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A NSFD Scheme for the Solving Fractional-Order Competitive Prey-Predator System

Mehran Namjoo[1,*] **and Sadegh Zibaei**[2]

[1] *Department of Mathematics, Vali–e–Asr University of Rafsanjan, Rafsanjan, Iran*
e-mail : namjoo@vru.ac.ir

[2] *Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran*
e-mail : z.sadegh133@gmail.com

Abstract In this paper, we introduce fractional–order into a model competitive Lotka–Volterra prey–predator system. We will discuss the stability analysis of this fractional system. The nonstandard finite difference (NSFD) scheme is implemented to study the dynamic behaviors in the fractional–order Lotka–Volterra system. Proposed nonstandard numerical scheme is compared with the forward Euler and fourth order Runge–Kutta methods. Numerical results show that the NSFD approach is easy and accurate for implementing when applied to fractional–order Lotka–Volterra model.

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1. Introduction

The study of biological systems has been developed through many years. In these systems it is common that state variables represent nonnegative quantities, such as concentrations, physical properties, the size of populations and the amount of chemical compounds [1]. These biological models are commonly based on systems of ordinary differential equations (ODEs). Exact solutions of these systems are rarely accessible and usually complicated; hence good approximations are required. The interspecies interaction is among the most intensively explored fields of biology. The increasing amount of realistic mathematical models in that area helps in understanding the population dynamics of analyzed biological systems. Mathematical models of predator–prey systems, characterized by decreasing growth rate of one of the interacting populations and increasing growth rate of the other, consist of the ODEs systems. The current technological advance has made it possible for humans to disturb the environmental balance in nature that may cause immense damages, such as species extinction or starvation. Therefore, understanding the

*Corresponding author.

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behaviour of the interaction between the species may help biologists and other related parties to prevent those events from happening. The real interaction of prey–predator in nature is complex and comprises both interspecies and external environmental factors. Therefore, several simplifications are usually assumed so that a basic model can be constructed and then developed or modified to approach the real system.

In most of the interactions modeled, all rate of changes are assumed to be time independent, which makes the corresponding systems autonomous. It is not always possible to find the exact solutions of the nonlinear models that have at least two ODEs. It is sometimes more useful to find numerical solutions of these type systems in order to programme easily and visualize the results. By applying a numerical method on a continuous differential equation system, it becomes a difference equation system, i.e., discrete time system. While applying these numerical methods, it is necessary that the new difference equation system provide the positivity conditions and exhibit the same quantitative behaviours of continuous system such as stability, bifurcation and chaos. It is well known that some traditional and explicit schemes such as forward Euler and Runge–Kutta are unsuccessful at generating oscillation, bifurcation, chaos and false steady states, despite using adaptive step size [2, 3] . Instead of classical methods, NSFD schemes can alternatively be used to obtain more qualitative results and remove numerical instabilities. These schemes are developed for compensating the weaknesses, such as numerical instabilities that may be caused by standard finite difference methods. The most important advantages of this scheme is that choosing a convenient denominator function instead of the step size h, better results can be obtained. If the step size h is chosen small enough, the obtained results do not change significantly but if the step size h gets larger this advantage comes into focus.

As is well known, in field of mathematical biology, the traditional Lotka–Volterra systems are very important mathematical models which describe multispecies population dynamics in a nonautonomous environment. The Lotka–Volterra equations are a system of differential equations of the following form:

$$\begin{cases} x' = ax - bxy, \\ y' = -cy + dxy, \\ x(0) = x_0, \quad y(0) = y_0, \end{cases}$$

where x and y are prey and predator, respectively. The parameter a stands for the prey growth rate in the absence of the predators, b is the capture rate of prey per predator, d is the rate at which each predator converts captured prey into predator births and c is the constant rate at which death occurs in the absence of prey. They showed that ditrophic food chains (i.e. prey–predator systems) permanently oscillate for any initial condition if the prey growth rate is constant and the predator functional response is linear. Thus the basic Lotka–Volterra model is represented by the system of the two previous differential equations.

