[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)]\}}, \\ \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)+d(x_n,Tx_n)d(x_n,Tx_{n-1})}{\max\{d(x_{n-1},Tx_n),d(Tx_{n-1},x_n)\}} \end{array} \right\} \\ &= \frac{1}{\delta}\max\left\{ \begin{array}{c} d(x_{n-1},x_n), \\ \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})d(x_n,x_n)}{\max\{k,s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\}}, \\ \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})d(x_n,x_n)}{\max\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}} \end{array} \right\} \\ &\le \frac{1}{\delta}d(x_{n-1},x_n). \qquad (2.20) \end{aligned}$$

Since $\delta > s \geq 1$, from the inequality (2.20), we have

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}. \qquad (2.21)$$

As in the proof of Theorem 2.2, we can prove that

$$\lim_{n\to\infty} d(x_n,Tx_n) = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \qquad (2.22)$$

Now, we claim that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \quad d(x_{p(n)},x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1},x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \qquad (2.23)$$

From the triangle inequality, for all $n \in \mathbb{N}$, we have the following inequality:

$$\epsilon \le d(x_{p(n)},x_{q(n)}) \le s[d(x_{p(n)},x_{q(n)+1}) + d(x_{q(n)+1},x_{q(n)})].$$

It follows from (2.22) that

$$\frac{\epsilon}{s} \le \liminf_{n\to\infty} d(x_{p(n)},x_{q(n)+1}). \qquad (2.24)$$

So, there exists $n_1 \in \mathbb{N}$ such that

$$\max\{d(x_{p(n)},Tx_{q(n)}),d(Tx_{p(n)},x_{q(n)})\} \geq d(x_{p(n)},Tx_{q(n)}) \geq \frac{\epsilon}{s}, \quad \forall n \geq n_1.$$

So from assumption of theorem, for all $n \geq n_1$ we have

$$\begin{aligned} d(x_{p(n)},x_{q(n)+1}) &= d(Tx_{p(n)-1},Tx_{q(n)}) \qquad (2.25) \\ &\le \varphi(M_{A,T}(x_{p(n)-1},x_{q(n)}))M_{A,T}(x_{p(n)-1},x_{q(n)}) \\ &\le \frac{1}{\delta}M_{A,T}(x_{p(n)-1},x_{q(n)}). \qquad (2.26) \end{aligned}$$

Observe that

$$
\begin{aligned}
&M_{A,T}(x_{p(n)-1}, x_{q(n)})\\
&= \max\left\{\begin{array}{c} d(x_{p(n)-1}, x_{q(n)}),\\ \frac{d(x_{p(n)-1},Tx_{p(n)-1})d(x_{p(n)-1},Tx_{q(n)})+d(x_{q(n)},Tx_{q(n)})d(x_{q(n)},Tx_{p(n)-1})}{\max\{k,s[d(x_{p(n)-1},Tx_{p(n)-1})+d(x_{q(n)},Tx_{q(n)})]\}},\\ \frac{d(x_{p(n)-1},Tx_{p(n)-1})d(x_{p(n)-1},Tx_{q(n)})+d(x_{q(n)},Tx_{q(n)})d(x_{q(n)},Tx_{p(n)-1})}{\max\{d(x_{p(n)-1},Tx_{q(n)}),d(Tx_{p(n)-1},x_{q(n)})\}}\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} d(x_{p(n)-1}, x_{q(n)}),\\ \frac{d(x_{p(n)-1},Tx_{p(n)-1})d(x_{p(n)-1},Tx_{q(n)})}{k}+\frac{d(x_{q(n)},Tx_{q(n)})d(x_{q(n)},Tx_{p(n)-1})}{k}\\ \frac{d(x_{p(n)-1},Tx_{p(n)-1})d(x_{p(n)-1},Tx_{q(n)})}{d(x_{p(n)-1},Tx_{q(n)})}+\frac{d(x_{q(n)},Tx_{q(n)})d(x_{q(n)},Tx_{p(n)-1})}{d(Tx_{p(n)-1},x_{q(n)})}\end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(x_{p(n)-1}, x_{q(n)}),\\ \frac{d(x_{p(n)-1},x_{p(n)})d(x_{p(n)-1},x_{q(n)+1})}{k}+\frac{d(x_{q(n)},x_{q(n)+1})d(x_{q(n)},x_{p(n)})}{k}\\ d(x_{p(n)-1}, x_{p(n)}) + d(x_{q(n)}, x_{q(n)+1})\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} d(x_{p(n)-1}, x_{q(n)}),\\ \frac{d(x_{p(n)-1},x_{p(n)})s[d(x_{p(n)-1},x_{q(n)})+d(x_{q(n)},x_{q(n)+1})]}{k}\\ +\frac{d(x_{q(n)},x_{q(n)+1})s[d(x_{q(n)},x_{p(n)-1})+d(x_{p(n)-1},x_{q(n)})]}{k}\\ d(x_{p(n)-1}, x_{p(n)}) + d(x_{q(n)}, x_{q(n)+1})\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} \epsilon,\\ \frac{d(x_{p(n)-1},x_{p(n)})s[\epsilon+d(x_{q(n)},x_{q(n)+1})]}{k}\\ +\frac{d(x_{q(n)},x_{q(n)+1})s[\epsilon+d(x_{p(n)-1},x_{p(n)})]}{k}\\ d(x_{p(n)-1}, x_{p(n)}) + d(x_{q(n)}, x_{q(n)+1})\end{array}\right\}
\end{aligned}
$$

