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Strong Convergence of Browder's Type Iterations in CAT(κ) Spaces

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Abstract The purpose of this paper is to present an existence and convergence theorem for a nonexpansive mapping on a complete geodesic space with curvature bounded above. A strong convergence theorem for Browder's type iterations of a nonexpansive mapping is derived. Moudafi's viscosity type methods with a spherical contraction are also discussed.

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1. Introduction

One of the most important tools for solving equations is the method of finding fixed points of mappings. The well-known Banach contraction principle guarantees that every contraction on a complete metric space has a unique fixed point. The existence of fixed point theorems for a nonexpansive mapping, as a relaxation notion of contractions, in CAT(κ) spaces was proved by Kirk [1, 2] for $\kappa \leq 0$, and by Espánola and Fernández-León [3] for $\kappa > 0$. Note that every CAT(κ') space is a CAT(κ) space whenever $\kappa' < \kappa$. In particular, results in CAT(0) spaces can immediately be applied to CAT(κ) spaces with $\kappa \leq 0$. Moreover, CAT(κ) spaces with positive κ can be treated as CAT(1) spaces by changing the scale of the space; see [4–15] for more details. It therefore suffices to focus only on CAT(1) spaces.

In 2011, Piątek [16] presented the following result in the setting of CAT(1) spaces.

Theorem 1.1. *Let X be a complete CAT(1) space, and $T : X \to X$ be a nonexpansive mapping such that $\mathrm{F}(T) := \{x \in X : x = Tx\} \neq \varnothing$. Let $u \in X$ and suppose that $d(u, \mathrm{F}(T)) \leq \pi/4$. Denote $q := P_{\mathrm{F}(T)}u$, where $P_{\mathrm{F}(T)}$ is the metric projection from X onto $\mathrm{F}(T)$. Then for each $t \in (0, 1)$, there exists a unique fixed point $x_t \in \overline{B}(q, \pi/4)$ of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$ on $\overline{B}(q, \pi/4)$, that is,*

$$x_t = tu \oplus (1-t)Tx_t.$$

In addition, the net $\{x_t\}$ converges strongly to $q := P_{\mathrm{F}(T)}u$ as $t \to 0^+$.

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In 2014, based on Browder's type convergence theorems [17], a nonexpansive semigroup could be reduced to a single nonexpansive mapping in the framework of CAT(1) spaces as follows:

Theorem 1.2. *Let X be a complete CAT(1) space, and $T : X \to X$ be a nonexpansive mapping such that $\mathrm{F}(T) \neq \varnothing$. Let $u \in X$ and suppose that $d(u, \mathrm{F}(T)) < \pi/4$. Let $\{x_n\}$ be a sequence in X defined by*

$$x_n = \alpha_n u \oplus (1 - \alpha_n)Tx_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(T)}u$.

In this paper, we present an existence and convergence theorem for a nonexpansive mapping on a complete CAT(1) space, which supplements Piątek's result. Furthermore, we prove a strong convergent theorem for Browder's type iterations of a nonexpansive mapping, which improve Theorem 1.2. Moudafi's viscosity type methods of a nonexpansive mapping with a spherical contraction are discussed in the framework of CAT(1) spaces. Finally, we consider results in the setting of CAT(κ) spaces with a real number κ.

2. Preliminaries

Let X be a geodesic space. A *geodesic triangle* $\triangle(u, v, w)$ consists of three points $u, v, w \in X$ and all the images of each geodesic part joining two of them. For a triangle $\triangle(u, v, w)$ in X satisfying $d(u, v) + d(v, w) + d(w, u) < 2\pi$, we can find the comparison triangle $\triangle(\overline{u}, \overline{v}, \overline{w})$ in the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$; that is, each corresponding edge has the same length as that of original triangle. If for any $p, q \in \triangle(u, v, w)$ and their corresponding comparison points $\overline{p}, \overline{q} \in \triangle(\overline{u}, \overline{v}, \overline{w})$, the inequality $d(p, q) \le d_{\mathbb{S}^2}(\overline{p}, \overline{q})$ holds, then we call X a *CAT(1) space.*

Let (X, d) be a CAT(1) space. Given a point $t \in [0, 1]$ and two points $v, w \in X$ such that $d(v, w) < \pi$, we use the notation $tv \oplus (1 - t)w$ for a unique point u in the unique geodesic segment $[v, w]$ such that

$$d(u, v) = (1 - t)d(v, w) \text{ and } d(u, w) = td(v, w).$$

A subset C of X is called *convex* if $tv \oplus (1-t)w \in C$ for all $v, w \in C$ such that $d(v, w) < \pi$.

