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Global Residue Harmonic Balance Method for Obtaining Higher-Order Accurate Solutions to the Strongly Nonlinear Oscillator

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Abstract In this paper, a suitable analytical technique has been introduced based on the principles of the homotopy perturbation method and the residue harmonic balance method. The proposed technique has been applied to obtain approximate higher-order angular frequencies and corresponding periodic solutions of the strongly nonlinear oscillator with a cubic and harmonic restoring force. Dissimilar other harmonic balance methods, all the earlier residual errors are presented in the approximate solutions to enhance the accuracy. The expressions of the frequency-amplitude relationship are obtained in a novel analytical way. It is highly remarkable that the second-order approximate solutions produce better than previously existing results and almost similar as compared with the corresponding numerical solutions (considered to be exact). The high accuracy and simple solution procedure are the merits of the proposed technique which could also be applied to other nonlinear oscillatory problems arising in science and engineering.

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1. Introduction

The study of strongly nonlinear oscillators is of great importance in the field of physics, applied mathematics, dynamics of structures, vibrations in nonlinear mechanics and engineering [1, 2]. Most phenomena in nature are nonlinear and they are described by nonlinear equations. In general, obtaining the exact solution of the nonlinear equations is tremendously difficult and this perception has led to intensive research over many

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decades. Recently, researchers are using either analytical techniques or numerical methods or a combination of both to obtain approximate solutions. A few nonlinear systems can be solved explicitly, and the numerical method, especially, the Runge-Kutta fourth-order method is frequently used to calculated approximate solutions. The numerical method is not always fruitful especially in the area of stiff differential equations, chaotic differential equations and hyperchaotic differential equations. It is a big challenge to the numerical method still now. In this situation, many researchers have been showing an intensifying interest in the field of analytical techniques to derive approximate solutions of the strongly nonlinear oscillatory problems. The most commonly used analytical techniques for solving nonlinear oscillatory problems is perturbation method [3–5], which is the most versatile tools available in nonlinear analysis of engineering problems, and they are constantly being developed and applied to ever more complex problems. However, the standard perturbation methods have many limitations, and they do not yield for strongly nonlinear oscillators.

As a result, to overcome the limitations of standard perturbation technique, many new analytical techniques have been investigated these days. Among of them, the energy balance method [6, 7], the optimal iteration method [8], the VIM-Pade´ technique [9], the algebraic method [10], the Quintication Method [11], He's frequency-amplitude formulation [12] and an iterative approach [13] have been paid much attention to determine approximate periodic solutions of strongly nonlinear oscillatory problems. In fact, to the best of our knowledge, in most of these methods, only the first-order approximation has been considered which does not give sufficient accuracy in the obtained results. The harmonic balance method (HBM) [14–18] is a general analytical technique for calculating approximate periodic solutions of strongly nonlinear oscillatory problems. However, a set of complicated higher-order nonlinear algebraic equations appear when the HBM is applied. In the case of the large amplitude of the oscillation, it is tremendously difficult and cumbersome to solve analytically these nonlinear algebraic equations.

To overcome these aforementioned issues, a suitable analytical technique namely the global residue harmonic balance method (GRHBM) has been investigated. The GRHBM was the first introduced by Ju and Xue [19, 20]. Afterwards, Ju [21, 22] has also used the GRHBM to obtain approximate periodic solutions to a nonlinear oscillator with discontinuity and Helmholtz-Duffing oscillator. Recently, Mohammadian [23], Mohammadian et al. [24, 25] and Cveticanin and Ismail [26] have applied the GRHBM to determine approximate periodic solutions to nonlinear oscillatory systems arising in engineering problems. In this paper, we have employed the GRHBM to attain approximate angular frequencies and corresponding periodic solutions to the strongly nonlinear oscillator with cubic and harmonic restoring force. The higher-order approximations (mainly second-order approximation) have been obtained. The comparison of the approximated results with previously existing and corresponding exact solutions have been shown. It is found that the proposed technique gives excellent agreement. A simple solution procedure with high accuracy in the results obtained from the benchmark problem reveals the novelty, reliability and wider applicability to the proposed analytical technique.

2. Review of the Global Residue Harmonic Balance Method

Consider a general second-order nonlinear oscillator with odd-nonlinearity [19, 20], which can be described as

$$f(x, \dot{x}, \ddot{x}) = 0, \qquad x(0) = A, \qquad \dot{x}(0) = 0. \tag{2.1}$$

where $\dot{x} = dx/dt$, A is the amplitude of the oscillations and $f(x, \dot{x}, \ddot{x})$ is a nonlinear analytical function.

Suppose ω is the angular frequency of the system which is further to be determined. By defining a new independent variable replacing the time variable $\tau = \omega t$, Eq. (2.1) can be transformed into:

$$f(x, \omega x', \omega^2 x'') = 0, \qquad x(0) = A, \qquad x'(0) = 0. \tag{2.2}$$

where $x' = dx/d\tau$.

