Thai Journal of Mathematics
Volume 18 Number 4 (2020)
Pages 1917– 1931

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Quadratic Transformations of Multivariate Semi-Copulas

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Abstract In this study, we introduce a construction of semi-copulas via a composition of multivariate semi-copulas with a quadratic polynomial. Obviously, such compositions will not always result in a semi-copula. Our main focus is to provide a characterization of such polynomials in terms of their coefficients. We found that the set of those coefficients forms a convex set with a linear boundary. We also found that several such transformations that are not a convex combination of semi-copulas.

MSC: 62H05
Keywords: semi-copula; quadratic transformation; quadratic construction; quadratic polynomial

Submission date: 25.09.2020 / Acceptance date: 13.11.2020

1. Introduction

In recent years, semi-copulas have attracted growing interest. Semi-copulas are used in several areas such as lifetime dependence [1, 2] and analysis related to aging function [3, 4]. Especially in statistics, survival analysis gives certain important methods which apply for analyzing the expected duration of times until one or more events happen. Furthermore, concepts of semi-copulas also appear in several separate works such as [5–8] and analytical aspects of semi-copulas are also examined in [9–11].

In literature (for instances, [9, 10, 12–16]), new constructions of semi-copulas are introduced to obtain varieties of semi-copulas. Several works focuses on transformations T_P which have the form

$$T_P(S_1, ..., S_k)(x, y) = P(x, y, S_1(x, y), ..., S_k(x, y))$$

where P is a polynomial. In other words, T_P transforms semi-copulas $S_1, \ldots, S_k$ into a new semi-copula $T_P(S_1, ..., S_k)$. Note that many constructions of other related objects have also been studied.

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Published by The Mathematical Association of Thailand.

In 2015, Kolesárová et al. [12] characterize linear functions $P(x,y,z)=ax+by+cz+d$ such that $T_P(A)$ remains an aggregation function for any bivariate aggregation function A. This occurs under the condition that P is a weighted arithmetic means, i.e., $P(x,y,z)=ax+by+cz$ where $a,b,c\geq 0$ and $a+b+c=1$. This fact can be extended to obtain that $T_P(A_1,...,A_k)$ is an aggregation function for any aggregation functions $A_1,\ldots,A_k$ if and only if the polynomial function P is an aggregation function. Also, the characterization of quadratic aggregation functions has been done in [18]. Moreover, Kolesárová et al. [12] also characterize quadratic polynomial functions P such that T_p is a transformation of bivariate semi-copulas and quasi-copulas. In particular, they show that the following statements are equivalent for any quadratic polynomial $P(x,y,z)$.

(1) the polynomial P can be written as
$$P(x,y,z)=cz^2+dxy-cxz-cyz+(1+c-d)z$$
where $0\leq d\leq 1, d-c\geq 0, 1+c-d\geq 0$ and $1-c-d\geq 0$;

(2) $T_P(Q)$ is a quasi-copula for any bivariate quasi-copula Q.

Similar characterization of copula transformations is also provided in [13]. For $k>1$, Wisadwongsa and Tasena [19] characterize quadratic polynomial functions P such that T_P is a transformation of two bivariate copulas. Later, this result has been extended to the case of any polynomial functions in [16]. However, all these results only provide a characterization in the case of bivariate (semi-, quasi-) copulas.

In this work, we are interested in characterizing quadratic transformation of multivariate semi-copulas, i.e., characterizing quadratic polynomial P such that

$$T_P(S_1,...,S_k)(x_1,...,x_n)=P(x_1,...,x_n,S_1(x_1,...,x_n),...,S_k(x_1,...,x_n))$$

remains a semi-copula for any semi-copulas $S_1,\ldots,S_k$. When k=1, this is also done in [17]. We are able to show that the set of such quadratic functions is convex with linear boundary. We also characterize its extreme points in the case $k\leq 2$. For the case $k=1$, we show that the set has exactly 2 extreme points for all $n>2$. This result is quite different from that of Kolesárová et al. [12] where they show that there are 4 extreme points when $n=2$. For the case $k=2$, we show that there are $5\cdot 2^n-2$ extreme points. For $k>2$, characterization of extreme points seem to be very complicated. In fact, we would conjecture that the number of extreme points is $O(k^n)$.

In the next section, we present basic notations and terms essential to this work. We also characterize quadratic transformations of semi-copulas. The characterization of the transformation T_P is provided in term of coefficients of quadratic polynomial P (see Theorem 2.8).

2. Quadratic Transformations of Semi-Copulas

Definition 2.1. [20] A function $S:[0,1]^n\to[0,1]$ is said to be an *n-dimensional semi-copula* (or a *semi-copula*) if it satisfies the followings:

(1) $S(x_1,...,x_n)=x_i$ if $x_j=1$ for all $j\neq i$;

(2) S is nondecreasing in each place; i.e., for each $(x_1,\ldots,x_n)\in[0,1]^n$ and any $y_i\in[0,1]$ with $x_i\leq y_i$,
$$S(x_1,...,y_i,...,x_n)-S(x_1,...,x_i,...,x_n)\geq 0.$$

An n-dimensional semi-copula with $n>2$ will also be called a multivariate semi-copulas.

We denote the collection of all n-dimensional semi-copulas by $\mathscr{S}_n$. Note that the set $\mathscr{S}_n$ actually contains maximum and minimum elements.

Example 2.2. The function $L : [0,1]^n \to [0,1]$ defined by

$$L(x_1, ..., x_n) = \begin{cases} x_i & \text{if } x_j = 1 \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

is a semi-copula. Another semi-copula is the function $M : [0,1]^n \to [0,1]$ defined via

$$M(x_1, ..., x_n) = \min\{x_1, ..., x_n\}.$$

Moreover, $L(x_1, ..., x_n) \leq S(x_1, ..., x_n) \leq M(x_1, ..., x_n)$ for all $S \in \mathscr{S}_n$.