The following lemma yields a crucial inequality in CAT(1) spaces, which plays an important role in this paper.

Lemma 2.1 ([18, Corollary 2.2])**.** *Let $t \in [0, 1]$ and u, v, w be three points in a* CAT(1) *space (X, d) such that $d(u, v) + d(v, w) + d(w, u) < 2\pi$. Then*

$$\begin{aligned}&\cos d(tv \oplus (1 - t)w, u) \sin d(v, w)\\ &\ge \cos d(v, u) \sin(td(v, w)) + \cos d(w, u) \sin((1 - t)d(v, w)).\end{aligned}$$

Throughout the rest of this section, we do assume that X is a complete CAT(1) space such that $d(u, v) < \pi/2$ for all $u, v \in X$.

Proposition 2.2. *Suppose that C is a closed and convex subset of X and $\{x_n\}$ is a sequence in X such that $\mathrm{rad}_C(\{x_n\}) := \inf_{x\in C} \sup_n d(x_n, x) < \pi/2$. Suppose that $g : C \to [0, 1]$ is defined by*

$$g(x) := \liminf_{n\to\infty} \cos d(x, x_n) \quad \textit{for all } x \in C.$$

Then g is upper semicontinuous and there exists a unique element $\widehat{x} \in C$ such that

$$g(\widehat{x}) = \max_{x \in C} g(x).$$

Moreover, if $x \in C$, then

$$g(x) \leq \cos d(x, \widehat{x}) \cdot g(\widehat{x}).$$

Proof. Since $\mathrm{rad}_C(\{x_n\}) < \pi/2$, we have $\alpha := \sup_{x \in C} g(x) > 0$. In particular, there exists a sequence $\{z_n\}$ in C such that $\lim_m g(z_m) = \alpha$. We prove that $\{z_m\}$ is a Cauchy sequence. To see this, we may assume that $z_m \neq z_k$ for all $m \neq k$. Since C is convex, it follows that $g\left(\frac{1}{2} z_m \oplus \frac{1}{2} z_k\right)$ is meaningful and not exceeding α. We apply the preceding lemma for z_m, z_k, x_n and $t := 1/2$ to obtain the following

$$\begin{aligned} &\cos d\left(\frac{1}{2} z_m \oplus \frac{1}{2} z_k, x_n\right) \sin d(z_m, z_k) \\ &\geq \cos d(z_m, x_n) \sin \frac{1}{2} d(z_m, z_k) + \cos d(z_k, x_n) \sin \frac{1}{2} d(z_m, z_k). \end{aligned}$$

Hence

$$\cos d\left(\frac{1}{2} z_m \oplus \frac{1}{2} z_k, x_n\right) \cdot 2 \cos \frac{1}{2} d(z_m, z_k) \geq \cos d(z_m, x_n) + \cos d(z_k, x_n).$$

This implies that

$$\begin{aligned} \alpha \cdot 2 \cos \frac{1}{2} d(z_m, z_k) &\geq g\left(\frac{1}{2} z_m \oplus \frac{1}{2} z_k\right) \cdot 2 \cos \frac{1}{2} d(z_m, z_k) \\ &= \liminf_{n \to \infty} \cos d\left(\frac{1}{2} z_m \oplus \frac{1}{2} z_k, x_n\right) \cdot 2 \cos \frac{1}{2} d(z_m, z_k) \\ &\geq \liminf_{n \to \infty} \cos d(z_m, x_n) + \liminf_{n \to \infty} \cos d(z_k, x_n). \end{aligned}$$