A slight variation of the Lotka–Volterra model, the competition model, is also quite useful in comparing the numerical approximation methods. The system of equations for the competition model introduces a ex^2 term representing the internal competition of the prey for limited external resources and a fy^2 term representing the competition among predators for the limited number of prey:

$$\begin{cases} x' = ax - bxy - ex^2, \\ y' = -cy + dxy - fy^2, \\ x(0) = x_0, \;\; y(0) = y_0. \end{cases} \tag{1.1}$$

This paper is organized as follows: In the next Section, we elaborate some basic definitions and properties of the Grünwald–Letnikov (GL) approximation and provides a brief overview of the important feature of the procedures for constructing NSFD schemes for systems of ODEs. In Section 3, we introduce fractional–order into the model that describes competitive Lotka–Volterra prey–predator system and also stability theorem and fractional Routh–Hurwitz stability conditions are given for the local asymptotic stability of the fractional systems. In Section 4, we will discuss the stability analysis of fractional–order system. In Section 5, we present the idea of NSFD scheme for solving the fractional–order competitive Lotka–Volterra prey–predator system. Finally in last Section, the theoretical results obtained in former section are compared with the other numerical methods and the simulated results are given.

2. Preliminaries

Fractional differential equations (FDEs) have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry and engineering [4]. In the recent years, the dynamic behaviors of fractional–order differential systems have received increasing attention. Although the concept of the fractional calculus was discussed in the same time interval of integer–order calculus, the complexity and the lack of applications postponed its progress till a few decades ago. Recently, most of the dynamical systems based on the integer–order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer–order modeling.

2.1. Grünwald–Letnikov Approximation

Derivatives of fractional–order have been introduced in several ways. In this paper we consider GL approach. The GL method of approximation for the one–dimensional fractional derivative is as follows [4]:

$$D^\alpha x(t) = f(t, x(t)), \qquad x(0) = x_0, \qquad t \in [0, t_f], \tag{2.1}$$

$$D^\alpha x(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{\alpha}{j} x(t - jh),$$

where $0 < \alpha \leq 1$, D^α denotes the fractional derivative and h is the step size and $[\frac{t}{h}]$ denotes the integer part of $\frac{t}{h}$. Therefore, Eq. (2.1) is discretized as follows:

$$\sum_{j=0}^{n} c_j^\alpha x_{n-j} = f(t_n, x_n), \qquad n = 1, 2, 3, \ldots$$

where $t_n = nh$ and c_j^α are the GL coefficients defined as:

$$c_j^\alpha = (1 - \frac{1+\alpha}{j}) c_{j-1}^\alpha, \qquad c_0^\alpha = h^{-\alpha}, \qquad j = 1, 2, 3, \ldots$$

2.2. Nonstandard Finite Difference Discretization

The initial foundation of NSFD schemes came from the exact finite difference schemes. These schemes are well developed by Mickens [5, 6] in the past decades. These schemes are developed for compensating the weaknesses such as numerical instabilities that may be caused by standard finite difference methods. Regarding the positivity, boundedness and monotonicity of solutions, NSFD schemes have a better performance over the standard finite difference schemes, due to its flexibility to construct a NSFD scheme that can preserve certain properties and structures, which are obeyed by the original equations. The advantages of NSFD schemes have been shown in many numerical applications. Gonzalez–Parra et al. [7] and Arenas et al. [8] developed NSFD schemes to solve population and biological models. Jordan [9] constructed NSFD schemes for heat transfer problems.
We now give an outline of the critical points which will allow the construction of NSFD discretizations for ODEs. Consider the autonomous ODE given by

$$x' = f(t, x, \lambda), \qquad x(0) = x_0, \qquad t \in [0, t_f],$$

where λ is a parameter and $f(t, x, \lambda)$ is, in general, a nonlinear function. For a discrete-time grid with step size, $\triangle t = h$, we replace the independent variable t by

$$t \approx t_n = nh, \qquad n = 0, 1, 2, \ldots, N$$

where $h = \frac{t_f}{N}$. The dependent variable $x(t)$ is replaced by

$$x(t) \approx x_n,$$

where x_n is the approximation of $x(t_n)$.

The first NSFD requirement is that the dependent functions should be modeled on the discrete–time computational grid. Particular examples of this include the following functions [5, 6].

$$\begin{cases} xy & \approx 2x_{n+1}y_n - x_{n+1}y_{n+1}, \\ x^2 & \approx x_{n+1}x_n, \\ x^3 & \approx (\dfrac{x_{n+1} + x_{n-1}}{2})x_n^2. \end{cases}$$

A standard way for representing a discrete first–derivative is given by

$$x' \cong \frac{x_{n+1} - x_n}{h}.$$

However, the NSFD scheme requires that x' has the more general representation

$$x' \cong \frac{x_{n+1} - x_n}{\phi},$$

where the denominator function, i.e. ϕ has the properties:

(i) $\phi(h) = h + O(h^2)$,

(ii) $\phi(h)$ is an increasing function of h,

(iii) $\phi(h)$ may depends on the parameters appearing in the differential equations.