Remark 2.3. Since every semi-copula S is nondecreasing, we get

$$0 \leq S(x_1, ..., 0, ..., x_n) \leq S(1, ..., 1, 0, 1, ..., 1) = 0.$$

In other words, $S(x_1, ..., x_n) = 0$ when $x_i = 0$ for some i.

Given a natural number k, we define a function $T_P : \mathscr{S}_n^k \to \mathscr{F}$ by

$$T_P(S_1, ..., S_k)(x_1, ..., x_n) = P\left(x_1, ..., x_n, S_1(x_1, ..., x_n), ..., S_k(x_1, ..., x_n)\right)$$

for each $(x_1, ..., x_n) \in [0,1]^n$ and $(S_1, ..., S_k) \in \mathscr{S}_n^k$ where P is a quadratic polynomial from $\mathbb{R}^{n+k}$ to $\mathbb{R}$ expressed as

$$P(x_1, ..., x_n, z_1, ..., z_k) = \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} z_p z_q + \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} x_i z_q + \sum_{p=1}^{k} c_q z_q + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} x_i x_j + \sum_{i=1}^{n} e_i x_i + f \tag{2.1}$$

where $a_{pq} = a_{qp}$ for all p, q.

Notice that $T_p(S_1, \ldots, S_k)$ is differentiable at $(x_1, \ldots, x_n) \in (0,1)^n$ whenever $S_1, \ldots, S_k$ are all differentiable at $(x_1, \ldots, x_n)$. This simply follows from the chain rule.

Definition 2.4. Let P be a quadratic polynomial. The function T_P is called a *transformation of k multivariate semi-copulas* if $T_P(S_1, ..., S_k)$ is a semi-copula whenever $S_1, ..., S_k$ are semi-copulas.

Examples of transformations of k multivariate semi-copulas are convex combinations $K_{\vec{c}}$ of k multivariate semi-copulas. This corresponds with the case of linear function P, that is, the case where the coefficients $a_{pq} = b_{iq} = d_{ij} = e_i = f = 0$. It is interesting to know whether an actual quadratic transformation exist and how we can characterize them. To do this, we will first prove the following lemmas.

Lemma 2.5. *Assume T_P is a transformation of k multivariate semi-copulas. We obtain the followings:*

(1) $f = e_i = d_{ij} = 0$ *for all* i, j;
(2) $\sum_{q=1}^{k} b_{iq} = -\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}$ *for all* i;
(3) $(n-1)\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} - \sum_{p=1}^{k} c_p + 1 = 0$;
(4) $b_{iq} \geq 0$ *for all* i, q;

(5) $c_q \geq 0$ *for all* q.

Proof. Assume that the function $T_P(S_1,...,S_k):[0,1]^n \to [0,1]$ is in the form

$$\begin{aligned} T_P(S_1,...,S_k)(x_1,...,x_n) &= \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}S_p(x_1,...,x_n)S_q(x_1,...,x_n) \\ &+ \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq}x_iS_q(x_1,...,x_n) + \sum_{p=1}^{k} c_qS_q(x_1,...,x_n) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_ix_j + \sum_{i=1}^{n} e_ix_i + f \end{aligned}$$

is a semi-copula for all semi-copulas $S_1,\ldots,S_k$. Following the fact that the value of a semi-copula is zero whenever one of its arguments is zero,

- $0 = T_P(S_1,\ldots,S_k)(0,\ldots,0) = f$;
- $0 = T_P(S_1,\ldots,S_k)(0,\ldots,0,x_i,0,\ldots,0) = d_{ii}x_ix_i + e_ix_i$;
- $0 = T_P(S_1,\ldots,S_k)(0,\ldots,0,x_i,0,\ldots,0,x_j,0,\ldots,0)$

$= d_{ii}x_i^2 + d_{jj}x_j^2 + d_{ij}x_ix_j + e_ix_i + e_jx_j$.

for all x_i and x_j between zero and one. Consequently, $d_{ij} = 0 = e_i = f$ for all i,j and the proof of 1. is done. Now we know that

$$\begin{aligned} T_P(S_1,...,S_k)(x_1,...,x_n) &= \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}S_p(x_1,...,x_n)S_q(x_1,...,x_n) \\ &+ \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq}x_iS_q(x_1,...,x_n) + \sum_{q=1}^{k} c_qS_q(x_1,...,x_n). \end{aligned}$$

Next, we will show 2. and 3. Since $T_P(S_1,...,S_k)(x_1,1,...,1) = x_1$ for all $x_1 \in [0,1]$, we have

$$x_1 = x_1^2\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} + x_1^2\sum_{q=1}^{k} b_{1q} + x_1\sum_{i=2}^{n}\sum_{q=1}^{k} b_{iq} + x_1\sum_{q=1}^{k} c_q,$$

$$1 = x_1\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} + x_1\sum_{q=1}^{k} b_{1q} + \sum_{i=2}^{n}\sum_{q=1}^{k} b_{iq} + \sum_{q=1}^{k} c_q, \tag{2.2}$$

and

$$1 = \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} + \sum_{q=1}^{k} b_{1q} + \sum_{i=2}^{n}\sum_{q=1}^{k} b_{iq} + \sum_{q=1}^{k} c_q. \tag{2.3}$$

Subtracting equation (2.2) from (2.3), we obtain that

$$0 = (1-x_1)\left[\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} + \sum_{q=1}^{k} b_{1q}\right]$$

for all $x_1 \in [0,1]$. Thus, $\sum_{q=1}^{k} b_{1q} = -\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}$. Similar to the previous argument, we obtain

$$\sum_{q=1}^{k} b_{iq} = -\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} \text{ for all } i.$$

This proves 2. Substitute $\sum_{q=1}^{k} b_{iq} = -\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}$ into the equations (2.3), we get

$$(n-1)\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} - \sum_{p=1}^{k} c_p + 1 = 0.$$

Thus, we obtain 3.