Now taking $m, k \to \infty$ gives $\lim_{m,k \to \infty} \cos \frac{1}{2} d(z_m, z_k) = 1$ because $\alpha > 0$. This implies that $\{z_m\}$ is a Cauchy sequence and hence $\lim_{m \to \infty} z_m = \widehat{x}$ for some $\widehat{x} \in C$. Since the cosine function is decreasing, it follows that g is upper semicontinuous and hence $g(\widehat{x}) = \alpha$. Now, we prove the uniqueness. Suppose that there exists another element $x' \in C$ such that $g(x') = \alpha$. We repeat the proof above for a sequence $\{w_m\}$ where $w_{2m-1} : \widehat{x}$ and $w_{2m} := x'$ for all $n \geq 1$. Since $\lim_{m \to \infty} g(w_m) = \alpha$, we can conclude that $\{w_n\}$ is a Cauchy sequence and this implies that $x' = \widehat{x}$. This completes the first assertion of the proposition.

Finally, let $x \in C$ and let $t \in (0, 1)$. It follows that $tx \oplus (1-t)\widehat{x} \in C$ and hence $g(tx \oplus (1-t)\widehat{x}) \leq g(\widehat{x})$. Note that

$$\begin{aligned} &\cos d(tx \oplus (1-t)\widehat{x}, x_n) \sin d(x, \widehat{x}) \\ &\geq \cos d(x, x_n) \sin t d(x, \widehat{x}) + \cos d(\widehat{x}, x_n) \sin(1-t) d(x, \widehat{x}). \end{aligned}$$

In particular, $g(\widehat{x}) \geq g(x) \sin t d(x, \widehat{x}) + g(\widehat{x}) \sin(1-t) d(x, \widehat{x})$. The desired inequality holds trivially if $x = \widehat{x}$. We now assume that $x \neq \widehat{x}$. Hence

$$g(\widehat{x}) \frac{1 - \sin(1-t) d(x, \widehat{x})}{t d(x, \widehat{x})} \geq g(x) \frac{\sin t d(x, \widehat{x})}{t d(x, \widehat{x})}.$$

Letting $t \downarrow 0$ gives the result. ∎

As a consequence of this result, we immediately have the following one.

Proposition 2.3. *Let C be a nonempty closed convex subset of X such that $d(v, C) := \inf_{w \in C} d(v, w) < \pi/2$ for all $v \in X$. Then the metric projection P_C from X onto C is well defined; that is, for each $v \in X$, there exists a unique point $P_C v \in C$ satisfying*

$$d(v, P_C v) = \inf_{w \in C} d(v, w).$$

If $u \in X$ and $w \in C$, then

$$w = P_C u \text{ if and only if } \cos d(u, v) \leq \cos d(u, w) \cos d(v, w) \quad \text{for all } v \in C,$$

where P_C is the metric projection from X onto C.

A mapping $T : X \to X$ is said to be:

- *spherically Lipschitz* if there exists a constant $L > 0$ such that
$$\sin \frac{d(Tv, Tw)}{2} \leq L \sin \frac{d(v, w)}{2} \quad \text{for all } v, w \in X.$$
- *spherical contraction* if it is spherically Lipschitz with Lipschitz constant $L < 1$.
- *nonexpansive* if it is spherically Lipschitz with Lipschitz constant $L = 1$, that is, $d(Tv, Tw) \leq d(v, w)$ for all $v, w \in X$.

The following lemma is extracted from [19, Proposition 3.4], and so the proof is omitted.

Lemma 2.4. *Let C be a nonempty closed convex subset of X and $M := \operatorname{diam} X < \pi/2$. Then the metric projection P_C is spherically Lipschitz with the Lipschitz constant $L = \sec M$, that is,*

$$\sin \frac{d(P_C v, P_C w)}{2} \leq \sec M \cdot \sin \frac{d(v, w)}{2} \quad \text{for all } v, w \in X.$$

The following lemmas are also required for our main results.

Lemma 2.5 ([20, Lemma 2.3])**.** *Let u, v, w be three points in a $\mathrm{CAT}(1)$ space (Y, d) such that $d(v, u) \leq \pi/2$ and $d(w, u) \leq \pi/2$, and let $t \in [0, 1]$. Then*

$$\cos d(tv \oplus (1-t)w, u) \geq t \cos d(v, u) + (1-t) \cos d(w, u).$$

Moreover, we have $d(tv \oplus (1-t)w, u) \leq \max\{d(v, u), d(w, u)\}$ [21, Lemma 3.4].