Letting $n \to \infty$ in the inequality above and using (2.22) we obtain

$$\limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)+1}) \leq \frac{\epsilon}{\delta}. \tag{2.27}$$

Thus from (2.24) and (2.27), we get $\frac{\epsilon}{s} \leq \frac{\epsilon}{\delta}$, a contradiction since $\delta > s$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. So by completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{2.28}$$

Now, we will show that $x^* = Tx^*$. Arguing by contradiction, we assume that $d(x^*, Tx^*) > 0$. From the triangle inequality, we have

$$d(x^*, Tx^*) \leq s[d(x^*, x_n) + d(x_n, Tx^*)].$$

It follows from (2.22) that

$$\frac{1}{s} d(x^*, Tx^*) \leq \liminf_{n\to\infty} d(x_n, Tx^*). \tag{2.29}$$

So, there exists $n_2 \in \mathbb{N}$ such that

$$\max\{d(x_n, Tx^*), d(Tx_n, x^*)\} \geq d(Tx_n, x^*) > \frac{1}{s} d(x^*, Tx^*), \quad \forall n \geq n_2.$$

So from assumption of theorem, for all $n \geq n_2$ we have

$$
\begin{aligned}
d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) &\leq \varphi(M_{A,T}(x_n, x^*))M_{A,T}(x_n, x^*)\\
&\leq \frac{1}{\delta} M_{A,T}(x_n, x^*).
\end{aligned}\tag{2.30}
$$

Observe that

$$\begin{aligned} M_{A,T}(x_n,y) &= \max\left\{\begin{array}{c} d(x_n,x^*), \frac{d(x_n,Tx_n)d(x_n,Tx^*)+d(x^*,Tx^*)d(x^*,Tx_n)}{\max\{k,s[d(x_n,Tx_n)+d(x^*,Tx^*)]\}}, \\ \frac{d(x_n,Tx_n)d(x_n,Tx^*)+d(x^*,Tx^*)d(x^*,Tx_n)}{\max\{d(x_n,Tx^*),d(Tx_n,x^*)\}} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d(x_n,x^*), \frac{d(x_n,Tx_n)s[d(x_n,x^*)+d(x^*,Tx^*)]+d(x^*,Tx^*)d(x^*,Tx_n)}{\max\{k,s[d(x_n,Tx_n)+d(x^*,Tx^*)]\}}, \\ \frac{d(x_n,Tx_n)s[d(x_n,x^*)+d(x^*,Tx^*)]}{d(x_n,Tx^*)} + \frac{d(x^*,Tx^*)d(x^*,Tx_n)}{d(Tx_n,x^*)} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d(x_n,x^*), \frac{d(x_n,Tx_n)s[d(x_n,x^*)+d(x^*,Tx^*)]+d(x^*,Tx^*)d(x^*,Tx_n)}{\max\{k,s[d(x_n,Tx_n)+d(x^*,Tx^*)]\}}, \\ \frac{d(x_n,Tx_n)s[d(x_n,x^*)+d(x^*,Tx^*)]}{d(x_n,Tx^*)} + \frac{d(x^*,Tx^*)d(x^*,Tx_n)}{d(Tx_n,x^*)} \end{array}\right\}. \end{aligned}$$