Note that coefficients of $z_p z_q$ and $z_q z_p$ on the quadratic polynomial P are the same by assumptions, that is, $a_{pq} = a_{qp}$ for all p, q. To show 4. and 5., let $S_q = M$ and $S_p = L$ for $p \neq q$. We have

$$\begin{aligned}
\frac{\partial T_P(S_1,...,S_k)}{\partial x_\alpha}(x_1,...,x_n) &= 2\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(x_1,...,x_n)\frac{\partial S_q}{\partial x_\alpha} + \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq}x_i\frac{\partial S_q}{x_\alpha} \\
&\quad + \sum_{q=1}^{k} b_{\alpha q} S_q(x_1,...,x_n) + \sum_{q=1}^{k} c_q \frac{\partial S_q}{\partial x_\alpha} \\
&= \sum_{q=1}^{k}\frac{\partial S_q}{\partial x_\alpha}\left(\sum_{p=1}^{k} 2a_{pq}S_p(x_1,...,x_n) + \sum_{i=1}^{n} b_{iq}x_i + c_q\right) \\
&\quad + \sum_{q=1}^{k} b_{\alpha q} S_q(x_1,...,x_n) \qquad (2.4)
\end{aligned}$$

whenever $0 < x_i < 1$ and $x_i \neq x_\alpha$ for all $i \neq \alpha$.

Since $T_P(S_1,...,S_k)$ is nondecreasing, we obtain $\frac{\partial T_P(S_1,...,S_k)}{\partial x_\alpha} \geq 0$ whenever the partial derivative exists. Combining this with the equation (2.4), we have

$$\begin{aligned}
0 &\leq \frac{\partial T_P(S_1,...,S_k)}{\partial x_\alpha}(x_1,...,x_n) \\
&= \frac{\partial M}{\partial x_\alpha}(x_1,...,x_n)\left(2a_{qq}M(x_1,...,x_n) + \sum_{i=1}^{n} b_{iq}x_i + c_q\right) + b_{\alpha q}M(x_1,...,x_n)
\end{aligned}$$

whenever $\frac{\partial M}{\partial x_\alpha}(x_1,\ldots,x_n)$ exists where $0 < x_i < 1$ and $x_i \neq x_\alpha$ for all $i \neq \alpha$. When $x_\alpha > \min\{x_1, x_2, \ldots, x_n\}$, for example, when $x_\alpha = 0.5$ and $x_i = 0.25$ for all $i \neq \alpha$, we have $\frac{\partial M}{\partial x_\alpha}(x_1,...,x_n) = 0$, and hence, $b_{\alpha q} \geq 0$. When $x_\alpha < \min\{x_1,\ldots,x_{\alpha-1},x_{\alpha+1},\ldots,x_n\}$, on the other hands, we have $\frac{\partial M}{\partial x_\alpha}(x_1,...,x_n) = 1$, and

$$0 \leq \frac{\partial T_P(S_1,...,S_k)}{\partial x_\alpha}(x_1,...,x_n) = \left(2a_{qq}x_\alpha + \sum_{i=1}^{n} b_{iq}x_i + c_q\right) + b_{\alpha q}x_\alpha$$

Letting $x_\alpha \to 0$ follows by $x_i \to 0$ for all $i \neq \alpha$ yields $c_q \geq 0$. ∎

Lemma 2.6. *Let a_{pq}, b_{iq} and c_q be arbitrary numbers. Assume that $b_{iq} \geq 0$ for all i, q. Denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The following statements are equivalent;*

(1) For any semi-copula $S_1, \ldots, S_k$ which are differentiable almost everywhere on $(0,1)^n$ and any $\vec{x} \in (0,1)^n$ such that $S_1, \ldots, S_k$ are differentiable at $\vec{x}$, we have

$$2\sum_{p=1}^{k} a_{pq} S_p(\vec{x}) + \sum_{i=1}^{n} b_{iq} x_i + c_q \geq 0$$

for each $q \in \{1, \ldots, k\}$;

(2) $c_q \geq 0$ and $2\sum_{p=1}^{k}(a_{pq} \wedge 0) + \sum_{i=1}^{n} b_{iq} + c_q \geq 0$ for each $q \in \{1, \ldots, k\}$;

(3) For any semi-copula $S_1, \ldots, S_k$ and any $\vec{x} \in [0,1]^n$, we have

$$2\sum_{p=1}^{k} a_{pq} S_p(\vec{x}) + \sum_{i=1}^{n} b_{iq} x_i + c_q \geq 0$$

for each $q \in \{1, \ldots, k\}$.