Lemma 2.6 ([22, Lemma 5.4])**.** *Let u, v, w be three points in a $\mathrm{CAT}(1)$ space (Y, d) such that $d(u, v) + d(v, w) + d(w, u) < 2\pi$. Let $x := tu \oplus (1-t)v$ and $y := tu \oplus (1-t)w$ for some $t \in [0, 1]$. If $d(u, v) \leq M$, $d(u, w) \leq M$, and $\sin((1-t)M) \leq \sin M$ for some $M \in (0, \pi)$, then*

$$d(x, y) \leq \frac{\sin(1-t)M}{\sin M} d(v, w).$$

Lemma 2.7 ([23, Lemma 2.5])**.** *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers, $\{b_n\}$ be a sequence of real numbers, and $\{\beta_n\}$ be a sequence of real numbers in $[0, 1]$ such that*

$$a_{n+1} \leq (1 - \beta_n) a_n + \beta_n b_n + c_n$$

for all $n \in \mathbb{N}$. If $\limsup_{n\to\infty} b_n \leq 0$, $\sum_{n=1}^{\infty} c_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n = \infty$, then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

From now on, we do assume that X is a complete CAT(1) space such that $d(u, v) < \pi/2$ for all $u, v \in X$. Recall that a sequence $\{v_n\} \subset X$ is said to be an *approximating fixed point sequence* of a mapping $T : X \to X$ if

$$\lim_{n\to\infty} d(v_n, Tv_n) = 0.$$

An approximating fixed point sequence plays an important role in the study of fixed points of nonexpansive mappings. We prove that every nonexpansive mapping admits an approximating fixed point sequence.

Proposition 3.1. *Let $T : X \to X$ be a nonexpansive mapping and $u \in X$ be fixed. Given a point $t \in (0, 1)$, define $S_t : X \to X$ by*

$$S_t x := tu \oplus (1-t)Tx \quad \text{for } x \in X.$$

Then S_t has a unique fixed point $x_t \in X$, that is,

$$x_t = tu \oplus (1-t)Tx_t. \tag{3.1}$$

In this case, we have $d(x_t, Tx_t) \to 0$ as $t \to 0^+$.

Proof. Let $x, y \in X$ and $t \in (0, 1)$. By Lemma 2.6, we obtain

$$\begin{aligned} d(S_t x, S_t y) &= d(tu \oplus (1-t)Tx, tu \oplus (1-t)Ty) \\ &\le \left(\sin\frac{(1-t)\pi}{2}\right) d(Tx, Ty) \\ &\le \left(\sin\frac{(1-t)\pi}{2}\right) d(x, y). \end{aligned}$$

Then S_t is a contraction. It follows from Banach contraction principle that there exists exactly one point $x_t \in X$ such that

$$x_t = tu \oplus (1-t)Tx_t.$$

Consequently, we have $\lim_{t\to 0^+} d(x_t, Tx_t) = \lim_{t\to 0^+} td(u, Tx_t) = 0$, and the proof is finished. ∎

As an immediate consequence of Proposition 3.1, we obtain the following result.

Corollary 3.2. *Every nonexpansive mapping $T : X \to X$ has an approximating fixed point sequence.*

Let ℓ_∞ denote the Banach space of bounded real sequences. Recall that a continuous linear functional μ on ℓ_∞ is said to be a *Banach limit* if $\|\mu\| = \mu(1, 1, \dots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

We now present the existence and convergence theorem for a nonexpansive mapping in complete CAT(1) spaces.

Theorem 3.3. *Let T, u be as in the preceding proposition. For each $t \in (0, 1)$ let x_t be a net given by (3.1). Then $\mathrm{F}(T) \neq \varnothing$ if and only if*

$$\mathrm{rad}_X(\{x_t\}) := \inf_{z\in X} \sup_{t\in(0,1)} d(x_t, z) < \pi/2.$$

In this case, the following statements hold:

(i) $q := \lim_{t\to 0^+} x_t$ *exists and it is a unique fixed point of* T *which is nearest to* u.