It follows from (2.22), (2.28), (2.30) and and Squeezing Theorem that

$$\lim_{n\to\infty} d(x_n,Tx^*) = 0. \tag{2.31}$$

Since

$$d(x^*,Tx^*) \leq s[d(x^*,x_n) + d(x_n,Tx^*)].$$

It follows from (2.28) and (2.31) that $d(x^*,Tx^*) = 0$. Now, we show that T has a unique fixed point. For this, we assume that y^* is another fixed point of T in X such that $d(x^*,y^*) > 0$. Therefore $\max\{d(x^*,Ty^*), d(Tx^*,y^*)\} = d(x^*,y^*) > 0$, so from assumption of theorem

$$\begin{aligned} d(x^*,y^*) = d(Tx^*,Ty^*) &\leq \varphi(M_{A,T}(x^*,y^*))M_{A,T}(x^*,y^*) \\ &\leq \frac{1}{\delta}\max\left\{\begin{array}{c} d(x^*,y^*), \frac{d(x^*,Tx^*)d(x^*,Ty^*)+d(y^*,Ty^*)d(y^*,Tx^*)}{\max\{k,s[d(x^*,Tx^*)+d(y^*,Ty^*)]\}}, \\ \frac{d(x^*,Tx^*)d(x^*,Ty^*)+d(y^*,Ty^*)d(y^*,Tx^*)}{\max\{d(x^*,Ty^*),d(Tx,y^*)\}} \end{array}\right\} \\ &= \frac{1}{\delta}d(x^*,y^*) < d(x^*,y^*), \end{aligned}$$

which is a contradiction. Hence, we conclude that x^* is a unique fixed point of T.
Cases 2. Assume that there exist $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ such that

$$\max\{d(x_m,Tx_n), d(Tx_m,x_n)\} = 0. \tag{2.32}$$

By condition (2.18), it follows that $d(Tx_m,Tx_n) = 0$ and hence $x_m = Tx_n = Tx_m = x_n$. This completes the proof of the existence of a fixed point of T. The uniqueness follows as in Case 1. ■

Corollary 2.6. *Let (X,d) be a complete b-metric space and T be a self mapping on X. Suppose that there exist $\delta \in (s,\infty)$, $k \in (0,\infty)$ and the function $\varphi : [0,\infty) \to [0,\frac{1}{\delta})$ such that*

$$d(Tx,Ty) \leq \begin{cases} \varphi(M_T(x,y))M_T(x,y), & if\ \ d(x,Ty) + d(Tx,y) \neq 0, \\ 0, & if\ \ d(x,Ty) + d(Tx,y) = 0, \end{cases}$$

where,

$$M_T(x,y) = \max\left\{\begin{array}{c} d(x,y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{k+sd(x,Tx)+sd(y,Ty)}, \\ \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)} \end{array}\right\}.$$

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof. Since $d(x,Ty)+d(Tx,y)>0$ implies that $\max\{d(x,Ty),d(Tx,y)\}>0$ and $d(x,Ty)+d(Tx,y)=0$ implies that $\max\{d(x,Ty),d(Tx,y)\}=0$, moreover $M_T(x,y)\leq M_{A,T}(x,y)$, so from Theorem 2.5 the proof is complete. ■

Definition 2.7. Let (X,d) be a b-metric space with constant $s\geq 1$. A mapping $T: X\to X$ is said to be a *generalized rational Geraghty contraction of type B* if there exist $\delta\in(s,\infty)$ and the function $\varphi:[0,\infty)\to[0,\frac{1}{\delta})$ such that

$$d(Tx,Ty)\leq\begin{cases}\varphi(M_{B,T}(x,y))M_{B,T}(x,y), & if\ \ M_1(x,y)\neq 0 \ and\ M_2(x,y)\neq 0,\\ 0, & if\ \ M_1(x,y)=0 \ or\ M_2(x,y)=0,\end{cases}\tag{2.33}$$

where,

$M_1(x,y)=\max\{d(x,y),d(x,Ty),d(Tx,y)\}$, $M_2(x,y)=\max\{d(x,Tx)+d(y,Ty),d(Tx,y)\}$,

and

$$M_{B,T}(x,y)=\max\left\{\ d(x,y),\frac{d(x,Tx)d(x,Ty)}{M_1(x,y)},\frac{d(x,Ty)d(x,y)}{sM_2(x,y)}\ \right\}.$$

Theorem 2.8. *Let (X,d) be a complete b-metric space with constant $s\geq 1$ and $T:X\to X$ be a generalized rational Geraghty contraction of type B. Then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof. Let $x_0=x\in X$. Put $x_{n+1}=Tx_n=T^{n+1}x_0$ for all $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. If, there exists $n\in\mathbb{N}$ such that $x_n=x_{n-1}$, then x_{n-1} is a fixed point of T. This completes the proof. Therefore, we suppose $x_n\neq x_{n-1}$ for all $n\in\mathbb{N}$. We shall divide the proof into two cases.