It is considered that the periodic solution does exist, it may be better to approximate the solution $x(\tau)$ by such a set of trigonometric functions:

$$\{\cos((2k-1)\tau), k = 1, 2, 3, ...\} \tag{2.3}$$

The initial approximate solution according to the Eq. (2.3), satisfying the initial conditions in Eq. (2.2), can be considered as:

$$x_{(0)}(\tau) = A\cos(\tau), \qquad \tau = \omega_{(0)} t, \tag{2.4}$$

and the parameter $\omega_{(0)}$ can be determined. Substituting Eq. (2.4) into Eq. (2.2), one could obtain the following residual:

$$R_0(\tau) = f(x_{(0)}, \omega_{(0)} x'_{(0)}, \omega^2_{(0)} x''_{(0)}). \tag{2.5}$$

If $R_0(\tau) = 0$, then $x_0(\tau)$ happens to be the exact solution. Generally, such a case will not arise for nonlinear problems.

Eq. (2.5) should not contain secular terms of $cos(\tau)$. Equating the coefficients of $cos(\tau)$ equal to zero, we can determine the unknown constant $\omega_{(0)}$ and taking it as the approximation ω_0. Hence, the zero-order approximation x_0 can be written as:

$$x_0(\tau) = A\cos(\tau), \qquad \tau = \omega_0 t. \tag{2.6}$$

This yields the initial residual

$$R_0(\tau) = f(x_0, \omega_0 x'_0, \omega_0^2 x''_0). \tag{2.7}$$

In the following, we consider the kth-order approximation ($k = 1, 2, 3, ...$) which can be shown as:

$$x(\tau) = x_{(k-1)}(\tau) + p x_k(\tau), \; \omega^2 = \omega^2_{(k-1)} + p\omega_k, \qquad k = 1, 2, 3, ..., \tag{2.8}$$

where p is the embedding parameter with values in the interval $[0, 1]$, and the kth-order approximate solutions of $x(\tau)$ and ω can be obtained by taking $p = 1$.

By using Eqs. (2.6) and (2.7), the first-order approximation and frequency can be written as:

$$x(\tau) = x_0(\tau) + p\, x_1(\tau), \qquad \omega^2 = \omega_0^2 + p\, \omega_1. \tag{2.9}$$

Inserting Eq. (2.6) into Eq. (2.2) and equating the coefficients of p, one could get

$$F_1(\tau, \omega_1, x_1(\tau)) \triangleq \left(\omega_1 \frac{\partial}{\partial(\omega^2)} + x''_1 \frac{\partial}{\partial x''} + x'_1 \frac{\partial}{\partial x'} + x_1 \frac{\partial}{\partial x} \right) f_0. \tag{2.10}$$

where $\partial f_0/\partial x$ denotes that $\partial f/\partial x$ is to be evaluated at the zero-order approximation after differentiation etc. It is noted that Eq. (2.10) is linear with respect to ω_1 and x_1. Noting that the solution has the form of Eq. (2.3), can be expressed as:

$$x_1(\tau) = C_1\left(\cos(\tau) - \cos(3\tau)\right) \tag{2.11}$$

From Eq. (2.11) into Eq. (2.10), we consider the following equation.

$$F_1\left(\tau, \omega_1, x_1(\tau)\right) + R_0(\tau) = 0. \tag{2.12}$$

Hence, all the residual errors of the zero-order approximation $R_0(\tau)$ are introduced into Eq. (2.12) which would improve the accuracy.

The left hand side of Eq. (2.12) should not contain the terms $cos(\tau)$ and $cos(3\tau)$ based on Galerkin technique. Letting their coefficients be zeros, one could obtain two linear equations containing two unknowns ω_1 and C_1. Then the two unknown constants can be solved easily. Thus, the first-order approximation can be obtained as:

$$x_{(1)}(\tau) = x_0(\tau) + x_1(\tau), \qquad \omega_{(1)}^2 = \omega_0^2 + \omega_1, \quad \tau = \omega_{(1)}t\ , \tag{2.13}$$

where $x_0(\tau)$ and $x_1(\tau)$ are denoted by Eqs. (2.6) and (2.11) respectively.

For high order approximation can be obtained by the iterate method in Eq. (2.8).

To determine the unknown parameters $a_{2i+1,k}(i = 2, ..., k)$ and ω_k, substituting Eq. (2.8) into Eq. (2.2) and collecting the coefficients of the p, one could yield:

$$F_k\left(\tau, \omega_k, x_k(\tau)\right) \triangleq \left(\omega_k \frac{\partial}{\partial(\omega^2)} + x_k'' \frac{\partial}{\partial x''} + x_k' \frac{\partial}{\partial x'} + x_k \frac{\partial}{\partial x}\right) f_{k-1}. \tag{2.14}$$

Eq. (2.14) is linear with respect to ω_k and x_k.

