



The Convergence Modes in Random Fuzzy Theory

Y. K. Liu and X. D. Dai

Abstract : Random fuzzy optimization problems include uncertain parameters defined only through probability and possibility distributions, they are inherently infinite-dimensional optimization problems that can rarely be solved directly. Thus, algorithms to solve such optimization problems must rely on intelligent computing and approximation schemes. This fact motivates us to discuss the modes of convergence in random fuzzy theory. Several new convergence concepts such as convergence almost uniform, and convergence in chance for sequences of random fuzzy variables were presented. Then the criteria for convergence almost sure, convergence almost uniform, and convergence in chance are established. Finally, the interconnections between convergence almost uniform and convergence almost sure, convergence almost uniform and convergence in chance, and convergence in chance and convergence almost sure are discussed. All these results can be regarded as the theoretical foundation of the intelligent computing and approximation schemes for random fuzzy optimization problems.

Keywords: Random fuzzy variable, mode of convergence; convergence criteria; random fuzzy programming

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1 Introduction

Random fuzzy theory is a combination of probability theory [14], and credibility theory [5, 9], it deals with a hybrid uncertain environment where linguistic and frequent nature coexist. Random fuzzy variable is an appropriate tool in this theory, it was introduced by Liu [4] to combine fuzziness and randomness in an optimization setting, e.g., random fuzzy dependent-chance programming [3], random fuzzy chance-constrained programming [10], and random fuzzy expected value model [11]. Because the random fuzzy optimization problems include random fuzzy variable parameters defined only through probability and possibility distributions, they are inherently infinite-dimensional optimization problems that can rarely be solved directly. Therefore, algorithms to solve such optimization prob-

lems must rely on intelligent computing and approximation schemes, which result in finite-dimensional optimization problems that can be tackled easily. This consideration motivates us to introduce several new modes of convergence in random fuzzy theory, which provide the theoretical foundation of intelligent computing and approximation schemes for uncertain programming [4, 8, 12].

The paper is organized as follows. First, in Section 2, we recall some concepts in uncertainty theory such as possibility space, credibility measure, random fuzzy variable, and the chance of a random fuzzy event. Section 3 presents the modes of convergence for sequences of random fuzzy variables, including uniform convergence, convergence almost uniform, convergence almost sure, convergence in chance, and convergence in distribution. The intent of Section 4 is to discuss the criteria for convergence almost sure, convergence almost uniform, and convergence in chance. The relations among convergence almost sure, convergence almost uniform, and convergence in chance are covered in Section 5, which generalize the results in probability theory and fuzzy measure theory [14, 16]. Finally, a brief summary is provided in Section 6.

2 Random Fuzzy Variables

Random fuzzy theory is an extension of probability theory [14], and possibility theory [1, 2, 13, 17, 18]. In this section, we review some concepts in this theory.

Given a universe Γ , an ample field [15] \mathcal{A} on Γ is a class of subsets of Γ that is closed under the formation of arbitrary unions, arbitrary intersections, and complement, and Pos is a possibility measure defined on \mathcal{A} .

Based on possibility measure, a self-dual set function Cr , called *credibility measure*, was defined as [7]:

$$\text{Cr}(A) = \frac{1}{2}(1 + \text{Pos}(A) - \text{Pos}(A^c)), A \in \mathcal{A} \quad (2.1)$$

where $A^c = \Gamma \setminus A$. The triplet $(\Gamma, \mathcal{A}, \text{Cr})$ is called a *credibility space* [6].

Definition 1. Let $(\Gamma, \mathcal{A}, \text{Cr})$ be a credibility space. A map X from Γ to \mathfrak{R} is called a fuzzy variable if for every $t \in \mathfrak{R}$,

$$\{\gamma \mid X(\gamma) \leq t\} \in \mathcal{A}. \quad (2.2)$$

The possibility distribution of the fuzzy variable X is defined as [6]:

$$\mu_X(t) = \min \{2\text{Cr}\{\gamma \mid X(\gamma) = t\}, 1\}, \quad t \in \mathfrak{R}. \quad (2.3)$$

Definition 2 ([4]). Let $(\Gamma, \mathcal{A}, \text{Cr})$ be a credibility space. A random fuzzy variable is a map $\xi : \Gamma \rightarrow \mathcal{R}_v$ such that for any Borel subset B of \mathfrak{R} ,

$$\xi^*(B)(\gamma) = \text{Pr} \{\omega \in \Omega \mid \xi_\gamma(\omega) \in B\} \quad (2.4)$$

is measurable with respect to γ in the sense of Definition 1, where \mathcal{R}_v is a collection of random variables defined on a probability space.