Cases 1. Assume that for all $m\in\mathbb{N}$ and $n\in\mathbb{N}_0$

$$M_1(x_m,x_n)\neq 0,\quad and\quad M_2(x_m,x_n)\neq 0.\tag{2.34}$$

Then from (2.33), we have

$$\begin{aligned}d(x_n,x_{n+1})&=d(Tx_{n-1},Tx_n)\\&\leq\varphi(M_{B,T}(x_{n-1},x_n))M_{B,T}(x_{n-1},x_n)\\&<\frac{1}{\delta}\max\left\{\begin{array}{c}d(x_{n-1},x_n),\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{\max\{d(x_{n-1},x_n),d(x_{n-1},Tx_n),d(Tx_{n-1},x_n)]\}},\\ \frac{d(x_{n-1},Tx_n)d(x_{n-1},x_n)}{s\max\{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n),d(Tx_{n-1},x_n)\}}\end{array}\right\}\\&=\frac{1}{\delta}\max\left\{\begin{array}{c}d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{\max\{d(x_{n-1},x_n),d(x_{n-1},x_{n+1}),d(x_n,x_n)]\}},\\ \frac{d(x_{n-1},x_{n+1})d(x_{n-1},x_n)}{s\max\{d(x_{n-1},x_n)+d(x_n,x_{n+1}),d(x_n,x_n)\}}\end{array}\right\}\\&\leq\frac{1}{\delta}\max\left\{\begin{array}{c}d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)},\\ \frac{s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]d(x_{n-1},x_n)}{s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}\end{array}\right\}\\&=\frac{1}{\delta}\max\{\ d(x_{n-1},x_n),d(x_n,x_{n+1})\ \}.\end{aligned}\tag{2.35}$$

If $d(x_n,x_{n+1})>d(x_{n-1},x_n)$, then

$$\max\{\ d(x_{n-1},x_n),d(x_n,x_{n+1})\ \}=d(x_n,x_{n+1}),$$

so (2.35) becomes

$$d(x_n,x_{n+1})\leq\frac{1}{\delta}d(x_n,x_{n+1}),$$

which is a contradiction (since $\delta > s \geq 1$). Thus, we conclude that

$$\max\left\{\ d(x_{n-1}, x_n), d(x_n, x_{n+1})\ \right\} = d(x_{n-1}, x_n). \tag{2.36}$$

It follows from (2.35) and (2.36) that

$$d(x_n, x_{n+1}) \leq \frac{1}{\delta} d(x_{n-1}, x_n) < d(x_{n-1}, x_n), \qquad \forall n \in \mathbb{N}. \tag{2.37}$$

As in the proof of Theorem 2.2, we can prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.38}$$

Now, we claim that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \quad d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{2.39}$$

From the triangle inequality, for all $n \in \mathbb{N}$, we have the following two inequalities:

$$\epsilon \leq d(x_{p(n)}, x_{q(n)}) \leq s[d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)})],$$

and

$$\epsilon \leq d(x_{p(n)}, x_{q(n)}) \leq s[d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)})].$$

It follows from (2.38) that

$$\frac{\epsilon}{s} \leq \liminf_{n\to\infty} d(x_{p(n)}, x_{q(n)+1}) \quad \textit{and} \quad \frac{\epsilon}{s} \leq \liminf_{n\to\infty} d(x_{p(n)+1}, x_{q(n)}). \tag{2.40}$$

So, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$d(x_{p(n)}, Tx_{q(n)}) \geq \frac{\epsilon}{s}, \quad \textit{and} \quad d(Tx_{p(n)}, x_{q(n)}) \geq \frac{\epsilon}{s}. \tag{2.41}$$

So for all $n \geq n_1$, we have

$$M_1(x_{p(n)}, x_{q(n)}) = \max\{d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\} \geq \frac{\epsilon}{s},$$

and

$$M_2(x_{p(n)}, x_{q(n)}) = \max\{d(x_{p(n)}, Tx_{p(n)}) + d(x_{q(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\} \geq \frac{\epsilon}{s}$$

It follows from assumption of theorem that, for all $n \geq n_1$,

$$\begin{aligned} d(x_{p(n)}, x_{q(n)+1}) &= d(Tx_{p(n)-1}, Tx_{q(n)}) && (2.42) \\ &\leq \varphi(M_{B,T}(x_{p(n)-1}, x_{q(n)}))M_{B,T}(x_{p(n)-1}, x_{q(n)}) \\ &\leq \frac{1}{\delta} M_{B,T}(x_{p(n)-1}, x_{q(n)}). && (2.43) \end{aligned}$$