(ii) *If* μ *is a Banach limit and* $\{y_n\}$ *is a and all approximating fixed point sequences of* T*, then* $\cos d(u,q) \geq \mu_n \cos d(u, y_n)$.

Proof. Suppose first that $\mathrm{F}(T) \neq \varnothing$. Fix $p \in F(T)$ and let $t \in (0,1)$. It follows from Lemma 2.5 and the nonexpansiveness of T that

$$\begin{aligned}\cos d(x_t, p) &= \cos d(tu \oplus (1-t)Tx_t, p)\\ &\geq t\cos d(u,p) + (1-t)\cos d(Tx_t, p)\\ &\geq t\cos d(u,p) + (1-t)\cos d(x_t, p).\end{aligned}$$

This implies that $\cos d(x_t, p) \geq \cos d(u,p)$. It follows that $d(x_t,p) \leq d(u,p) < \pi/2$, and hence $\mathrm{rad}_X(\{x_t\}) < \pi/2$.

Conversely, suppose that $\mathrm{rad}_X(\{x_t\}) < \pi/2$. Let $\{t_k\}$ be any sequence in $(0,1)$ such that $\lim_{k\to\infty} t_k = 0$ and we define $g : X \to [0,1]$ by

$$g(z) := \liminf_{k\to\infty} \cos d(x_{t_k}, z) \quad \text{for all } z \in X.$$

Then Proposition 2.2 guaruntees that there exists a unique element $\widehat{z} \in X$ such that $g(\widehat{z}) = \max_{z\in C} g(z)$. Finally, we prove that $\widehat{z} = T\widehat{z}$. To see this, it suffices to prove that $g(T\widehat{z}) = g(\widehat{z})$. In fact, it follows from the following argument

$$\begin{aligned}g(\widehat{z}) \geq g(T\widehat{z}) &= \liminf_{k\to\infty} \cos d(x_{t_k}, T\widehat{z})\\ &\geq \liminf_{k\to\infty} \cos(d(x_{t_k}, Tx_{t_k}) + d(Tx_{t_k}, T\widehat{z}))\\ &\geq \liminf_{k\to\infty} \cos(d(x_{t_k}, Tx_{t_k}) + d(x_{t_k}, \widehat{z}))\\ &= \liminf_{k\to\infty} \cos d(x_{t_k}, \widehat{z}) = g(\widehat{z}).\end{aligned}$$

This implies that $\widehat{z} = T\widehat{z}$, and hence $\mathrm{F}(T) \neq \varnothing$.

To prove Statements (i) and (ii), it suffices to show that

$$\cos d(u, \widehat{z}) \geq \mu_n \cos d(u, y_n)$$

for all Banach limits μ and for all approximating fixed point sequence $\{y_n\}$. In fact, it follows from this statement that $\widehat{z}$ is a unique fixed point of T which is nearest to u.

To this end, let $\{y_n\} \subset X$ be such that $\lim_{n\to\infty} d(y_n, Ty_n) = 0$ and let μ be a Banach limit. We may assume that $Tx_{t_k} \neq u$ for all k. (Otherwise, we obtain that $\widehat{z} = u$ and we are done.) By Lemma 2.1, we have

$$\begin{aligned}&\cos d(y_n, x_{t_k}) \sin d(u, Tx_{t_k})\\ &= \cos d(y_n, t_k u \oplus (1-t_k)Tx_{t_k}) \sin d(u, Tx_{t_k})\\ &\geq \cos d(y_n, u) \sin t_k d(u, Tx_{t_k}) + \cos d(y_n, Tx_{t_k}) \sin(1-t_k) d(u, Tx_{t_k})\\ &\geq \cos d(y_n, u) \sin t_k d(u, Tx_{t_k})\\ &\quad + \cos\left(d(y_n, Ty_n) + d(Ty_n, Tx_{t_k})\right) \sin(1-t_k) d(u, Tx_{t_k})\\ &\geq \cos d(y_n, u) \sin t_k d(u, Tx_{t_k})\\ &\quad + \cos\left(d(y_n, Ty_n) + d(y_n, x_{t_k})\right) \sin(1-t_k) d(u, Tx_{t_k}).\end{aligned}$$