Definition 3 ([10]). Let ξ be a random fuzzy variable, and B a Borel subset of \mathfrak{R} . Then the (mean) chance, denoted Ch , of an event $\xi \in B$ is defined as

$$\begin{aligned} \text{Ch}\{\xi \in B\} \\ = \int_0^1 \text{Cr}\{\gamma \in \Gamma \mid \Pr\{\omega \in \Omega \mid \xi_\gamma(\omega) \in B\} \geq p\} dp. \end{aligned} \quad (2.5)$$

3 Modes of Convergence

In random fuzzy theory, we are interested in the following modes of convergence.

Definition 4. A sequence $\{\xi_n\}$ of random fuzzy variables is said to converge almost surely to a random fuzzy variable ξ , denoted by $\xi_n \xrightarrow{a.s.} \xi$, if there exist $E \in \mathcal{A}, F \in \Sigma$ with $\text{Cr}(E) = \text{Pr}(F) = 0$ such that for every $(\gamma, \omega) \in \Gamma \setminus E \times \Omega \setminus F$,

$$\lim_{n \rightarrow \infty} \xi_{n,\gamma}(\omega) \rightarrow \xi_\gamma(\omega).$$

Definition 5. A sequence $\{\xi_n\}$ of random fuzzy variables is said to converge uniformly to a random fuzzy variable ξ on $\Gamma \times \Omega$, denoted by $\xi_n \xrightarrow{u} \xi$, if

$$\lim_{n \rightarrow \infty} \sup_{(\gamma, \omega) \in \Gamma \times \Omega} |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| = 0.$$

Definition 6. A sequence $\{\xi_n\}$ of random fuzzy variables is said to converge almost uniformly to a random fuzzy variable ξ , denoted by $\xi_n \xrightarrow{a.u.} \xi$, if there exist two nonincreasing sequences $\{E_m\} \subset \mathcal{A}, \{F_m\} \subset \Sigma$ with $\lim_m \text{Cr}(E_m) = \lim_m \text{Pr}(F_m) = 0$ such that for each $m = 1, 2, \dots$, we have $\xi_n \xrightarrow{u} \xi$ on $\Gamma \setminus F_m \times \Omega \setminus E_m$.

Definition 7. A sequence $\{\xi_n\}$ of random fuzzy variables is said to converge in (mean) chance Ch to a random fuzzy variable ξ , denoted by $\xi_n \xrightarrow{\text{Ch}} \xi$, if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \text{Ch}\{|\xi_n - \xi| \geq \varepsilon\} = 0.$$

Let ξ be a random fuzzy variable. The (mean) chance function of ξ is denoted by $G_\xi(t) = \text{Ch}\{\xi \geq t\}$, $t \in \mathfrak{R}$. It is evident that G_ξ is a nonincreasing $[0, 1]$ -valued function.

Let $\{F_n\}$ and F be nonincreasing real-valued functions. The sequence $\{F_n\}$ is said to converge weakly to F , denoted by $F_n \xrightarrow{w} F$, if $F_n(t) \rightarrow F(t)$ for all continuity points t of F .

Definition 8. Let G_{ξ_n} and G_ξ be chance functions of random fuzzy variables ξ_n and ξ , respectively. The sequence $\{\xi_n\}$ is said to converge in distribution to ξ , denoted by $\xi_n \xrightarrow{d} \xi$, if $G_{\xi_n} \xrightarrow{w} G_\xi$.

4 Criteria for Convergence

The following proposition gives the criterion for convergence almost sure.