Observe that, from (2.41), we have

$$\begin{aligned} M_{B,T}(x_{p(n)-1}, x_{q(n)}) &= \max\left\{ \begin{array}{c} d(x_{p(n)-1}, x_{q(n)}), \\ \frac{d(x_{p(n)-1}, Tx_{p(n)-1}) d(x_{q(n)}, Tx_{q(n)})}{\max\{d(x_{p(n)-1}, x_{q(n)}), d(x_{p(n)-1}, Tx_{q(n)}), d(Tx_{p(n)-1}, x_{q(n)})\}}, \\ \frac{d(x_{p(n)-1}, Tx_{q(n)}) d(x_{p(n)-1}, x_{q(n)})}{s\max\{d(x_{p(n)-1}, Tx_{p(n)-1}) + d(x_{q(n)}, Tx_{q(n)}), d(Tx_{p(n)-1}, x_{q(n)})\}} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(x_{p(n)-1}, x_{q(n)}), \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{\max\{d(x_{p(n)-1}, x_{q(n)}), d(x_{p(n)-1}, x_{q(n)+1}), d(x_{p(n)}, x_{q(n)})\}}, \\ \frac{d(x_{p(n)-1}, x_{q(n)+1}) d(x_{p(n)-1}, x_{q(n)})}{s\max\{d(x_{p(n)-1}, x_{p(n)}) + d(x_{q(n)}, x_{q(n)+1}), d(x_{p(n)}, x_{q(n)})\}} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d(x_{p(n)-1}, x_{q(n)}), \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{d(x_{p(n)}, x_{q(n)})}, \\ \frac{d(x_{p(n)-1}, x_{q(n)+1}) d(x_{p(n)-1}, x_{q(n)})}{sd(x_{p(n)}, x_{q(n)})} \end{array} \right\}. \end{aligned}$$

So, from (2.39), we get

$$\begin{aligned} M_{B,T}(x_{p(n)-1}, x_{q(n)}) &\leq \max\left\{ \begin{array}{c} \epsilon, \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{\epsilon}, \\ \frac{d(x_{p(n)-1}, x_{q(n)+1})\epsilon}{s\epsilon} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} \epsilon, \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{\epsilon}, \\ \frac{s[d(x_{p(n)-1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})]\epsilon}{s\epsilon} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} \epsilon, \\ \frac{d(x_{p(n)-1}, x_{p(n)}) d(x_{q(n)}, x_{q(n)+1})}{\epsilon}, \\ \frac{s[\epsilon + d(x_{q(n)}, x_{q(n)+1})]\epsilon}{s\epsilon} \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$ in the inequality above and using (2.38) and (2.48) we obtain

$$\limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)+1}) \leq \frac{\epsilon}{\delta}. \tag{2.44}$$

Thus from (2.40) and (2.44), we get $\frac{\epsilon}{s} \leq \frac{\epsilon}{\delta}$, a contradiction since $\delta > s$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. So by completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{2.45}$$

Now, we will show that $x^* = Tx^*$. Arguing by contradiction, we assume that $d(x^*, Tx^*) > 0$. From the triangle inequality, we have

$$d(x^*, Tx^*) \leq s[d(x^*, x_n) + d(x_n, Tx^*)].$$

It follows from (2.38) that

$$\frac{1}{s} d(x^*, Tx^*) \leq \liminf_{n\to\infty} d(x_n, Tx^*). \tag{2.46}$$

So, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_n, Tx^*) \geq \frac{1}{s} d(x^*, Tx^*), \quad \forall n \geq n_2. \tag{2.47}$$

So for all $n \geq n_2$, we have

$$M_1(x_n, x^*) = \max\{d(x_n, x^*), d(x_n, Tx^*), d(Tx_n, x^*)\} \geq \frac{1}{s} d(x^*, Tx^*),$$

and

$$M_2(x_n, x^*) = \max\{d(x_n, Tx) + d(x^*, Tx^*), d(Tx_n, x^*)\} \geq d(x^*, Tx^*) > 0$$

So from assumption of theorem, for all $n \geq n_2$ we have

$$\begin{aligned} d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) &\leq \varphi(M_{B,T}(x_n, x^*))M_{B,T}(x_n, x^*) \\ &\leq \frac{1}{\delta} M_{B,T}(x_n, x^*). \end{aligned} \tag{2.48}$$