Proof. Assume that 1. holds. For each index p, we set $S_p = \begin{cases} L & \text{if } a_{pq} > 0 \\ M & \text{if } a_{pq} \leq 0 \end{cases}$. Then $S_1, \ldots, S_k$ are differentiable on the set of $\vec{x} = (x_1, \ldots, x_n)$ such that $0 < x_i < 1$ and $x_i \neq x_j$ for all $i \neq j$. Therefore, they are differentiable almost everywhere. Moreover,

$$2\sum_{p=1}^{k} a_{pq} S_p(\vec{x}) + \sum_{i=1}^{n} b_{iq} x_i + c_q = 2\sum_{p=1}^{k} (a_{pq} \wedge 0) M(\vec{x}) + \sum_{i=1}^{n} b_{iq} x_i + c_q$$

for all such $\vec{x}$. Set $x \in (0,1)$ and

$$\vec{x}_m = \left(x + \frac{1}{nm}(1-x), x + \frac{2}{nm}(1-x), \ldots, x + \frac{1}{m}(1-x)\right).$$

Then we must have
$2\sum_{p=1}^{k}(a_{pq} \wedge 0) M(\vec{x}_m) + \sum_{i=1}^{n} b_{iq}\left(x + \frac{i}{nm}(1-x)\right) + c_q > 0.$

Let $n \to \infty$ yields $2\sum_{p=1}^{k}(a_{pq} \wedge 0) M(x, x, \ldots, x) + x\sum_{i=1}^{n} b_{iq} + c_q > 0.$
Let $x \to 0$ and $x \to 1$ yields 2. as desired.

Next, assume that 2. holds. Because of the maximum property of the semi-copula M, and the assumption $b_{iq} \geq 0$ for all i, we have

$$\begin{aligned}
&2\sum_{p=1}^{k} a_{pq} S_p(\vec{x}) + \sum_{i=1}^{n} b_{iq} x_i + c_q \\
&\quad \geq 2\sum_{p=1}^{k} (a_{pq} \wedge 0) S_p(\vec{x}) + \sum_{i=1}^{n} b_{iq} x_i + c_q \\
&\quad \geq 2\sum_{p=1}^{k} (a_{pq} \wedge 0) M(\vec{x}) + \sum_{i=1}^{n} b_{iq} M(\vec{x}) + c_q \\
&\quad = M(\vec{x})\left(2\sum_{p=1}^{k} (a_{pq} \wedge 0) + \sum_{i=1}^{n} b_{iq} + c_q\right) + (1 - M(\vec{x}))\, c_q \\
&\quad \geq 0
\end{aligned}$$

where $S_1, \ldots, S_k$ are semi-copulas and $\vec{x} \in [0,1]^n$.

Finally, it is obvious that 3. implies 1. ■

Theorem 2.7. *Let P be a quadratic polynomial. If T_P is a transformation of k multivariate semi-copulas, then the following statements hold:*

(1) $f = e_i = d_{ij} = 0$ *for all* i, j*;*
(2) $\sum_{q=1}^{k} b_{iq} = -\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}$ *for all* i*;*
(3) $(n-1)\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} - \sum_{p=1}^{k} c_p + 1 = 0$*;*
(4) $b_{iq} \geq 0$ *for all* i *and* q*;*
(5) $c_q \geq 0$ *for all* q*;*
(6) $2\sum_{p=1}^{k}(a_{pq} \wedge 0) + \sum_{i=1}^{n} b_{iq} + c_q \geq 0$ *for all* q*.*

Proof. Assume that T_P is a transformation of k multivariate semi-copulas. By Lemma 2.5, we will show only 6. Let $S_1, S_2, \ldots, S_k$ be semi-copulas which are differentiable almost everywhere on $(0,1)^n$ and $\vec{a} = (a_1, a_2, \ldots, a_n) \in (0,1)^n$ such that $S_1, \ldots, S_k$ are differentiable at $\vec{a}$. Thus, $T_P(S_1, \ldots, S_k)$ is a semi-copula by the assumption. Since P is a quadratic function, the transformation $T_p(S_1, \ldots, S_k)$ must be differentiable at $\vec{a}$. Moreover, $\frac{\partial S_1}{\partial x_\alpha}(\vec{a}), \frac{\partial S_2}{\partial x_\alpha}(\vec{a}), \ldots, \frac{\partial S_k}{\partial x_\alpha}(\vec{a})$ and $\frac{\partial T_p(S_1,\ldots,S_k)}{\partial x_\alpha}$ exist for all $\alpha = 1, 2, \ldots n$.

Let $q \in \{1, 2, \ldots, k\}$ and $0 < \varepsilon < \frac{1}{2}\left(\min_i a_i \;\wedge\; \left[1 - \max_i a_i\right]\right)$. For each $p \neq q$, set

$$S_{p,\varepsilon}(\vec{x}) = \begin{cases} M(\vec{x}) & \text{if } \max_i x_i = 1 \\ S_p\left(\vec{a} - \varepsilon\vec{1}\right) & \text{if } \vec{a} - \varepsilon\vec{1} \leq \vec{x} < \vec{1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $S_{p,\varepsilon}$ are semi-copulas which satisfy $\frac{\partial S_{p,\varepsilon}}{\partial x_\alpha}(\vec{a}) = 0$ and

$$\begin{aligned} 0 \leq & \frac{\partial T_P\left(S_{1,\varepsilon}, \ldots, S_{q-1,\varepsilon},\ S_q,\ S_{q+1,\varepsilon}, \ldots, S_{k,\varepsilon}\right)}{\partial x_\alpha}(\vec{a}) \\ = & \frac{\partial S_q}{\partial x_\alpha}(\vec{a}) \left(\sum_{p \neq q} 2a_{pq} S_p\left(\vec{a} - \varepsilon\vec{1}\right) + 2a_{qq} S_q(\vec{a}) + \sum_{i=1}^{n} b_{iq} a_i + c_q\right) \\ & + b_{\alpha q} S_q(\vec{a}) + \sum_{p \neq q} b_{\alpha p} S_p(\vec{a} - \varepsilon\vec{1}). \end{aligned}$$