Examples of denominator function that satisfy condition (i) are

$$h, \qquad sin(h), \qquad 1 - e^{-h}, \qquad \frac{1 - e^{-\lambda h}}{\lambda},$$

and so forth. The paper of Mickens [5] gives a general procedure for determining $\phi(h)$ for systems of ODEs. An example of the NSFD discretization process is its application to the decay equation

$$x' = -\lambda x,$$

where λ is a constant. The discretization scheme is as follows [5]

$$\frac{x_{n+1} - x_n}{\phi} = -\lambda x_n, \qquad \phi(h, \lambda) = \frac{1 - e^{-\lambda h}}{\lambda}.$$

Another example is given by

$$x' = \lambda_1 x - \lambda_2 x^2,$$

where the NSFD scheme is

$$\frac{x_{n+1} - x_n}{\phi} = \lambda_1 x_n - \lambda_2 x_{n+1} x_n, \qquad \phi(h, \lambda_1) = \frac{e^{\lambda_1 h} - 1}{\lambda_1}.$$

It should be noted that the NSFD schemes for these two ODEs are exact in the sense that $x_n = x(t_n)$ for all values of $h > 0$. In general, for an ODE with polynomial terms

$$x' = ax + (NL), \qquad NL \equiv \text{Nonlinear terms},$$

the NSFD discretization for the linear expressions is given by Mickens [5]

$$\frac{x_{n+1} - x_n}{\phi} = ax_n + (NL)_n,$$

where the denominator function is

$$\phi(h, a) = \frac{e^{ah} - 1}{a}.$$

It follows that if x' is to a function of x which does not have a linear term, then the denominator function is just h, i.e., $\phi(h) = h$.
By applying this technique and using the GL discretization method, it yields the following relations

$$x_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha} x_{n+1-j} + f(t_{n+1}, x_{n+1})}{c_0^{\alpha}}, \qquad n = 0, 1, 2, ...$$

where $c_0^{\alpha} = \phi(h)^{-\alpha}$.

3. Fractional–Order Competitive Lotka–Volterra System

In this section, we introduce fractional–order into the model (1.1) of Lotka–Volterra system. The new system is described by the following set of fractional ODEs of order α_1, α_2:

$$\begin{cases} D^{\alpha_1} x(t) = ax - bxy - ex^2, \\ D^{\alpha_2} y(t) = -cy + dxy - fy^2, \\ 0 < \alpha_i \leq 1, \qquad i = 1, 2 \end{cases} \tag{3.1}$$

with initial conditions

$$x(0) = x_0, \qquad y(0) = y_0.$$

In order to analyze the stability of the model, stability theorems on fractional–order systems and fractional Routh–Hurwitz stability conditions for fractional–order differential

equations are introduced. The first stability theorem has been given for incommensurate fractional–order systems.

Theorem 3.1 ([10]). *Consider the incommensurate fractional–order system*

$$D^{\alpha}x(t) = f(x(t)), \qquad x(0) = x_0, \tag{3.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in (0, 1]$ *for* $i = 1, 2, \ldots, n$ *and* $x \in \mathbb{R}^n$. *The equilibrium points of system Eqs.* (3.2) *are calculated by solving the equation:*

$$f(x) = 0.$$

These points are locally asymptotically stable if all eigenvalues λ *of the Jacobian matrix* $J \equiv \frac{\partial f}{\partial x}$ *evaluated at the equilibrium points satisfy:*

$$|arg(\lambda)| > \frac{\alpha^{\star}\pi}{2}, \qquad \alpha^{\star} = \max(\alpha_1, \ldots, \alpha_n), \qquad i = 1, \ldots, n.$$

Theorem 3.2 ([11]). *Consider the commensurate fractional–order system Eqs.* (3.2), *i.e.,* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha^{\star}$. *If all eigenvalues of the Jacobian matrix of an equilibrium point satisfy:*

$$|arg(\lambda)| > \frac{\alpha^{\star}\pi}{2},$$

then, the fractional system is locally asymptotically stable at the equilibrium point.