Observe that from (2.47), for all $n \geq n_2$ we have

$$\begin{aligned} M_{B,T}(x_n, x^*) &= \max\left\{ \begin{array}{c} d(x_n, x^*), \frac{d(x_n,Tx_n)d(x^*,Tx^*)}{\max\{d(x_n,x^*),d(x_n,Tx^*),d(Tx_n,x^*)\}}, \\ \frac{d(x_n,Tx^*)d(x_n,x^*)}{s\max\{d(x_n,Tx_n)+d(x^*,Tx^*),d(Tx_n,x^*)\}} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(x_n, x^*), \frac{d(x_n,x_{n+1})d(x^*,Tx^*)}{\max\{d(x_n,x^*),d(x_n,Tx^*),d(x_{n+1},x^*)\}}, \\ \frac{d(x_n,Tx^*)d(x_n,x^*)}{s\max\{d(x_n,x_{n+1})+d(x^*,Tx^*),d(x_{n+1},x^*)\}} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d(x_n, x^*), \frac{d(x_n,x_{n+1})d(x^*,Tx^*)}{d(x_n,Tx^*)}, \\ \frac{d(x_n,Tx^*)d(x_n,x^*)}{s[d(x_n,x_{n+1})+d(x^*,Tx^*)]} \end{array} \right\} \\ &\leq \max\left\{ d(x_n, x^*), s\frac{d(x_n,x_{n+1})d(x^*,Tx^*)}{d(x^*,Tx^*)}), \frac{s[d(x_n,x^*)+d(x^*,Tx^*)]d(x_n,x^*)}{d(x_n,x_{n+1})+d(x^*,Tx^*)} \right\} \end{aligned}$$

It follows from (2.38), (2.45), (2.48) and Squeezing Theorem that

$$\limsup_{n\to\infty} d(x_n, Tx^*) = 0. \tag{2.49}$$

It follows from (2.46) and (2.49) that $d(x^*, Tx^*) = 0$. Now, we show that T has a unique fixed point. For this, we assume that y^* is another fixed point of T in X such that $d(x^*, y^*) > 0$. Therefore

$$\max\{d(x^*, y^*), d(x^*, Ty^*), d(Tx^*, y^*)\} \geq d(x^*, y^*) > 0$$

and

$$\max\{d(x^*, Tx) + d(y^*, Ty^*), d(Tx^*, y^*)\} \geq d(Tx^*, y^*) > 0,$$

so from assumption of theorem

$$\begin{aligned} d(x^*, y^*) = d(Tx^*, Ty^*) &\leq \varphi(M_{B,T}(x^*, y^*))M_{B,T}(x^*, y^*) \\ &\leq \frac{1}{\delta} \max\left\{ \begin{array}{c} d(x^*, y^*), \frac{d(x^*,Tx^*)d(y^*,Ty^*)}{\max\{d(x^*,y^*),d(x^*,Ty^*),d(Tx^*,y^*)]\}}, \\ \frac{d(x^*,Ty^*)d(x^*,y^*)}{s\max\{d(x^*,Tx^*)+d(y^*,Ty^*),d(Tx^*,y^*)\}} \end{array} \right\} \\ &= \frac{1}{\delta} d(x^*, y^*) < d(x^*, y^*), \end{aligned}$$

which is a contradiction. Hence, we conclude that x^* is a unique fixed point of T.
Cases 2. Assume that there exist $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ such that

$$\max\{d(x_m, x_n), d(x_m, Tx_n), d(Tx_m, x_n)\} = 0$$
or
$$\max\{s[d(x_m, Tx_m) + d(x_n, Tx_n)], d(Tx_m, x_n)\} = 0. \tag{2.50}$$

By condition (2.33), it follows that $d(Tx_m, Tx_n) = 0$ and hence $x_m = Tx_n = Tx_m = x_n$. This completes the proof of the existence of a fixed point of T. The uniqueness follows as in Case 1. ∎

Corollary 2.9. *Let (X, d) be a complete b-metric space and T be a self mapping on X. Suppose that there exist $\delta \in (s, \infty)$ and the function $\varphi : [0, \infty) \to [0, \frac{1}{\delta})$ such that*

$$d(Tx, Ty) \leq \begin{cases} \varphi(N_T(x, y))N_T(x, y), & if \;\; N_1(x, y) \neq 0 \text{ and } N_2(x, y) \neq 0, \\ 0, & if \;\; N_1(x, y) = 0 \text{ or } N_2(x, y) = 0, \end{cases}$$

where,

$$N_1(x, y) = d(x, y) + d(x, Ty) + d(Tx, y), \quad N_2(x, y) = sd(x, Tx) + sd(y, Ty) + d(Tx, y),$$

and

$$N_T(x, y) = \max\left\{ d(x, y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{N_1(x,y)}, \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{N_2(x,y)} \right\}.$$