Letting $\varepsilon \to 0$ yields

$$\frac{\partial S_q}{\partial x_\alpha}(\vec{a}) \left(\sum_{p=1}^{k} 2a_{pq} S_p(\vec{a}) + \sum_{i=1}^{n} b_{iq} a_i + c_q\right) + b_{\alpha q} S_q(\vec{a}) + \sum_{p \neq q} b_{\alpha p} S_p(\vec{a}) \geq 0. \quad (2.5)$$

Now, set $S_{q,m,\alpha}(\vec{x}) = \left(\frac{m-1}{m} S_q + \frac{1}{m}\Pi\right)(\vec{x}) \cdot \phi_{m,\alpha}(\vec{x})$
for all $\vec{x} \in [0,1]^n$ where

$$\phi_{m,\alpha}(\vec{x}) = \begin{cases} 1 & \text{if } \max_i x_i = 1 \\ \left[\left(m\left(x_\alpha - \left(1-\frac{1}{m}\right)a_\alpha\right) + \left(1-\frac{1}{m}\right)(1-a_\alpha)\right) \vee 0\right] \wedge 1 & \text{otherwise.} \end{cases}$$

Since $\phi_{m,\alpha}$ is nondecreasing and $S_{q,m,\alpha}$ is also a semi-copula. Moreover, we get $S_{q,m,\alpha}(\vec{a}) \to S_q(\vec{a})$ while

$$\frac{\partial S_{q,m,\alpha}}{\partial x_\alpha}(\vec{a}) = \frac{m}{2}\left(\frac{m-1}{m}S_q(\vec{a}) + \frac{1}{m}\Pi(\vec{a})\right) + \frac{1}{2}\frac{\partial\left(\frac{m-1}{m}S_q + \frac{1}{m}\Pi\right)}{\partial x_\alpha}(\vec{a}) \to \infty$$

when $m \to \infty$ where $\prod(\vec{a}) = a_1 a_2 \cdots a_n$. Replacing S_q by $S_{q,m,\alpha}$ into the inequality (2.5) yields

$$\frac{\partial S_{q,m,\alpha}}{\partial x_\alpha}(\vec{a})\left(\sum_{p\neq q} 2a_{pq}S_p(\vec{a}) + 2a_{qq}S_{q,m,\alpha}(\vec{a}) + \sum_{i=1}^{n} b_{iq}a_i + c_q\right) \geq -b_{\alpha q}S_{q,m,\alpha}(\vec{a}) - \sum_{p\neq q} b_{\alpha p}S_p(\vec{a})$$

Therefore,

$$\begin{aligned}\sum_{p=1}^{k} 2a_{pq}S_p(\vec{a}) + \sum_{i=1}^{n} b_{iq}a_i + c_q &= \lim_{m\to\infty} \sum_{p\neq q} 2a_{pq}S_p(\vec{a}) + 2a_{qq}S_{m,\alpha}(\vec{a}) \\ &\quad + \sum_{i=1}^{n} b_{iq}a_i + c_q \\ &\geq \lim_{m\to\infty} \frac{-b_{\alpha q}S_{q,m,\alpha}(\vec{a}) - \sum_{p\neq q} b_{\alpha p}S_p(\vec{a})}{\frac{\partial S_{q,m,\alpha}}{\partial x_\alpha}(\vec{a})} \\ &= 0.\end{aligned}$$

By Lemma 2.6, we get 6. ■

Next, we will prove the converse of Theorem 2.7. In other words, will show that conditions 1.-6. are sufficient to guarantee that T_P is a transformation of k multivariate semi-copulas.

Theorem 2.8. *Let P be a quadratic polynomial. Then T_P is a transformation of k multivariate semi-copulas if the following statements hold:*

(1) $f = e_i = d_{ij} = 0$ *for all* i, j;
(2) $\sum_{q=1}^{k} b_{iq} = -\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq}$ *for all* i;
(3) $(n-1)\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} - \sum_{p=1}^{k} c_p + 1 = 0$;
(4) $b_{iq} \geq 0$ *for all* i *and* q;
(5) $c_q \geq 0$ *for all* q;
(6) $2\sum_{p=1}^{k}(a_{pq} \wedge 0) + \sum_{i=1}^{n} b_{iq} + c_q \geq 0$ *for all* q.

Proof. Assume that 1.-6. hold. To show that T_P is a transformation of k multivariate semi-copulas, let $S_1, \ldots, S_k$ be semi-copulas and $x_\alpha \in [0,1]$. We consider

$$T_P(S_1,...,S_k)(1,...,1,x_\alpha,1,...,1) = x_\alpha^2 \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} + x_\alpha^2 \sum_{q=1}^{k} a_{\alpha q} + x_\alpha \sum_{i=1, i\neq\alpha}^{n} \sum_{q=1}^{k} b_{iq} + x_\alpha \sum_{q=1}^{k} c_q.$$

Since $-\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} = \sum_{q=1}^{k} b_{\beta q}$ for all $\beta = 1,...,n$, we get

$$\begin{aligned} T_P(S_1,...,S_k)(1,...,1,x_\alpha,1,...,1) &= x_\alpha \left(\sum_{i=1, i\neq\alpha}^{n} \sum_{q=1}^{k} b_{iq} + \sum_{q=1}^{k} c_q \right) \\ &= x_\alpha \left(\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} + \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} + \sum_{q=1}^{k} c_q \right) \\ &= x_\alpha. \end{aligned}$$