Substituting the $x_{(k-1)}(\tau)$ and $\omega_{(k-1)}(\tau)$ from Eq. (2.8) into Eq. (2.2), one yields the following residual:

$$R_{k-1}(\tau) = f\left(x_{(k-1)}, \omega_{(k-1)}x'_{(k-1)}, \omega^2_{(k-1)}x''_{(k-1)}\right). \tag{2.15}$$

Considering the following equation

$$F_k\left(\tau, \omega_k, x_k(\tau)\right) + R_{k-1}(\tau) = 0. \tag{2.16}$$

Eliminating the secular terms of $cos(\tau)$, $cos(3\tau)$, ..., and $cos((2k+1)\tau)$, there is the same number of linear equations for the same number of unknowns a_{3k}, a_{5k}, . . . ,$a_{2k+1,k}$ and ω_k.

Then, the kth-order approximate analytical solution and frequency can be obtained as:

$$x_k(\tau) = x_{k-1}(\tau) + x_k(\tau), \quad \omega^2_{(k)} = \omega^2_{(k-1)} + \omega_k. \tag{2.17}$$

3. Formulation of the Problem

A strongly nonlinear oscillator with a cubic and harmonic restoring force represents a system consisting of a mass resting on a spring with cubic and quintic nonlinearity as shown in Figure 1, where M is the mass, K is the linear spring stiffness coefficient, $b\ sin(x)$ is the driving force and $x(t)$ is the system response.

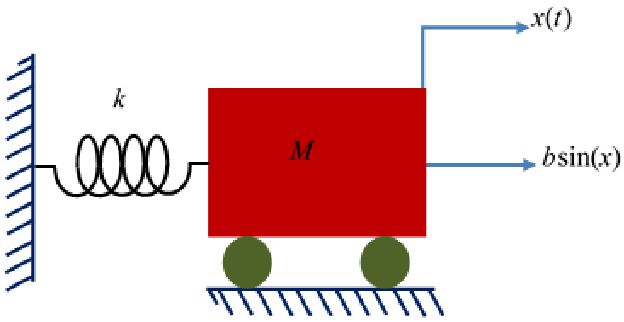

FIGURE 1. Geometric Structure of the Problem.

The strongly nonlinear oscillator with cubic and harmonic restoring force modelled by the following governing nonlinear differential equation [7–9, 15] is as follows

$$\ddot{x} + x + ax^3 + b\ \sin x, \qquad x(0) = A, \quad \dot{x}(0) = 0, \tag{3.1}$$

where a and b are constants.

Re-write Eq. (3.1) by introducing the independent variable $\tau = \omega t$ can be written as:

$$\omega^2 x'' + x + ax^3 + b\left(x + \frac{x^3}{4} + \frac{x^5}{24}\right) = 0. \tag{3.2}$$

3.1. Zeroth-Order Analytical Approximation

A practical example of the strongly nonlinear oscillator with cubic and harmonic restoring force is presented to illustrate the solution step, accuracy and effectiveness of the global residue harmonic balance method.

Firstly, the zero-order approximate solution can be assumed as:

$$x_0(\tau) = A\cos(\tau). \tag{3.3}$$

Substituting Eq. (3.3) into Eq. (3.2) and equating the coefficient of $\cos(\tau)$ equal to zero, the zero-order analytical approximate solution of Eq. (3.2), can be obtained as:

$$x = A\cos\left(\sqrt{\frac{8 + 6aA^2 + 8b - bA^2}{8}}t\right). \tag{3.4}$$

Now, substituting Eq. (3.3) into Eq. (3.2), we obtain the residual error for the zero-order approximation as:

$$R_0(\tau) = \frac{1}{24}\left(6aA^2 - bA^3\right)\cos(3\tau). \tag{3.5}$$

3.2. First-Order Analytical Approximation

In order to obtain the first-order analytical approximation, let us consider:

$$x(\tau) = A\cos(\tau) + p\ \left(C_1\left(\cos(\tau) - \cos(3\tau)\right)\right), \qquad \omega^2 = \omega_0^2 + p\,\omega_1, \tag{3.6}$$

where C_1 and ω_1 are two unknown constants will be determined later.

Now, Substituting Eq. (3.6) into Eq. (3.2) and considering the coefficients of p, we obtain a function as $F_1(\tau, \omega_1, x_1)$. Let us consider the following equation:

$$F_1(\tau, \omega_1, x_1) + R_0(\tau) = 0, \tag{3.7}$$

In Eq. (3.7), equating the coefficients of $\cos(\tau)$ and $\cos(3\tau)$ equal to zero, one could obtain two linear equations. Now solving these equations, two unknown constants C_1 and ω_1, can be obtained as:

$$C_1 = -\frac{A^3(6a-b)}{24(8+6aA^2+8b-bA^2)}, \quad \omega_1 = -\frac{A^4(6a-b)^2}{192(8+6aA^2+8b-bA^2)}. \tag{3.8}$$