This implies that

$$\sin d(u, Tx_{t_k})\mu_n \cos d(y_n, x_{t_k}) \geq \sin t_k d(u, Tx_{t_k})\mu_n \cos d(y_n, u) + \sin(1-t_k)d(u, Tx_{t_k})\mu_n \cos d(y_n, x_{t_k}).$$

Hence, we have

$$\frac{\sin d(u, Tx_{t_k}) - \sin(1-t_k)d(u, Tx_{t_k})}{t_k d(u, Tx_{t_k})}\mu_n \cos d(y_n, x_{t_k}) \geq \frac{\sin t_k d(u, Tx_{t_k})}{t_k d(u, Tx_{t_k})}\mu_n \cos d(y_n, u).$$

Note that

$$\lim_{k\to\infty} \frac{\sin d(u, Tx_{t_k}) - \sin(1-t_k)d(u, Tx_{t_k})}{t_k d(u, Tx_{t_k})} = \lim_{k\to\infty} \frac{2\cos(1-\frac{t_k}{2})d(u, Tx_{t_k}) \sin\frac{t_k}{2}d(u, Tx_{t_k})}{t_k d(u, Tx_{t_k})} = \cos d(u, \widehat{z}).$$

Hence

$$\cos d(u, \widehat{z})\mu_n \cos d(y_n, \widehat{z}) \geq \mu_n \cos d(y_n, u).$$

Consequently, $\cos d(u, \widehat{z}) \geq \mu_n \cos d(y_n, u)$, as desired. ∎

Corollary 3.4. *Let $T : X \to X$ be a nonexpansive mapping and $f : X \to X$ be a spherical contraction with a constant k. Suppose that $M := \operatorname{diam} X < \pi/2$ and $k < \cos M$. Define a net $\{x_t\}$ in X by*

$$x_t = tf(x_t) \oplus (1-t)Tx_t \quad \text{for } t \in (0,1).$$

Then the net $\{x_t\}$ converges strongly to the point $\hat{x} = P_F f(\hat{x})$.

Proof. Note that $\mathrm{F}(T)$ is a nonempty closed and convex subset of X. Given a point $t \in (0,1)$, define $S_t : X \to X$ by

$$S_t x := tf(x) \oplus (1-t)Tx \quad \text{for } x \in X.$$

Let $x, y \in X$. Note that $d(f(x), f(y)) \leq kd(x,y)$. By Lemma 2.6, we have

$$\begin{aligned} d(S_t x, S_t y) &\leq d(S_t x, tf(y) \oplus (1-t)Tx) + d(tf(y) \oplus (1-t)Tx, S_t y) \\ &\leq \frac{\sin tM}{\sin M} d(f(x), f(y)) + \frac{\sin(1-t)M}{\sin M} d(Tx, Ty) \\ &\leq \frac{k \sin tM + \sin(1-t)M}{\sin M} d(x,y) \\ &\leq \frac{\cos M \sin tM + \sin M \cos tM - \cos M \sin tM}{\sin M} d(x,y) \\ &= \cos tM \cdot d(x,y). \end{aligned}$$

Then S_t is a contraction, and hence, x_t is well-defined. It follows from Lemma 2.4 that

$$\sin \frac{d(P_{\mathrm{F}(T)}f(x), P_{\mathrm{F}(T)}f(y))}{2} \leq \frac{1}{\cos M} \sin \frac{d(f(x), f(y))}{2} \leq \frac{k}{\cos M} \sin \frac{d(x,y)}{2} \leq \sin \frac{kd(x,y)}{2\cos M}.$$

This implies that $P_{\mathrm{F}(T)} \circ f$ is a contraction, so there exists a unique fixed point $\hat{x}$ of $P_{\mathrm{F}(T)} \circ f$. Define a net $\{y_t\}$ by

$$y_t = tf(\hat{x}) \oplus (1-t)Ty_t \quad \text{for all } t \in (0,1).$$

The strong convergence of $\{y_t\}$ to $\hat{x}$ is assured by Theorem 3.3 (i). For each $t \in (0,1)$, it follows from Lemma 2.6 that