Next, we will use 6. and Lemma 2.6 (3) to show that $T_P(S_1,..,S_k)$ is nondecreasing in each place, i.e.,

$$\begin{aligned} \Delta_\alpha T_P(S_1,...,S_k) =& T_P(S_1,...,S_k)(x_1,...,x_\alpha+\varepsilon,...,x_n) \\ &- T_P(S_1,...,S_k)(x_1,...,x_\alpha,...,x_n) \\ \geq& 0 \end{aligned}$$

whenever $(x_1,...,x_\alpha,...,x_n) \in [0,1]^n$ and $0 \leq \varepsilon \leq 1 - x_\alpha$.
Denote $\vec{x}_\varepsilon := (x_1,...,x_\alpha+\varepsilon,...,x_n)$ and $\vec{x} := (x_1,...,x_\alpha,...,x_n)$.
Since $a_{pq} = a_{qp}$ for all p and q, we obtain

$$\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}_\varepsilon) S_q(\vec{x}) = \sum_{q=1}^{k}\sum_{p=1}^{k} a_{qp} S_q(\vec{x}_\varepsilon) S_p(\vec{x}) = \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_q(\vec{x}_\varepsilon) S_p(\vec{x}).$$

Consequently,

$$\begin{aligned} &\Delta_\alpha T_P(S_1,...,S_k) = T_P(S_1,...,S_k)(\vec{x}_\varepsilon) - T_P(S_1,...,S_k)(\vec{x}) \\ &= \left[\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}_\varepsilon) S_q(\vec{x}_\varepsilon) + \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} x_i S_q(\vec{x}_\varepsilon) + \varepsilon \sum_{q=1}^{k} b_{\alpha q} S_q(\vec{x}_\varepsilon) \right. \\ &\quad \left. + \sum_{q=1}^{k} c_q S_q(\vec{x}_\varepsilon) \right] - \left[\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}) S_q(\vec{x}) + \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} x_i S_q(\vec{x}) \right. \\ &\quad \left. + \sum_{q=1}^{k} c_q S_q(\vec{x}) \right] \end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}_\varepsilon) S_q(\vec{x}_\varepsilon) - \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}_\varepsilon) S_q(\vec{x}) \right. \\
&\quad \left. + \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}_\varepsilon) S_q(\vec{x}) - \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}) S_q(\vec{x}) \right] \\
&\quad + \left[\sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} x_i \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right]\right] \\
&\quad + \left[\sum_{q=1}^{k} c_q \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right]\right] + \varepsilon \sum_{q=1}^{k} b_{\alpha q} S_q(\vec{x}_\varepsilon) \\
&= \left[\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}_\varepsilon) \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right] + \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_q(\vec{x}_\varepsilon) S_p(\vec{x}) \right. \\
&\quad \left. - \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} S_p(\vec{x}) S_q(\vec{x}) \right] + \left[\sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} x_i \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right]\right] \\
&\quad + \left[\sum_{q=1}^{k} c_q \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right]\right] + \varepsilon \sum_{q=1}^{k} b_{\alpha q} S_q(\vec{x}_\varepsilon) \\
&= \sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} \left[S_p(\vec{x}_\varepsilon) + S_p(\vec{x})\right] \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right] \\
&\quad + \sum_{i=1}^{n}\sum_{q=1}^{k} b_{iq} x_i [S_q(\vec{x}_\varepsilon) - S_q(\vec{x})] \\
&\quad + \sum_{q=1}^{k} c_q [S_q(\vec{x}_\varepsilon) - S_q(\vec{x})] + \varepsilon \sum_{q=1}^{k} b_{\alpha q} S_q(\vec{x}_\varepsilon) \\
&\geq \sum_{q=1}^{k} \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right] \left[\sum_{p=1}^{k} a_{pq} \left[S_p(\vec{x}_\varepsilon) + S_p(\vec{x})\right] + \sum_{i=1}^{n} b_{iq} x_i + c_q \right] \\
&\quad + \sum_{q=1}^{k} \varepsilon b_{\alpha q} [S_q(\vec{x}_\varepsilon) - S_q(\vec{x})] \\
&\geq \sum_{q=1}^{k} \left[S_q(\vec{x}_\varepsilon) - S_q(\vec{x})\right] \left[2 \sum_{p=1}^{k} (a_{pq} \wedge 0) S_p(\vec{x}_\varepsilon) + \sum_{i=1}^{n} b_{iq} x_i + \varepsilon b_{\alpha q} + c_q \right] \\
&\geq 0.
\end{aligned}
$$

Therefore, $T_P(S_1, ..., S_k)$ is a semi-copula. ∎

Remark 2.9. In the case of univariate transformation, that is, the case $k = 1$, conditions 2. and 4. reduce to $b_{i1} = -a_{11} \geq 0$ while conditions 3. and 5. reduce to $c_1 = 1 + (n - 1)a_{11} \geq 0$. Combining this fact with condition 6. yields $1 + a_{11} \geq 0$ which is redundant since $n > 1$. Therefore, we can conclude that univariate quadratic transformation must be in the form

$$T_a(S) = \frac{an}{n-1} \cdot S \cdot \text{Mean} - \frac{a}{n-1} S^2 + (1-a) S$$

where $0 \le a \le 1$ and $\text{Mean}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$ is the average function. Note that T_0 is the identity map and hence all T_a are convex combinations of the identity map T_0 and $T_1 = \frac{1}{n-1} T_0 (n \cdot \text{Mean} - T_0)$. This situation is already different from the univariate quadratic transformation of bivariate semi-copula. In the latter case, it has been proved that there are exactly 4 transformations which are not convex combinations of others (see [13]).

As mentioned before, we are interested in the case of actual quadratic transformations. This case only appears when the coefficient $a_{pq} < 0$ for some p, q. Otherwise, conditions 2. and 5. will force $b_{iq} = 0 = a_{pq}$ for all i, p, q. Then conditions 3. and 5. imply that T_P is simply a convex combination of semi-copulas. In the case of univariate transformations, this reduces to T_0 and T_1 defined above. For multivariate transformations, the actual quadratice transformations are given in the following forms.