Therefore, the first-order approximate solution of Eq. (3.2) can be expressed as:

$$\begin{aligned} x(\tau) &= \left(A - \frac{A^3(6a-b)}{24(8+6aA^2+8b-bA^2)}\right)\cos(\tau) \\ &\quad + \frac{A^3(6a-b)}{24(8+6aA^2+8b-bA^2)}\cos(3\tau), \end{aligned} \tag{3.9}$$

and the angular frequency is:

$$\omega = \sqrt{\frac{8+6aA^2+8b-bA^2}{8} - \frac{A^4(6a-b)^2}{192(8+6aA^2+8b-bA^2)}}. \tag{3.10}$$

3.3. Second-Order Analytical Approximation

Let us consider a second-order approximation solution in the following form:

$$\begin{aligned} x(\tau) &= (A + C_1)\cos(\tau) - C_1\cos(3\tau) \\ &\quad + p\ \left(C_2(\cos(\tau) - \cos(3\tau)) + C_3(\cos(\tau) - \cos(5\tau))\right), \end{aligned} \tag{3.11}$$

$$\omega^2 = \omega_0^2 + \omega_1 + p\omega_2,$$

where C_2, C_3, and ω_2, are three unknown constants which are further to be determined.

By utilizing the same mathematical manipulation (as discussed in the previous section), one could obtain the residual error $R_1(\tau)$, the approximate frequency ω_2 and the constants C_2, and C_3 are respectively:

$$R_1 = -\frac{3}{4}aA^2C_2 + \frac{1}{8}bA^2C_2 - \frac{3}{4}aAC_2^2 + \frac{1}{8}bAC_2^2 \tag{3.12}$$

$$
\begin{aligned}
\omega_2 \quad = \quad & -(A^4(6a-b)\Delta_1(A^2\Delta_1-24\Delta_2)(180a^2A^{12}\Delta_1^4-60aA^{12}b\Delta_1^4+5A^{12}b^2\Delta_1^4 \\
& -10368a^2A^{10}\Delta_1^3\Delta_2+3456aA^{10}b\Delta_1^3\Delta_2-288A^{10}b^2\Delta_1^3\Delta_2+1512aA^{10}\Delta_1^4\Delta_2 \\
& -252A^{10}b\Delta_1^4\Delta_2+69120aA^6\Delta_1^2\Delta_2^2+248832a^2A^8\Delta_1^2\Delta_2^2-11520A^6b\Delta_1^2\Delta_2^2 \\
& +69120aA^6b\Delta_1^2\Delta_2^2-248832a^2A^8\Delta_1^2\Delta_2^2-11520A^6b^2\Delta_1^2\Delta_2^2+6912A^8b^2\Delta_1^2\Delta_2^2 \\
& -50112aA^8\Delta_1^3\Delta_2^2+8352A^8b\Delta_1^3\Delta_2^2+1296A^8\Delta_1^4\Delta_2^2-1658880aA^4\Delta_1\Delta_2^3 \\
& +276480A^4b\Delta_1\Delta_2^3-1658880aA^4b\Delta_1\Delta_2^3+829440aA^6b\Delta_1\Delta_2^3 \\
& +276480A^4b^2\Delta_1\Delta_2^3-69120A^6b^2\Delta_1\Delta_2^3+276480A^4\Delta_1^2\Delta_2^3+544320aA^6\Delta_1^2\Delta_2^3 \\
& +276480A^4b\Delta_1^2\Delta_2^3-90720A^6b\Delta_1^2\Delta_2^3+5308416\Delta_2^2+15925248aA^2\Delta_2^4 \\
& +8957952a^2A^4\Delta_2^4+10616832b\Delta_2^4-2654208A^2b\Delta_2^4+15925248aA^2b\Delta_2^4 \\
& -2985984aA^4b\Delta_2^4+5308416b^2\Delta_2^4-2654208A^2b^2\Delta_2^4+248832A^4b^2\Delta_2^4 \\
& +1202688aA^4\Delta_1\Delta_2^4-200448A^4b\Delta_1\Delta_2^4-62208A^4\Delta_1^2\Delta_2^4-6635520\Delta_2^5 \\
& -13934592aA^2\Delta_2^5-6635520b\Delta_2^5+2322432A^2b\Delta_2^5+746496\Delta_2^6)) \\
& /(96\Delta_2(4140a^2A^{14}\Delta_1^5-1380aA^{14}b\Delta_1^5+115A^{14}b^2\Delta_1^5-176256a^2A^{12}\Delta_1^4\Delta_2 \\
& +58752aA^{12}b\Delta_1^4\Delta_2-4896A^{12}b^2\Delta_1^4\Delta_2+82800aA^{12}\Delta_1^5\Delta_2-13800A^{12}b\Delta_1^5\Delta_2 \\
& +1188864aA^8\Delta_1^3\Delta_2^2+2623104a^2A^{12}\Delta_1^3\Delta_2^2-198144A^8b\Delta_1^3\Delta_2^2 \\
& +1188864aA^8b\Delta_1^3\Delta_2^2-874368aA^{10}b\Delta_1^3\Delta_2^2-198144A^8b^2\Delta_1^3\Delta_2^2 \\
& +72864A^{10}b^2\Delta_1^3\Delta_2^2-1619136aA^{10}\Delta_1^4\Delta_2^2+269856A^{10}b\Delta_1^4\Delta_2^2 \\
& -21565440aA^6\Delta_1^2\Delta_2^3-10202112a^2A^8\Delta_1^2\Delta_2^3+3594240A^6b\Delta_1^2\Delta_2^3 \\
& -21565440aA^6b\Delta_1^2\Delta_2^3+3400704aA^8b\Delta_1^2\Delta_2^3+3594240A^6b^2\Delta_1^2\Delta_2^3 \\
& -283392A^8b^2\Delta_1^2\Delta_2^3+11059200A^6\Delta_1^3\Delta_2^3+6804864aA^8\Delta_1^3\Delta_2^3 \\
& +11059200A^6b\Delta_1^3\Delta_2^3-1134144A^8b\Delta_1^3\Delta_2^3+1555200A^8\Delta_1^4\Delta_2^3 \\
& +84934656A^2\Delta_1\Delta_2^4+95551488aA^4\Delta_1\Delta_2^4-53747712a^2A^66\Delta_1\Delta_2^4 \\
& +169869312A^2b\Delta_1\Delta_2^4-15925248A^4b\Delta_1\Delta_2^4+95551488aA^4b\Delta_1\Delta_2^4 \\
& +17915904aA^6b\Delta_1\Delta_2^4+84934656A^2b^2\Delta_1\Delta_2^4-15925248A^4b^2\Delta_1\Delta_2^4 \\
& -1492992A^6b^2\Delta_1\Delta_2^4+45121536A^4\Delta_1^2\Delta_2^4+81658368aA^6\Delta_1^2\Delta_2^4 \\
& +45121536A^4b\Delta_1^2\Delta_2^4-13609728A^6b\Delta_1^2\Delta_2^4+254803968\Delta_2^5+573308928aA^2\Delta_2^5 \\
& +286654464a^2A^4\Delta_2^5+509607936b\Delta_2^5-95551488A^2b\Delta_2^5+573308928aA^2b\Delta_2^5 \\
& -95551488aA^4b\Delta_2^5+254803968b^2\Delta_2^5-95551488A^2b^2\Delta_2^5+7962624A^4b^2\Delta_2^5 \\
& -265420800A^2\Delta_1\Delta_2^5-211009536aA^4\Delta_1\Delta_2^5-265420800A^2b\Delta_1\Delta_2^5 \\
& +35168256A^4b\Delta_1\Delta_2^5-74649600A^4\Delta_1^2\Delta_2^5-1082916864\Delta_2^6 \\
& -1027178496aA^2\Delta_2^6-1082916864b\Delta_2^6+171196416A^2b\Delta_2^6+895795200\Delta_2^7))
\end{aligned}
$$