$$\begin{aligned} d(x_t, y_t) &\le \frac{\sin tM}{\sin M} d(f(x_t), f(\hat{x})) + \frac{\sin(1-t)M}{\sin M} d(Tx_t, Ty_t) \\ &\le \frac{\sin tM}{\sin M} \left(d(f(x_t), f(y_t)) + d(f(y_t), f(\hat{x})\right) + \frac{\sin(1-t)M}{\sin M} d(x_t, y_t) \\ &\le \frac{k \sin tM}{\sin M} d(y_t, \hat{x}) + \left(\frac{k \sin tM}{\sin M} + \frac{\sin(1-t)M}{\sin M}\right) d(x_t, y_t), \end{aligned}$$

which implies that

$$\left(\frac{\sin M - \sin(1-t)M}{tM} - \frac{k \sin tM}{tM}\right) d(x_t, y_t) \le \frac{k \sin tM}{tM} d(y_t, \hat{x}).$$

Since $k < \cos M$, we have

$$\lim_{t \to 0^+} \left(\frac{\sin M - \sin(1-t)M}{tM} - \frac{k \sin tM}{tM}\right) = \cos M - k > 0.$$

Consequently, we obtain $\lim_{t \to 0^+} d(x_t, y_t) = 0$ and the proof is finished. ■

Next, we present a Browder's type convergence theorem for a nonexpansive mapping in X.

Theorem 3.5. *Let $T : X \to X$ be a nonexpansive mapping such that $\mathrm{F}(T) \neq \varnothing$. Let $u \in X$ be fixed and $\{x_n\}$ be a sequence in X defined by*

$$x_n = \alpha_n u \oplus (1 - \alpha_n) T x_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(T)} u$.

Proof. Let $U_n x := \alpha_n u \oplus (1-\alpha_n) T x$ for all $x \in X$. By Proposition 3.1, there exists a unique element $x_n \in X$ such that

$$x_n = \alpha_n u \oplus (1 - \alpha_n) T x_n.$$

Then we have $\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} \alpha_n d(u, Tx_n) = 0$. Note that $d(x_n, p) \le d(u, p) < \pi/2$ for all $p \in \mathrm{F}(T)$, and hence,

$$\mathrm{rad}_X(\{x_n\}) := \inf_{z \in X} \sup_{n \in \mathbb{N}} d(x_n, z) < \pi/2.$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and μ be a Banach limit. Define $g : X \to [0,1]$ by

$$g(z) := \lim_{n \to \infty} \cos d(x_n, z) \quad \text{for all } z \in X.$$

It follows from Proposition 2.2 that there exists a unique element $\widehat{x} \in X$ such that $g(\widehat{x}) = \max_{x \in X} g(x)$ and $g(\widehat{x}) \cos d(x, \widehat{x}) \ge g(x)$ for all $x \in X$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and T is nonexpansive, we have $g(T\widehat{x}) \ge g(\widehat{x})$ and hence $\widehat{x} = T\widehat{x}$. Let $p \in \mathrm{F}(T)$. We now consider the following estimate

$$\begin{aligned} &\cos d(p, x_n) \sin d(u, Tx_n) \\ &= \cos d(p, \alpha_n u \oplus (1-\alpha_n) T x_n) \sin d(u, Tx_n) \\ &\ge \cos d(p, u) \sin \alpha_n d(u, Tx_n) + \cos d(p, Tx_n) \sin(1-\alpha_n) d(u, Tx_n) \\ &\ge \cos d(p, u) \sin \alpha_n d(u, Tx_n) + \cos d(p, x_n) \sin(1-\alpha_n) d(u, Tx_n). \end{aligned}$$

In particular,

$$\frac{\sin d(u,Tx_n) - \sin(1-\alpha_n)d(u,Tx_n)}{\alpha_n d(u,Tx_n)} \cos d(p,x_n) \geq \frac{\sin \alpha_n d(u,Tx_n)}{\alpha_n d(u,Tx_n)} \cos d(p,u).$$