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof. Since $N_1(x, y) \neq 0$ and $N_2(x, y) \neq 0$ implies that $M_1(x, y) \neq 0$ and $M_2(x, y) \neq 0$ and $N_T(x, y) \leq M_T(x, y)$, so from Theorem 2.8 the proof is complete. ∎

Example 2.10. Let $X = \{0, 1, 4\}$ and define a metric d on X by $d(x, y) = (x - y)^2$. Then, (X, d) is a complete b-metric space with $s = \frac{16}{10}$. Let $k \in [1, \infty)$, $\delta = \frac{17}{10}$ and $\varphi\colon [0, \infty) \to [0, \frac{1}{\delta})$ be defined by $\varphi(t) = \frac{10}{17}(\frac{10}{11})^t$. Let $T\colon X \to X$ be defined by

$$T0 = T1 = 1, \quad T4 = 0.$$

First observe that

$$\frac{1}{2}d(x, Tx) < d(x, y) \Leftrightarrow [(x = 0, y = 1) \vee (x = 0, y = 4) \vee (x = 1, y = 4)].$$

Now we consider the following cases:
Case1. Let $x = 0, y = 1$, then $d(T0, T1) = d(1, 1) = 0$ and

$$M_T(0, 1) = \max\left\{d(0, 1), \frac{d(0, 1)d(1, 1)}{\max\{k, d(0, 1)\}}, \frac{d(0, 1)d(1, 1)}{\max\{k, d(1, 1)\}}\right\} = 1.$$

Case2. Let $x = 0, y = 4$, then $d(T0, T4) = d(1, 0) = 1$ and

$$M_T(0, 4) = \max\left\{d(0, 4), \frac{d(0, 1)d(4, 0)}{\max\{k, d(0, 4)\}}, \frac{d(0, 1)d(4, 0)}{\max\{k, d(1, 0)\}}\right\} = 16.$$

Case3. Let $x = 1 \wedge y = 4$, then $d(T1, T4) = d(1, 0) = 1$ and

$$M_T(1, 4) = \max\left\{d(1, 4), \frac{d(1, 1)d(4, 0)}{\max\{k, d(1, 4)\}}, \frac{d(1, 1)d(4, 0)}{\max\{k, d(1, 0)\}}\right\} = 9.$$

In all cases, we have $d(Tx,Ty) \le \varphi(M_T(x,y))M_T(x,y)$. This proves that T satisfies in all assumptions of Theorem 2.2 and thus it has a unique fixed point, which is $x^* = 1$.

Example 2.11. Let $X = \{0, 1, 4\}$ and define a metric d on X by

$$d(x,y) = \begin{cases} 0, & if\ x = y, \\ \mid x - y \mid^{-1} & if\ x \neq y. \end{cases}$$

Then, (X, d) is a complete b-metric space with $s = \frac{12}{7}$. Let $k \in (0,1)$, $\delta = \frac{17}{10}$ and $\varphi\colon [0,\infty) \to [0, \frac{1}{\delta})$ be defined by $\varphi(t) = \frac{17}{11}(2)^{-t}$. Let $T\colon X \to X$ be defined by

$$T0 = T4 = 0, \quad T1 = 4.$$

First observe that

$$\max\{d(x,Ty), d(Tx,y)\} \neq 0 \Leftrightarrow [(x = 0, y = 1) \vee (x = 0, y = 4) \vee (x = 1, y = 4)].$$

Now we consider the following cases:

Case1. Let $x = 0, y = 1$, then

$$d(T0,T1) = d(0,4) = \frac{1}{4},$$

and

$$M_{A,T}(0,1) = \max\left\{ \begin{array}{c} d(0,1), \frac{d(0,0)d(0,4)+d(1,4)d(1,0)}{\max\{k,\frac{12}{7}[d(0,0)+d(1,4)]\}}, \\ \frac{d(0,0)d(0,4)+d(1,4)d(1,0)}{\max\{d(0,4),d(0,1)\}} \end{array} \right\} = 1.$$