Definition 2.10. A quadratic transformation T_P is said to be *proper* if the polynomial P is in the form

$$P(x_1, \ldots, x_n, z_1, \ldots, z_k) = \frac{1}{n-1} \left(\sum_{i=1}^{n} \sum_{q=1}^{k} b_{iq} x_i z_q - \sum_{p=1}^{k} \sum_{q=1}^{k} a_{pq} z_p z_q \right)$$

where $b_{iq} \ge 0$ for all i and q, $\sum_{p=1}^{k} \sum_{q=1}^{k} a_{pq} = \sum_{q=1}^{k} b_{iq} = 1$ for all i, and $\sum_{i=1}^{n} b_{iq} - 2 \sum_{p=1}^{k} a_{pq} \vee 0 \ge 0$ for all q.

Theorem 2.11. *Any quadratic transformation can be written as a convex sum of a proper quadratic transformation and some* $K_{\vec{d}}$.

Proof. Let T_P be a quadratic transformation with its coefficients satisfy conditions in Theorem 3.4. Since $\sum_{p=1}^{k} \sum_{q=1}^{k} a_{pq} = -\sum_{q=1}^{k} b_{iq} \le 0$ and $c_q \ge 0$ for all q, $0 \le \sum_{q=1}^{k} c_q \le 1$. If $\sum_{q=1}^{k} c_q = 0$, then T_P is proper. If $\sum_{q=1}^{k} c_q = 1$, then $a_{pq} = b_{iq} = 0$ for all i, p, q. Therefore, $T_P = K_{\vec{c}}$. In the case of $0 < \sum_{q=1}^{k} c_q < 1$, set $t = \sum_{q=1}^{k} c_q$,

$$\vec{d} = \frac{1}{\sum_{q=1}^{k} c_q} (c_1, \ldots, c_k),$$

and

$$Q(x_1, \ldots, x_n, z_1, \ldots, z_k)$$
$$= \left(\sum_{i=1}^{n} \sum_{q=1}^{k} \frac{b_{iq}}{\left(1 - \sum_{q=1}^{k} c_q\right)} x_i z_q - \sum_{p=1}^{k} \sum_{q=1}^{k} \frac{-a_{pq}}{\left(1 - \sum_{q=1}^{k} c_q\right)} z_p z_q \right).$$

Then T_Q is a proper quadratic transformation and $T_P = (1-t) T_Q + t K_{\vec{d}}$. ∎

Note that there is only one proper quadratic transformation of a semi-copula but there are infinitely many proper quadratic transformations of k semi-copulas when $k > 1$. In fact, it can be shown that the set of proper quadratic transformations of k semi-copulas is convex with $O(k^n)$ extreme points. We will demonstrate this fact in the case that $k = 2$. In this case, the polynomial P such that T_P is a proper quadratic transformation of 2 semi-copulas must be in the form

$$P(x_1, \ldots, x_n, z_1, z_2) = \frac{1}{n-1}\left[z_1 \sum_{i=1}^{n} b_i x_i + z_2 \sum_{i=1}^{n} (1-b_i)\, x_i \right.$$
$$\left. -a_{11} z_1^2 - (1 - a_{11} - a_{22})\, z_1 z_2 - a_{22} z_2^2 \right]$$

with the conditions that $b_i \in [0,1]$ for all i,

$$[2a_{11} \vee 0] + [(1 - a_{11} - a_{22}) \vee 0] \leq \sum_{i=1}^{n} b_i,$$

and

$$[2a_{22} \vee 0] + [(1 - a_{11} - a_{22}) \vee 0] \leq n - \sum_{i=1}^{n} b_i.$$

Theorem 2.12. *There are exactly* $5 \cdot 2^n - 4$ *extreme points on the set of proper quadratic transformations of* 2 *semi-copulas. In fact, the corresponding quadratic polynomials of these extreme points are*

$$P_{I,c,d}(x_1, \ldots, x_n, z_1, z_2) = \frac{1}{n-1}\left(z_1 \sum_{i\in I} x_i + z_2 \sum_{i \notin I} x_i - c z_1^2 - (1-c-d)\, z_1 z_2 - d z_2^2\right)$$

where $I \subseteq \{1, \ldots, n\}$ *and* (c,d) *belongs to the set*

$$\left\{\left(\tfrac{|I|}{2}, \tfrac{n-|I|}{2}\right), \left(\tfrac{1-n-|I|}{2}, \tfrac{1+n+|I|}{2}\right), \left(\tfrac{n-1}{2}, \tfrac{n+1}{2} - |I|\right), (0, 1-|I|), \left(\tfrac{|I|}{2}, 1 - \tfrac{|I|}{2}\right)\right\}$$

when the number of elements $|I|$ *of* I *is between* 0 *and* $\frac{n}{2}$, *otherwise,* (c,d) *belongs to the set*

$$\left\{\left(\tfrac{n-|I|}{2}, \tfrac{|I|}{2}\right), \left(\tfrac{1+n+|I|}{2}, \tfrac{1-n-|I|}{2}\right), \left(\tfrac{n+1}{2} - |I|, \tfrac{n-1}{2}\right), (1-|I|, 0), \left(1 - \tfrac{|I|}{2}, \tfrac{|I|}{2}\right)\right\}.$$

Proof. First, we will show that any such P can be written as a convex combination of $P_{I,c,d}$ where $2c \vee 0 + (1-c-d) \vee 0 \leq |I|$, and $2d \vee 0 + (1-c-d) \vee 0 \leq n - |I|$.