Proposition 1. *Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. Then $\xi_n \xrightarrow{a.s.} \xi$ if and only if for every $\varepsilon > 0$,*

$$\text{Ch} \left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\} \right) = 0. \quad (4.1)$$

Proof. First, it is easy to check that $\xi_n \xrightarrow{a.s.} \xi$ iff the limit $\xi_{n,\gamma} \xrightarrow{a.s.} \xi_\gamma$ holds with credibility 1 (w.c.1), i.e.,

$$\text{Cr} \left\{ \gamma \mid \xi_{n,\gamma} \xrightarrow{a.s.} \xi_\gamma \right\} = 1.$$

Therefore, there is $E \in \mathcal{A}$ with $\text{Cr}(E) = 0$ such that for every $\gamma \in \Gamma \setminus E$,

$$\xi_{n,\gamma} \xrightarrow{a.s.} \xi_\gamma,$$

i.e., for every $\varepsilon > 0$,

$$\text{Pr} \left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{ \omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon \} \right) = 0$$

with credibility 1, which is equivalent to

$$\text{Ch} \left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\} \right) = 0.$$

The proof is complete. \square

The criterion for convergence almost uniform is established by the following proposition.

Proposition 2. *Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. If $\xi_n \xrightarrow{a.u.} \xi$, then for every $\varepsilon > 0$, the following limit holds w.c.1*

$$\lim_{m \rightarrow \infty} \text{Pr} \left(\bigcup_{n=m}^{\infty} \{ \omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon \} \right) = 0. \quad (4.2)$$

Conversely, if γ is a discrete fuzzy variable assuming finite number of values, then Eq. (4.2) implies $\xi_n \xrightarrow{a.u.} \xi$.

Proof. If

$$\xi_n \xrightarrow{a.u.} \xi,$$

then there exist two nonincreasing sequences $\{E_m\} \subset \mathcal{A}$, $\{F_m\} \subset \Sigma$ with $\lim_m \text{Cr}(E_m) = \lim_m \text{Pr}(F_m) = 0$ such that for each $m = 1, 2, \dots$, $\xi_n \xrightarrow{u.} \xi$ on $\Gamma \setminus E_m \times \Omega \setminus F_m$.

Let

$$E = \bigcap_{m=1}^{\infty} E_m.$$

Then $\text{Cr}(E) = 0$, and for every $\gamma \in \Gamma \setminus E$, there is a positive integer m_γ such that $\gamma \in \Gamma \setminus E_{m_\gamma}$. Since $\{E_m\}$ is nonincreasing, we have $\gamma \in \Gamma \setminus E_m$ whenever $m \geq m_\gamma$. Therefore, there is a subsequence $\{F_m, m \geq m_\gamma\}$ of $\{F_m\}$ such that for each $m_\gamma, m_\gamma + 1, \dots$, the sequence $\{\xi_{n,\gamma}\}$ converges to ξ_γ uniformly on F_m , i.e.,

$$\xi_{n,\gamma} \xrightarrow{a.u.} \xi_\gamma$$

with credibility 1.

We now show that $\xi_{n,\gamma} \xrightarrow{a.u.} \xi_\gamma$ w.c.1 implies Eq. (4.2).

In fact, for any $\delta > 0$, there exists $F_\gamma \in \Sigma$ with $\text{Pr}(F_\gamma) < \delta$ such that $\{\xi_{n,\gamma}\}$ converges to ξ_γ uniformly on $\Omega \setminus F_\gamma$. Thus, for every $\varepsilon > 0$, there exists a positive integer $m(\varepsilon, \gamma)$ such that for all $\omega \in \Omega \setminus F_\gamma$,

$$|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| < \varepsilon$$

whenever $n \geq m$. Therefore, one has

$$\Omega \setminus F_\gamma \subset \bigcap_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| < \varepsilon\},$$

or

$$\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \subset F_\gamma,$$

which implies

$$\begin{aligned} \text{Pr}\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) \\ \leq \text{Pr}(F_\gamma) < \delta. \end{aligned}$$

Letting $\delta \rightarrow 0$, we have

$$\lim_{m \rightarrow \infty} \text{Pr}\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) = 0,$$

which verifies Eq. (4.2).