$$
\begin{aligned}
C_2 \quad = \quad & (A^7(6a-b)\Delta_1^2(A^2\Delta_1-24\Delta_2)(102aA^6\Delta_1^2-17A^6b\Delta_1^2-3600aA^4\Delta_1\Delta_2 \\
& +600A^4b\Delta_1\Delta_2+108A^4\Delta_1^2\Delta_2+20736\Delta_2^2+43200aA^2\Delta_2^2+20736b\Delta_2^2 \\
& -7200A^2b\Delta_2^2-2592\Delta_2^3))/(4140a^2A^{14}\Delta_1^5-1380aA^{14}b\Delta_1^5+115A^{14}b^2\Delta_1^5 \\
& -176256a^2A^{12}\Delta_1^2\Delta_2+58752aA^{12}b\Delta_1^4\Delta_2-4896A^{12}b^2\Delta_1^4\Delta_2+82800aA^{12}\Delta_1^5\Delta_2 \\
& -13800A^{12}b\Delta_1^5\Delta_2+1188864aA^8\Delta_1^3\Delta_2^2+2623104a^2A^{10}\Delta_1^3\Delta_2^2-198144A^8b\Delta_1^3\Delta_2^2 \\
& +1188864aA^8b\Delta_1^3\Delta_2^2-874368aA^{10}b\Delta_1^3\Delta_2^2-198144A\hat{}8b^2\Delta_1^3\Delta_2^2
\end{aligned}
$$