The Jacobian matrix of the system Eqs. (3.1) at equilibrium point $E = (x, y)$ is

$$J(E) = \begin{pmatrix} a - by - 2ex & -bx \\ dy & -c + dx - 2fy \end{pmatrix}, \tag{3.3}$$

with the characteristic polynomial P

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2. \tag{3.4}$$

Using the results of [12], for $n = 2$ the conditions for $|arg(\lambda)| > \frac{\alpha\pi}{2}$ are either Routh–Hurwitz conditions or

$$a_1 < 0, \qquad 4a_2 > a_1^2, \qquad |tan^{-1}(\frac{\sqrt{4a_2 - a_1^2}}{a_1})| > \alpha\frac{\pi}{2},$$

also $a_2 > 0$, is a necessary condition for $|arg(\lambda)| > \frac{\alpha^{\star}\pi}{2}$.
In the next section we discuss the asymptotic stability of the equilibrium point E of the system Eqs. (3.1).

4. Stability Analysis of the Equilibrium Points

In this section we deal with asymptotically stability of the equilibrium points of system Eqs. (3.1). To evaluate the equilibrium points of system Eqs. (3.1), let

$$\begin{aligned} ax - bxy - ex^2 &= 0, \\ -cy + dxy - fy^2 &= 0. \end{aligned} \tag{4.1}$$

Solving Eqs. (4.1), we can get for the system Eqs. (3.1) four equilibrium points:

$$E_0 = (0, 0), \qquad E_1 = (0, -\frac{c}{f}), \qquad E_2 = (\frac{a}{e}, 0), \qquad E_3 = (\frac{af + bc}{db + fe}, \frac{ad - ce}{db + fe}).$$

Only the equilibrium points E_0, E_2 and E_3 have real biological meaning. The asymptotically stability conditions of these equilibrium points are as follows:

(i) The Jacobian matrix of the system Eqs. (3.1) at the equilibrium point $E_0 = (0, 0)$ is

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}. \tag{4.2}$$

The characteristic equation of the Jacobian matrix (4.2) at the equilibrium point E_0 is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = c - a, \qquad a_2 = -ac.$$

Eigenvalues of the matrix J at the equilibrium point E_0 are

$$\lambda_1 = a, \qquad \lambda_2 = -c.$$

Since $a_2 < 0$, hence by using Routh–Hurwitz stability conditions mentioned in the previous section the equilibrium point E_0 is unstable.

(ii) The Jacobian matrix (3.3) at the equilibrium point $E_2 = (\frac{a}{e}, 0)$ is

$$J(\frac{a}{e}, 0) = \begin{pmatrix} -a & -\frac{ba}{e} \\ 0 & \frac{ad-ce}{e} \end{pmatrix}. \tag{4.3}$$

In this case the characteristic equation is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = -\frac{ad - ce - ae}{e}, \qquad a_2 = -\frac{(ad - ce)a}{e}.$$

Therefore, the eigenvalues of the Jacobian matrix (4.3) corresponding to the equilibrium point E_2 are

$$\lambda_1 = -a, \qquad \lambda_2 = \frac{ad - ce}{e}.$$

Now, if $ad < ce$ then $a_2 > 0$, and consequently the eigenvalues λ_1 and λ_2 are less than zero. Therefore by Theorem 3.2 and Routh–Hurwitz stability conditions a necessary and sufficient condition, for stability the equilibrium point E_2 is $ad < ce$.

(iii) The Jacobian matrix (3.3) at the equilibrium point $E_3 = (\frac{af+bc}{db+fe}, \frac{ad-ce}{db+fe})$ is

$$J(\frac{af + bc}{db + fe}, \frac{ad - ce}{db + fe}) = \begin{pmatrix} -\frac{e(af+bc)}{bd+ef} & -\frac{b(af+cb)}{bd+ef} \\ \frac{(ad-ce)d}{bd+ef} & -\frac{f(ad-ce)}{bd+ef} \end{pmatrix},$$

with the characteristic equation

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = \frac{f(ad - ce) + e(bc + af)}{bd + ef}, \qquad a_2 = \frac{(ad - ce)(af + bc)}{bd + ef}.$$