Let $P_1 = P$ and $I_1 = \{i \mid b_i > 0\}$. If $I_1 = \emptyset$, then $P = P_{I_1, a_{11}, a_{22}}$ and we are done. Suppose that $I_1 \neq \emptyset$, let $b = \min\{b_i \mid i \in I_1\}$ and

$$Q_1 = \begin{cases} P_{I_1, |I_1|/2, -(|I_1|-1)/2} & \text{if } |I_1| = n \\ P_{I_1, (|I_1|-1)/2, -(|I_1|-1)/2} & \text{if } |I_1| < n. \end{cases}$$

Also, set $P_2 = \frac{1}{1-b}(P_1 - bQ_1)$. Then

$$\begin{aligned}P_2\left(x_1,\ldots,x_n,z_1,z_2\right) &= \frac{1}{n-1}\left(z_1\sum_{i=1}^{n}\frac{b_i-b}{1-b}x_i+z_2\left(\sum_{i=1}^{n}\frac{1-b_i}{1-b}x_i\right)\right)\\ &\quad -\frac{1}{n-1}\left(\left(\frac{a_{11}-b\left|I_1\right|/2}{1-b}\right)z_1^2\right.\\ &\quad \left.+\left(1-\frac{a_{11}+a_{22}-b/2}{1-b}\right)z_1z_2\right)\\ &\quad -\frac{1}{n-1}\left(\frac{a_{22}-b/2+b\left|I_1\right|/2}{1-b}\right)z_2^2\end{aligned}$$

in the case that $|I_1| = n$ and

$$\begin{aligned}&P_2\left(x_1,\ldots,x_n,z_1,z_2\right)\\ &= \frac{1}{n-1}\left(z_1\sum_{i\in I_1}\frac{b_i-b}{1-b}x_i+z_2\left(\sum_{i\in I_1}\frac{1-b_i}{1-b}x_i+\sum_{i\notin I_1}x_i\right)\right)\\ &\quad -\frac{1}{n-1}\left(\left(\frac{a_{11}+b/2-b\left|I_1\right|/2}{1-b}\right)z_1^2+\left(1-\frac{a_{11}+a_{22}}{1-b}\right)z_1z_2\right)\\ &\quad -\frac{1}{n-1}\left(\frac{a_{22}-b/2+b\left|I_1\right|/2}{1-b}\right)z_2^2\end{aligned}$$

in the otherwise. It can be checked that both T_{Q_1} and T_{P_2} are proper quadratic transformations and $P = bQ_1 + (1-b)\,P_2$. Moreover, $I_2 = \left\{i \mid \frac{b_i-b}{1-b} > 0\right\}$ is a proper subset of I_1. Thus, we may repeat this process, say, for m number of times with P_{i+1} in place of P_i until we have $I_m = \emptyset$. It follows that P must be a convex combination of some Q_i. Specifically,

$$P = b_{(1)}Q_1 + \sum_{i=i}^{m} b_{(i)} \prod_{j=1}^{i-1}\left(1-b_{(j)}\right)Q_i$$

where $b_{(1)} < b_{(2)} < \ldots < b_{(m)}$ is the ordering of $b_1,\ldots,b_n$.

From the above fact, we can see that any extreme point of this set must have either zero or one as coefficients of all x_iz_p. To find the extreme points of this set, it is then sufficient to find extreme points of the set

$$\left\{P_{I,c,d} \mid 2c\vee 0+(1-c-d)\vee 0\le |I|,\ \text{and } 2d\vee 0+(1-c-d)\vee 0\le n-|I|\right\}$$

where $I \subseteq \{1,\ldots,n\}$ is fixed. This can simply be done by graphical method. See Figure 1 for the case $|I| \le \frac{n}{2}$. The case $|I| > \frac{n}{2}$ simply follows by switching c and d.

For $|I| = 0$ or n, the number of intersection points reduced to 3 instead of 5. Thus, there are totally $5\cdot 2^n - 4$ extreme points. ■

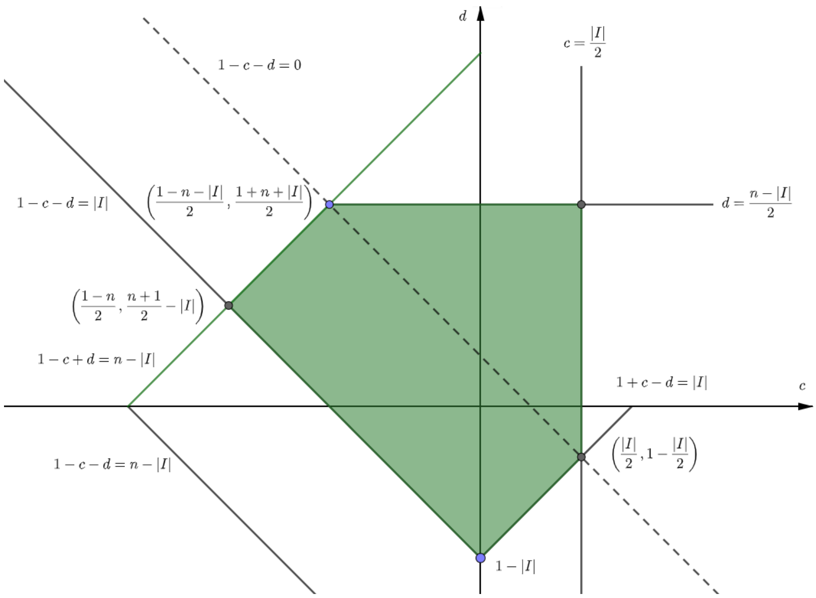

FIGURE 1. Area of possible c and d when $|I| \leq \frac{n}{2}$.

Acknowledgements

This research was supported by Chiang Mai University.

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