$$+72864A^{10}b^2\Delta_1^3\Delta_2^2 - 1619136aA^{10}\Delta_1^4\Delta_2^2 + 269856A^{10}b\Delta_1^4\Delta_2^2$$
$$-21565440aA^6\Delta_1^2\Delta_2^3 - 10202112a^2A^8\Delta_1^2\Delta_2^3 + 3594240A^6b\Delta_1^2\Delta_2^3$$
$$-21565440aA^6b\Delta_1^2\Delta_2^3 + 3400704aA^8b\Delta_1^2\Delta_2^3 + 3594240A^6b^2\Delta_1^2\Delta_2^3$$
$$-283392A^8b^2\Delta_1^2\Delta_2^3 + 11059200A^6\Delta_1^2\Delta_2^3 + 6804864aA^8\Delta_1^2\Delta_2^3$$
$$+11059200A^6b\Delta_1^2\Delta_2^3 - 1134144A^8b\Delta_1^2\Delta_2^3 + 1555200A^8\Delta_1^4\Delta_2^3$$
$$+84934656A^2\Delta_1\Delta_2^4 + 95551488aA^4\Delta_1\Delta_2^4 - 53747712a^2A^6\Delta_1\Delta_2^4$$
$$+169869312A^2b\Delta_1\Delta_2^4 - 15925248A^4b\Delta_1\Delta_2^4 + 95551488aA^4b\Delta_1\Delta_2^4$$
$$+17915904aA^6b\Delta_1\Delta_2^4 + 84934656A^2b^2\Delta_1\Delta_2^4 - 15925248A^4b^2\Delta_1\Delta_2^4$$
$$-1492992A^6b^2\Delta_1\Delta_2^4 + 45121536A^4\Delta_1^2\Delta_2^4 + 81658368aA^6\Delta_1^2\Delta_2^4$$
$$+45121536A^4b\Delta_1^2\Delta_2^4 - 13609728A^6b\Delta_1^2\Delta_2^4 + 254803968\Delta_2^5$$
$$+573308928aA^2\Delta_2^5 + 286654464a^2A^4\Delta_2^5 + 509607936b\Delta_2^5$$
$$-95551488A^2b\Delta_2^5 + 573308928aA^2b\Delta_2^5 - 95551488aA^4b\Delta_2^5$$
$$+254803968b^2\Delta_2^5 - 95551488A^2b^2\Delta_2^5 + 7962624A^4b^2\Delta_2^5$$
$$-265420800A^2\Delta_1\Delta_2^5 - 211009536aA^4\Delta_1\Delta_2^5 - 265420800A^2b\Delta_1\Delta_2^5$$
$$+35168256A^4b\Delta_1\Delta_2^5 - 74649600A^4\Delta_1^2\Delta_2^5 - 1082916864\Delta_2^6 - 1027178496aA^2\Delta_2^6$$
$$-1082916864b\Delta_2^6 + 171196416A^2b\Delta_2^6 + 895795200\Delta_2^7)$$

$$C_3 = -(A^5(6a-b)\Delta_1(A^2\Delta_1 - 24\Delta_2)(138aA^8\Delta_1^3 - 23A^8b\Delta_1^3 - 2880aA^6\Delta_1^2\Delta_2$$
$$+480A^6b\Delta_1^2\Delta_2 + 18432A^2\Delta_1\Delta_2^2 + 19008aA^4\Delta_1\Delta_2^2 + 18432A^2b\Delta_1\Delta_2^2$$
$$-3168A^4b\Delta_1\Delta_2^2 + 2592A^4\Delta_1^2\Delta_2^2 + 55296\Delta_2^2 + 41472aA^2\Delta_2^2 + 55296b\Delta_2^3$$
$$-6912A^2b\Delta_2^3 - 62208\Delta_2^4))/(4140a^2A^{14}\Delta_1^5 - 1380aA^{14}b\Delta_1^5 + 115A^{14}b^2\Delta_1^5$$
$$-176256a^2A^{12}\Delta_1^4\Delta_2 + 58752aA^{12}b\Delta_1^4\Delta_2 - 4896A^{12}b^2\Delta_1^4\Delta_2 + 82800aA^{12}\Delta_1^5\Delta_2$$
$$-13800A^{12}b\Delta_1^5\Delta_2 + 1188864aA^8\Delta_1^3\Delta_2^2 + 2623104a^2A^{10}\Delta_1^3\Delta_2^2$$
$$-198144A^8b\Delta_1^3\Delta_2^2 + 1188864aA^8b\Delta_1^3\Delta_2^2 - 874368aA^{10}b\Delta_1^3\Delta_2^2$$
$$-198144A^8b^2\Delta_1^3\Delta_2^2 + 72864A^{10}b^2\Delta_1^3\Delta_2^2 - 1619136aA\hat{}10\Delta_1^4\Delta_2^2$$
$$+269856A\hat{}10b\Delta_1^4\Delta_2^2 - 21565440aA^6\Delta_1^2\Delta_2^3 - 10202112a^2A^8\Delta_1^2\Delta_2^3$$
$$+3594240A^6b\Delta_1^2\Delta_2^3 - 40aA^6b\Delta_1^2\Delta_2^3 + 3400704aA^8b\Delta_1^2\Delta_2^3$$
$$+3594240A^6b^2\Delta_1^2\Delta_2^3 - 283392A^8b^2\Delta_1^2\Delta_2^3 + 11059200A^6\Delta_1^3\Delta_2^3$$
$$+6804864aA^8\Delta_1^3\Delta_2^3 + 11059200A^6b\Delta_1^3\Delta_2^3 - 1134144A\hat{}8b\Delta_1^3\Delta_2^3$$
$$+1555200A^8\Delta_1^4\Delta_2^3 + 84934656A^2\Delta_1\Delta_2^4 + 95551488aA^4\Delta_1\Delta_2^4$$
$$-53747712a^2A^6\Delta_1\Delta_2^4 + 169869312A^2b\Delta_1\Delta_2^4 - 15925248A^4b\Delta_1\Delta_2^4$$
$$+95551488aA^4b\Delta_1\Delta_2^4 + 17915904aA^6b\Delta_1\Delta_2^4 + 84934656A^2b^2\Delta_1\Delta_2^4$$
$$-15925248A^4b^2\Delta_1\Delta_2^4 - 1492992A^6b^2\Delta_1\Delta_2^4 + 45121536A^4\Delta_1^2\Delta_2^4$$
$$+81658368aA^6\Delta_1^2\Delta_2^4 + 45121536A^4b\Delta_1^2\Delta_2^4 - 13609728A^6b\Delta_1^2\Delta_2^4$$
$$+254803968\Delta_2^5 + 573308928aA^2\Delta_2^5 + 286654464a^2A^4\Delta_2^5$$
$$+509607936b\Delta_2^5 - 95551488A^2b\Delta_2^5 + 573308928aA^2b\Delta_2^5$$
$$-95551488aA^4b\Delta_2^5 + 254803968b^2\Delta_2^5 - 95551488A^2b^2\Delta_2^5$$
$$+7962624A^4b^2\Delta_2^5 - 265420800A^2\Delta_1\Delta_2^5 - 211009536aA^2\Delta_1\Delta_2^5$$