Note that if $ad > ce$ then $a_2 > 0$ and $a_1 > 0$. We now consider the following two cases:
(i) If $4a_2 - a_1^2 \leq 0$ then

$$\lambda_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} < 0, \qquad \lambda_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} < 0.$$

(ii) If $4a_2 - a_1^2 > 0$ then

$$\lambda_1 = \frac{-a_1 + i\sqrt{4a_2 - a_1^2}}{2}, \qquad \lambda_2 = \frac{-a_1 - i\sqrt{4a_2 - a_1^2}}{2}.$$

Since $a_1 > 0$, hence by Theorem 3.2 and Routh–Hurwitz stability conditions a necessary and sufficient condition for stability the equilibrium point E_3 is $ad > ce$.

5. NSFD Scheme for Fractional–Order Competitive Lotka–Volterra Model

In this section we use NSFD scheme to get numerical solution of Lotka–Volterra model. By using definition of GL derivative and use NSFD for the model we have:

$$\begin{aligned} \sum_{j=0}^{n+1} c_j^{\alpha_1} x_{n+1-j} &= ax_n - bx_{n+1}y_n - ex_{n+1}x_n, \\ \sum_{j=0}^{n+1} c_j^{\alpha_2} y_{n+1-j} &= -cy_{n+1} + dx_{n+1}y_n - fy_{n+1}y_n. \end{aligned} \tag{5.1}$$

Comparing Eqs. (5.1) with system Eqs. (3.1), we note the following statements:
(i) The linear and nonlinear terms on the right–hand side of first equation in system (3.1) are in the form

$$x \approx x_n, \qquad -xy \approx -x_{n+1}y_n, \qquad x^2 \approx -x_{n+1}x_n.$$

(ii) The linear and nonlinear terms on the right–hand side of second equation in system (3.1) are

$$-y \approx -y_{n+1}, \qquad xy \approx x_{n+1}y_n, \qquad y^2 \approx -y_{n+1}y_n.$$

Invoking some algebraic manipulations on Eqs. (5.1), the following relations are obtained

$$\begin{aligned} x_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} x_{n+1-j} + ax_n}{c_0^{\alpha_1} + by_n + ex_n}, \\ y_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} y_{n+1-j} + dx_{n+1}y_n}{c_0^{\alpha_2} + c + fy_n}, \end{aligned}$$

where

$$c_0^{\alpha_1} = \phi_1(h)^{-\alpha_1}, \qquad c_0^{\alpha_2} = \phi_2(h)^{-\alpha_2},$$

with [13–16]

$$\phi_1(h) = \frac{e^{ah} - 1}{a}, \qquad \phi_2(h) = \frac{e^{ch} - 1}{c}.$$

6. Numerical Results

Analytical studies always remain in complete without numerical verification of the results. In this section, numerical results from the implementation of NSFD scheme for the fractional–order Lotka–Volterra predator–prey system are presented. The approximate solutions displayed in Figs. 1 and 2 for the step size $h = 0.5$ and different $0 < \alpha_i \leq 1$, for $i = 1, 2$ and parameters $a = 1$, $b = 0.2$, $c = 0.5$, $d = 0.04$, $e = 0.01$ and $f = 0.005$ with the initial conditions $x(0) = 5$ and $y(0) = 2$, for simulation time 200s.

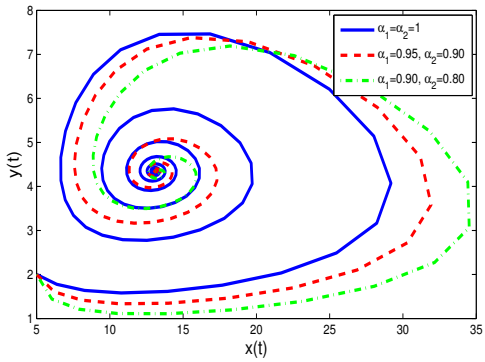

Figure 1. Plot of populations x, y for different α_1 and α_2 with step size $h = 0.5$.