Case2. Let $x = 0, y = 4$, then

$$d(T0,T4) = d(0,0) = 0$$

and

$$M_{A,T}(0,4) = \max\left\{ \begin{array}{c} d(0,4), \frac{d(0,0)d(0,0)+d(4,0)d(4,0)}{\max\{k,\frac{12}{7}[d(0,0)+d(4,0)]\}}, \\ \frac{d(0,0)d(0,0)+d(4,0)d(4,0)}{\max\{d(0,0),d(0,4)\}} \end{array} \right\} = \frac{1}{4}.$$

Case3. Let $x = 1 \wedge y = 4$, then

$$d(T1,T4) = d(4,0) = \frac{1}{4}$$

and

$$M_{A,T}(1,4) = \max\left\{ \begin{array}{c} d(1,4), \frac{d(1,4)d(1,0)+d(4,0)d(4,0)}{\max\{k,\frac{12}{7}[d(1,0)+d(4,0)]\}}, \\ \frac{d(1,0)d(1,0)+d(4,0)d(4,0)}{\max\{d(1,0),d(4,4)\}} \end{array} \right\} = \frac{17}{16}.$$

In all cases, we have $d(Tx,Ty) \le \varphi(M_T(x,y))M_T(x,y)$. This proves that T satisfies in all assumptions of Theorem 2.5 and thus it has a unique fixed point, which is $x^* = 0$.

References

[1] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012) Article No. 94.

[2] H.H. Alsulami, E. Karapinar, H. Piri, Fixed points of generalized F-Suzuki type contraction in complete b-metric spaces, Discrete Dyn. Nat. Soc. 2015 (2015) https://doi.org/10.1155/2015/969726.

[3] H.H. Alsulami, E. Karapinar, H. Piri, Fixed points of modified F-contractive mappings in complete metric-like spaces, J. Funct. Spaces 2015 (2015) http://dx.doi.org/10.1155/2015/270971.

[4] M. Edelstein, On fixed and periodic points under contractive mappings, J. Lond. Math. Soc. 37 (1962) 74–79.

[5] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal. 71 (2009) 5313–5317.

[6] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl. 2014 (2014) Article No. 38.

[7] M.A. Geraghty, On contractive maps, Proc. of Amer. Math. Soc. 40 (1930) 604–608.

[8] H. Huang, L. Paunovic, S. Radenovic, On some new fixed point results for rational Geraghty contractive mappings in ordered bmetric spaces, J. Nonlinear Sci. Appl. 8 (2015) 800–807.

[9] B. Fisher, On a theorem of Khan, Riv. Math. Univ. Parma. 4 (1978) 135–137.

[10] M.S. Khan, A fixed point theorem for metric spaces, Rend. Inst. Math. Univ. Trieste 8 (1976) 69–72.

[11] R.J. Shahkoohi, A. Razani, Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces, J. Inequal. Appl. 2014 (2014) Article No. 373.

[12] F. Zabihi, A. Razani, Fixed point theorems for hybrid rational Geraghty contractive mappings in ordered b-metric spaces, J. Math. Appl. 2014 (2014) https://doi.org/10.1155/2014/929821.

[13] D. *Dukić*, Z. Kadelburg, S. *Radenović*, Fixed points of Geraghty-type mappings in various generalized metric spaces, Abstr. Appl. Anal. 2011 (2011) https://doi.org/10.1155/2011/561245.

[14] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica Et Informatica Universitatis Ostraviensis 1 (1993) 5–11.

[15] P. Kumam, N.V. Dung, V.T.L. Hang, Some equivalences between cone b-metric spaces and b-metric spaces, Abstr. Appl. Anal. 2013 (2013) https://doi.org/10.1155/2013/573740.

[16] P. Sookprasert, P. Kumam, D. Thongtha, W. Sintunavarat, Extension of almost generalized weakly contractive mappings in rectangular b-metric spaces and fixed point results, Afr. Mat. 28 (2017) 271–278.

[17] A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca 64 (2014) 941–960.

[18] M.A. Khamski, N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal. 73 (9) (2010) 3123–3129.

[19] M. Boriceanu, M. Bota, A. Petrusel, Mutivalued fractals in b-metric spaces, Cen. Eur. J. Math. 8 (2) (2010) 367–377.

[20] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Univ. Modena 46 (1998) 263–276.

[21] V. Ozturk, D. Turkoglu, Common fixed point theorems for mappings satisfying (E:A)-property in b-metric spaces, J. Nonlinear Sci. Appl. 8 (6) (2015) 1127–1133.

[22] M. Bota, A. Molnar, V. Csaba, On Ekelands variational principle in b -metric spaces, Fixed Point Theory 12 (2011) 21–28.