$$-265420800A^2b\Delta_1\Delta_2^5 + 35168256A^4b\Delta_1\Delta_2^5 - 74649600A^4\Delta_1^5\Delta_2^2$$
$$-1082916864\Delta_2^6 - 1027178496aA^2\Delta_2^6 - 1082916864b\Delta_2^6$$
$$+171196416A^2b\Delta_2^6 + 895795200\Delta_2^7)$$

where

$$\Delta_1 = 6a - b, \ \ \Delta_2 = 8 + 6aA^2 + 8b - A^2b.$$

Therefore, the second-order approximate solution of Eq. (3.2), can be written as:

$$x(\tau) = (A + C_1 + C_2 + C_3)\cos(\tau) - (C_1 + C_2)\cos(3\tau) - C_3\cos(5\tau), \tag{3.13}$$

and the second-order approximate angular frequency is:

$$\omega = \sqrt{\omega_0^2 + \omega_1 + p\,\omega_2}. \tag{3.14}$$

4. Numerical Results and Discussion

To demonstrate and verify the accuracy of the proposed technique, we have compared the approximated periodic solutions with existing and exact solutions. Table 1 to Table 2; give the comparison between approximated solutions with existing and exact solutions for different values of initial amplitude A and the constants a and b take the value as unity. A comparison between the approximated solutions and the phase plan trajectory with Runge-Kutta fourth-order solutions (consider to be exact) are plotted in Figures (2-4) for different values of initial amplitude A and the constants a and b . It is noteworthy that the second-order approximate solutions are almost similar to the exact solutions. Excellent accuracy and the straightforward solution procedure that guaranteed us the proposed analytical technique is better applicable and highly efficient for solving strongly nonlinear problems arising in science and engineering.

Table 1. The comparison between first, second order approximate solutions with the existing and corresponding Runge-Kutta forth-order solutions for $a = 1, b = 1$ and $A_0 = \frac{\pi}{18}$

t	1^{st}(HBM)[15]	1^{st} [present]	2^{nd} (HBM)[15]	2^{nd} [present]	x_{RK4}
0	0.174532	0.174533	0.174532	0.174532	0.174532
0.5	0.132306	0.132218	0.132217	0.132217	0.132217
1	0.026059	0.0260194	0.026019	0.026019	0.026019
1.5	−0.092797	−0.0926925	−0.092692	−0.092692	−0.092692
2	−0.166751	−0.166728	−0.166728	−0.166728	−0.166728
2.5	−0.160017	−0.159977	−0.159977	−0.159977	−0.159977
3	−0.075853	−0.0757579	−0.075757	−0.075757	−0.075757
3.5	0.045014	0.0449468	0.044946	0.044946	0.044946
4	0.144100	0.144027	0.144027	0.144027	0.144027
4.5	0.173458	0.173456	0.173455	0.173455	0.173455
5	0.118883	0.118785	0.118785	0.118785	0.118785

TABLE 2. The comparison between first, second order approximate solutions with the existing and corresponding Runge-Kutta forth-order solutions for $a = 1, b = 1$ and $A_0 = \frac{\pi}{6}$.