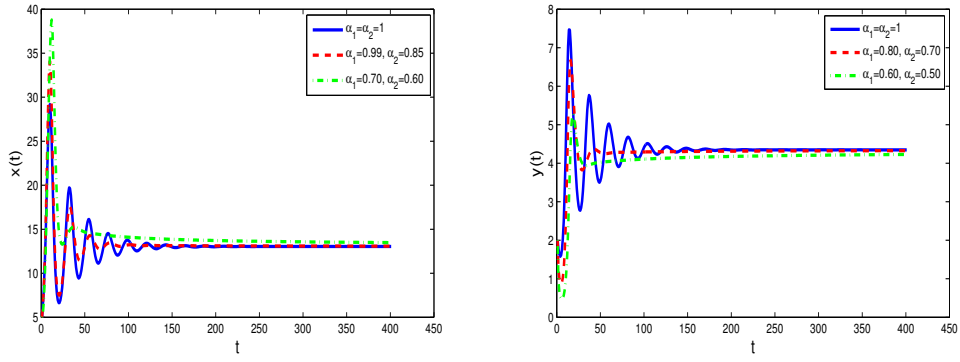

Figure 2. Plot of populations x, y over time for different α_1 and α_2 with step size $h = 0.5$.

The approximate solutions displayed in Figs. 3 and 4 for the step size $h = 1.5$ and different $0 < \alpha_i \leq 1$, for $i = 1, 2$ and parameters $a = 0.5$, $b = 0.002$, $c = 0.1$, $d = 0.001$, $e = 0.0001$ and $f = 0.0002$ with the initial conditions $x(0) = 250$ and $y(0) = 100$, for simulation time 300s.

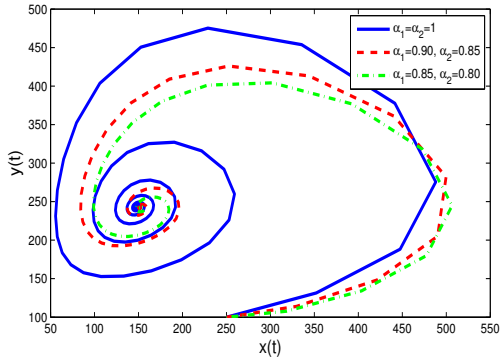

FIGURE 3. Plot of populations x, y for different α_1 and α_2 with step size $h = 1.5$.

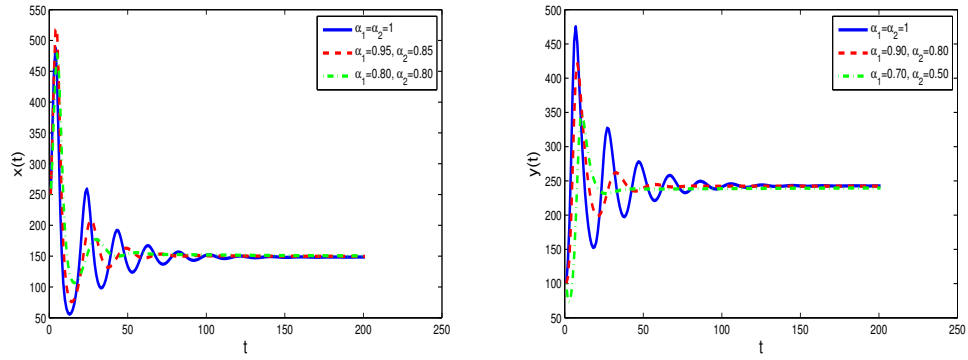

FIGURE 4. Plot of populations x, y over time for different α_1 and α_2 with step size $h = 1.5$.

The approximate solutions displayed in Fig. 5 for the step size $h = 2$ and different $0 < \alpha_i \leq 1$, for $i = 1, 2$ and parameters $a = 0.5$, $b = 0.002$, $c = 0.1$, $d = 0.001$, $e = 0.01$ and $f = 0.0002$ with the initial conditions $x(0) = 250$ and $y(0) = 100$, for simulation time 200s.

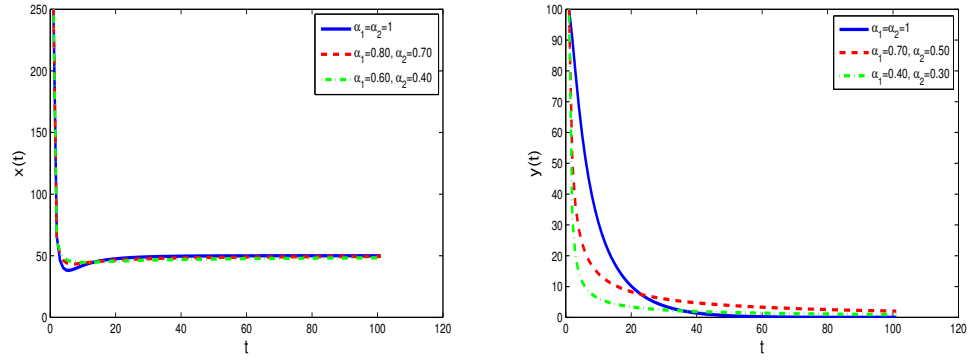

FIGURE 5. Plot of populations x, y over time for different α_1 and α_2 with step size $h = 2$.