Note that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\sin d(u,Tx_n) - \sin(1-\alpha_n)d(u,Tx_n)}{\alpha_n d(u,Tx_n)} \\
&= \liminf_{n\to\infty} \frac{2\cos(1-\frac{\alpha_n}{2})d(u,Tx_n)\sin\frac{\alpha_n}{2}d(u,Tx_n)}{\alpha_n d(u,Tx_n)} \\
&\leq \liminf_{n\to\infty} \cos d(u,Tx_n) = \liminf_{n\to\infty} \cos d(u,x_n) = g(u).
\end{aligned}$$

This implies that

$$g(u)\liminf_{n\to\infty} \cos d(p,x_n) \geq \cos d(p,u).$$

In particular, since $\widehat{x} = T\widehat{x}$, we have

$$g(u)\liminf_{n\to\infty} \cos d(\widehat{x},x_n) \geq \cos d(\widehat{x},u).$$

Note that

$$g(\widehat{x})\cos d(u,\widehat{x}) \geq g(u).$$

Hence $\liminf_{n\to\infty} \cos d(\widehat{x},x_n) = 1$. In particular, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \widehat{x}$. This implies that

$$\cos d(u,\widehat{x})\cos d(p,\widehat{x}) \geq \cos d(p,u) \quad \text{for all } p \in \mathrm{F}(T).$$

Hence $\widehat{x} = P_{\mathrm{F}(T)}u$. The conclusion now follows from the double extract subsequence principle. ∎

Now, we can extend the above theorem to viscosity approximations. To be precise, the following theorem yields an implication of the Browder's type convergence theorem on viscosity approximations in the setting of CAT(1) spaces.

Corollary 3.6. *Let $T : X \to X$ be a nonexpansive mapping and $f : X \to X$ be a spherical contraction with a constant k. Suppose that $M := \operatorname{diam} X < \pi/2$ and $k < \cos M$. Define a sequence $\{x_n\}$ in X by*

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to the point $q = P_F f(q)$.

Proof. The proof of this corollary is very similar to that of Corollary 3.4, so the proof is omitted. ∎

Remark 3.7. The geometric properties of CAT(1) spaces are much more complicated than that of CAT(0) spaces. Here, we extend and supplement some results as follows.

(1) Our Theorem 3.3 extends Lemma 2.2 of [24] from the framework of CAT(0) spaces to that of CAT(1) spaces. In addition, the statement (i) of Theorem 3.3 is more natural than Theorem 3.4 of [16] because the condition $d(x_1, F(T)) \leq \pi/4$ is unnatural to assume for finding a fixed point of the mapping T.

(2) Our Corollary 3.4 entends Theorem 3.1 of [25] from the setting of CAT(0) spaces to that of CAT(1) spaces without the nice projection property.

4. Results in CAT(κ) Spaces

Throughout this section, we assume that X is a complete CAT(κ) space with a real number κ such that $d(u,v) < D_\kappa/2$ for all $u,v \in X$, where $D_\kappa = \infty$ if $\kappa \le 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$.

Note that $(X, \sqrt{\kappa}d)$ is a complete CAT(1) space for any $\kappa > 0$ and every CAT(κ') space is a CAT(κ) space whenever $\kappa' < \kappa$. As a consequence, Theorems 3.3 and 3.5 can be applied to the following two results, respectively.

Theorem 4.1. *Let $T : X \to X$ be a nonexpansive mapping and fix $u \in X$. For each $t \in (0,1)$ let x_t be a net given by $x_t = tu \oplus (1-t)Tx_t$. Then $\mathrm{F}(T) \neq \varnothing$ if and only if*

$$\mathrm{rad}_X(\{x_t\}) := \inf_{z\in X} \sup_{t\in(0,1)} d(x_t, z) < D_\kappa/2.$$

In this case, the following statements hold:

(i) *$q := \lim_{t\to 0^+} x_t$ exists and it is a unique fixed point of T which is nearest to u.*
(ii) *If μ is a Banach limit and $\{y_n\}$ is a and all approximating fixed point sequences of T, then $\cos d(u,q) \ge \mu_n \cos d(u,y_n)$.*

Theorem 4.2. *Let $T : X \to X$ be a nonexpansive mapping such that $\mathrm{F}(T) \neq \varnothing$. Let $u \in X$ be fixed and $\{x_n\}$ be a sequence in X defined by*

$$x_n = \alpha_n u \oplus (1-\alpha_n)Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(T)}u$.

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