t	1^{st}(HBM)[15]	1^{st} [present]	2^{nd} (HBM)[15]	2^{nd} [present]	x_{RK4}
0	0.523598	0.523599	0.523598	0.523599	0.523598
0.5	0.387797	0.385529	0.385528	0.38552	0.385519
1	0.050838	0.0502743	0.050273	0.050277	0.050276
1.5	−0.312492	−0.309726	−0.309725	−0.309721	−0.309721
2	−0.513726	−0.513435	−0.513434	−0.513432	−0.513432
2.5	−0.448480	−0.44704	−0.447039	−0.447031	−0.447029
3	−0.150597	−0.149066	−0.149063	−0.149071	−0.149069
3.5	0.225402	0.222673	0.222674	0.222673	0.222674
4	0.484482	0.483411	0.483411	0.483402	0.483403
4.5	0.492250	0.491655	0.491654	0.49165	0.491648
5	0.244678	0.242603	0.242600	0.242607	0.242603

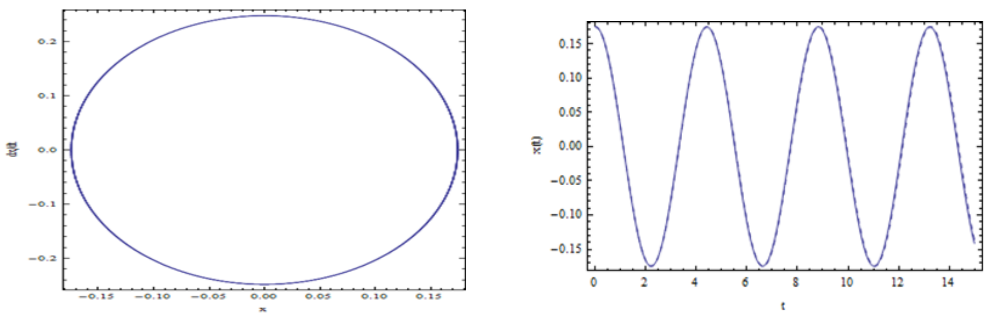

FIGURE 2. A comparison of the approximate solutions and the phase plan trajectory (dashed line) with corresponding exact solutions (solid line) for $A = \pi/18, a = 1$ and $b = 1$.

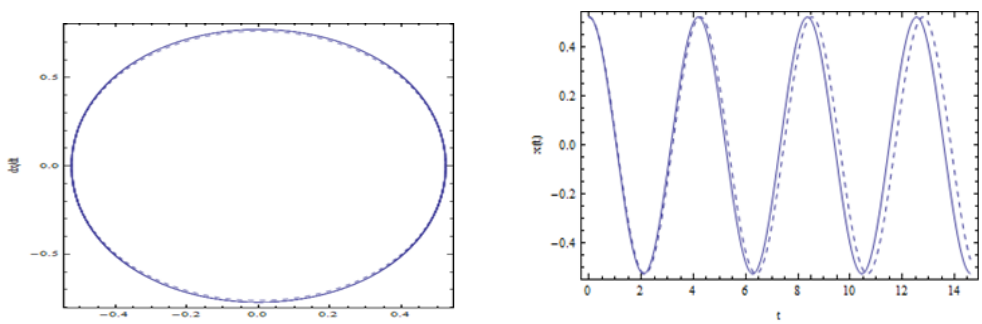

FIGURE 3. A comparison of the approximate solutions and the phase plan trajectory (dashed line) with corresponding exact solutions (solid line) for $A = \pi/6, a = 1$ and $b = 1$.

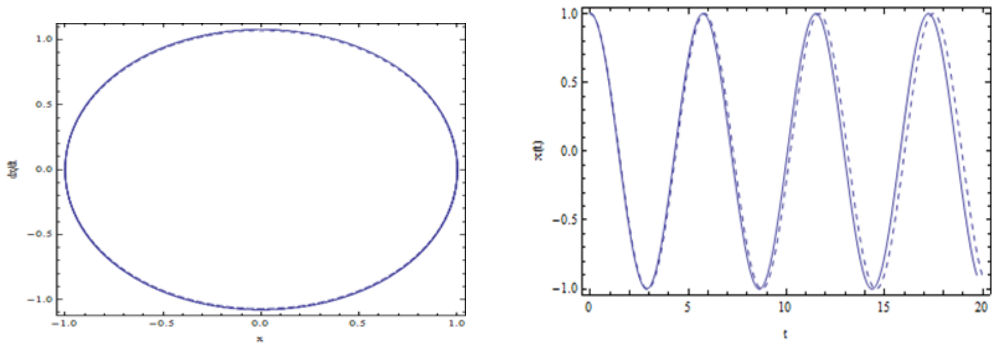

FIGURE 4. A comparison of the approximate solutions and the phase plan trajectory (dashed line) with corresponding exact solutions (solid line) for $A = 1, a = 0.1$ and $b = 0.1$.

5. Conclusions

A suitable analytical technique namely the global residue harmonic balance method has been used to determine approximate periodic solutions of the strongly nonlinear oscillator with a cubic and harmonic restoring force. In this problem, the second-order approximate solutions show an excellent agreement as compared with the corresponding exact solutions. The solution procedure of the proposed technique is straightforward and simple. High accuracy in results and simple solution procedure reveal the versatility of the proposed technique. Based on the above findings it can be concluded that the proposed technique can be considered as a very potential and an efficient alternative to the existing methods for approximating strongly nonlinear oscillatory problems.

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