In Tables 1 and 2, for different step sizes h, the qualitative stability results, obtained by NSFD scheme, of the equilibrium points E_2 and E_3 are respectively compared to classical methods such as forward Euler and fourth order Runge-Kutta methods. If step size h is chosen small enough, the results of the proposed NSFD scheme are similar with the results of the other two numerical methods, but if the step size h is chosen larger, the efficiency of NSFD scheme is clearly seen. From Tables 1 and 2, it follows that CPU times of the NSFD method is less than the CPU times forward Euler and fourth order Runge–Kutta methods.

TABLE 1. Qualitative results of the equilibrium point E_3 for different time step sizes, t= 0–100.

h	Euler	CPU time	Runge-Kutta	CPU time	NSFD	CPU time
0.001	Convergence	0.011167	Convergence	0.027123	Convergence	0.001486
0.1	Convergence	0.010702	Convergence	0.026794	Convergence	0.001301
0.2	Convergence	0.010605	Convergence	0.025951	Convergence	0.000815
0.5	Divergence	–	Convergence	0.024713	Convergence	0.000302
1	Divergence	–	Convergence	0.024423	Convergence	0.000153
10	Divergence	–	Divergence	–	Convergence	0.000020
100	Divergence	–	Divergence	–	Convergence	0.000007

TABLE 2. Qualitative results of the equilibrium point E_2 for different time step sizes, t= 0–100.

h	Euler	CPU time	Runge-Kutta	CPU time	NSFD	CPU time
0.001	Convergence	0.001586	Convergence	0.003324	Convergence	0.001820
0.1	Convergence	0.000922	Convergence	0.002973	Convergence	0.000153
0.2	Convergence	0.000868	Convergence	0.002926	Convergence	0.000079
0.5	Divergence	–	Convergence	0.002682	Convergence	0.000035
1	Divergence	–	Divergence	–	Convergence	0.000020
10	Divergence	–	Divergence	–	Convergence	0.000007
100	Divergence	–	Divergence	–	Convergence	0.000005

In Figs. 6 and 7, the NSFD solutions of x and y converge to equilibrium points E_3 and E_2 as simulated and also forward Euler and Runge–Kutta methods are compared with NSFD scheme graphically. All the numerical calculations and simulations are performed by using Matlab programme. In conclusion, the efficiency of the proposed NSFD scheme is investigated and compared with other numerical methods.

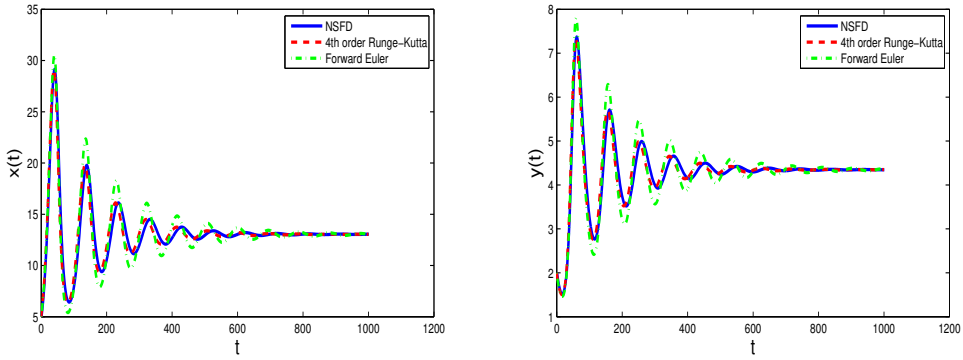

FIGURE 6. Numerical solutions of forward Euler and fourth order Runge–Kutta and NSFD methods converge to the equilibrium point E_3 with time step size $h = 0.1$ for $\alpha_1 = \alpha_2 = 1$.

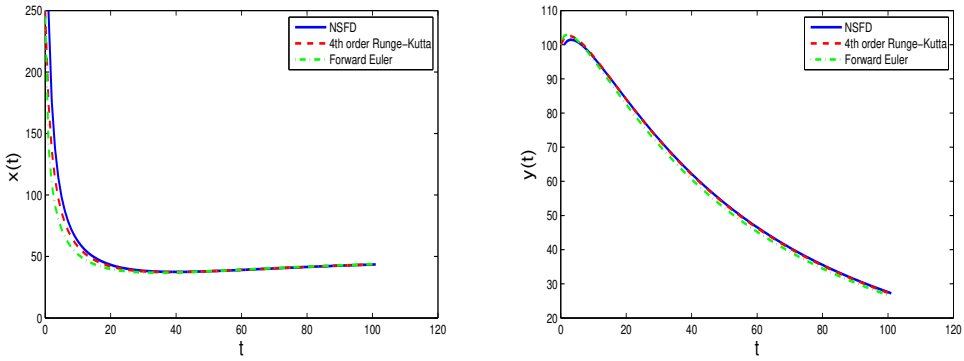

FIGURE 7. Numerical solutions of forward Euler and fourth order Runge–Kutta and NSFD methods converge to the equilibrium point E_2 with time step size $h = 0.2$ for $\alpha_1 = \alpha_2 = 1$.

7. Conclusion

In this paper we study the fractional–order competitive Lotka–Volterra model. The stability of equilibrium points are studied. Numerical solutions of this model are given. The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena.

References

[1] J.D. Murray, Mathematical Biology, Springer, New York, 2003.

[2] R.E. Mickens, Discretizations of nonlinear differential equations using explicit nonstandard methods, J. Comput. Appl. Math. 110 (1999) 181–185.

[3] L.W. Roger, Local stability of Eulers and Kahans methods, J. Difference Equ. Appl. 10 (6) (2004) 601–614.

[4] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[5] R.E. Mickens, Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition, Numer. Methods Partial Differential Equations 23 (3) (2007) 672–691.

[6] R.E. Mickens, Applications of Nonstandard Finite Difference Schemes, World Scientific, Singapore, 2000.

[7] G. Gonzalez–Parra, A.J. Arena, B.M. Chen–Charpentier, Combination of nonstandard schemes and Richardsons extrapolation to improve the numerical solution of population models, Math. Comput. Modelling 52 (7–8) (2010) 1030–1036.

[8] A.J. Arenas, G. Gonzalez–Parra, B.M. Chen–Charpentier, A nonstandard numerical scheme of predictor–corrector type for epidemic models, Comput. Math. Appl. 59 (12) (2010) 3740–3749.

[9] P.M. Jordan, A nonstandard finite difference scheme for nonlinear heat transfer in a thin finite rod, J. Difference Equ. Appl. 9 (11) (2003) 1015–1021.

[10] A.A. Mehmat, A. Secer, M. Bayram, Stability, synchronization control and numerical solution of fractional Shimizu-Morioka dynamical system, Appl. Math. Inf. Sci. 8 (14) (2014) 1699-1705.

[11] D. Matignon, Stability result on fractional differential equations with applications to control processing, Computational Engineering in Systems Applications 2 (1996) 963–968.

[12] E. Ahmed, A.M.A. El–Sayed, H.A.A. El–Saka, On some Routh–Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, Modern Phys. Lett. A 358 (2006).

[13] S. Zibaei, M. Namjoo, A NSFD scheme for Lotka–Volterra food web model, Iran. J. Sci. Technol. Trans. A Sci. 38 (4) (2014) 399–414.

[14] M. Zeinadini, M. Namjoo, A numerical method for discrete fractional-order chemostat model derived from nonstandard numerical scheme, Bull. Iranian Math. Soc. 43 (5) (2017) 1165–1182.

[15] M. Zeinadini, M. Namjoo, Approximation of fractional-order chemostat model with nonstandard finite-difference scheme, Hace. J. Math. Stat. 46 (3) (2017) 477–490.

[16] S. Zibaei, M. Namjoo, A nonstandard finite-difference scheme for solving fractional-order model of HIV-1 infection of CD 4^+ T-cells, Iran. J. Math. Chem. 6 (2015) 145–160.