Conversely, suppose Eq. (4.2) is valid, we prove

$$\xi_n \xrightarrow{a.u.} \xi.$$

By supposition, assume that γ has the following possibility distribution

$$\gamma \sim \begin{pmatrix} \gamma_1, & \gamma_2, & \cdots, & \gamma_N \\ p_1, & p_2, & \cdots, & p_N \end{pmatrix}$$

with $p_i > 0$ and $\max_{i=1}^N p_i = 1$. Since for each $i = 1, 2, \dots, N$, we have

$$\lim_{m \rightarrow \infty} \Pr \left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n, \gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq \varepsilon\} \right) = 0.$$

Then for every $\delta \in (0, 1)$, and each $k = 1, 2, \dots$, there exists a positive integer m_k such that for $i = 1, 2, \dots, N$,

$$\Pr \left(\bigcup_{n=m_k}^{\infty} \{\omega \in \Omega \mid |\xi_{n, \gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\} \right) < \delta/2^{k+i}$$

Letting

$$F = \bigcup_{i=1}^N \bigcup_{k=1}^{\infty} \bigcup_{n=m_k}^{\infty} \{\omega \in \Omega \mid |\xi_{n, \gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\},$$

then

$$\begin{aligned} \Pr(F) &= \Pr \left(\bigcup_{i=1}^N \bigcup_{k=1}^{\infty} \bigcup_{n=m_k}^{\infty} \{\omega \mid |\xi_{n, \gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\} \right) \\ &\leq \sum_{i=1}^N \sum_{k=1}^{\infty} \Pr \left(\bigcup_{n=m_k}^{\infty} \{\omega \mid |\xi_{n, \gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\} \right) \\ &< \delta. \end{aligned}$$

In addition, for each $k = 1, 2, \dots$, one has

$$\sup_{1 \leq i \leq N} \sup_{\omega \in \Omega \setminus F} |\xi_{n, \gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| < 1/k$$

whenever $n \geq m_k$, which implies

$$\xi_n \xrightarrow{a.u.} \xi.$$

The proof is complete. \square

The next proposition deals with the criterion for convergence in chance.

Proposition 3. Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. Then $\xi_n \xrightarrow{\text{Ch}} \xi$ if and only if for every $\varepsilon > 0$,

$$\Pr\{\omega \mid |\xi_{n, \gamma}(\omega) - \xi_{\gamma}(\omega)| \geq \varepsilon\} \xrightarrow{\text{Cr}} 0.$$

Proof. Assume that

$$\xi_n \xrightarrow{\text{Ch}} \xi,$$

then for every $\varepsilon > 0$ and $\eta > 0$,

$$\begin{aligned} &\text{Ch}\{|\xi_n - \xi| \geq \varepsilon\} \\ &= \int_0^1 \text{Cr}\{\gamma \mid \Pr\{|\xi_{n, \gamma}(\omega) - \xi_{\gamma}(\omega)| \geq \varepsilon\} \geq p\} dp \\ &\geq \eta \text{Cr}\{\gamma \mid \Pr\{|\xi_{n, \gamma}(\omega) - \xi_{\gamma}(\omega)| \geq \varepsilon\} \geq \eta\}, \end{aligned}$$

which implies

$$\Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \xrightarrow{\text{Cr}} 0.$$

On the other hand, if

$$\Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \xrightarrow{\text{Cr}} 0,$$

then for every $p \in (0, 1]$, one has

$$\lim_{n \rightarrow \infty} \text{Cr}\{\gamma \mid \Pr\{|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq p\} = 0.$$

Applying the bounded convergence theorem of integral sequence, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \text{Ch}\{|\xi_n - \xi| \geq \varepsilon\} \\ &= \int_0^1 \lim_{n \rightarrow \infty} \text{Cr}\{\gamma \mid \Pr\{|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq p\} dp \\ &= 0, \end{aligned}$$

which completes the proof. \square

5 Interconnections among Convergence

The following theorem compares convergence a.u. and convergence a.s..

Theorem 1. *Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. If $\xi_n \xrightarrow{a.u.} \xi$, then $\xi_n \xrightarrow{a.s.} \xi$.*

Proof. Assume $\xi_n \xrightarrow{a.u.} \xi$. By Proposition 2, we have

$$\lim_{m \rightarrow \infty} \Pr\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) = 0$$

with credibility 1. By the upper semicontinuity of probability, we have

$$\Pr\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) = 0$$

with credibility 1, which implies

$$\text{Ch}\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) = 0.$$

It follows from Proposition 1 that

$$\xi_n \xrightarrow{a.s.} \xi.$$

The proof is complete. \square

The following theorem compares convergence a.u. and convergence in chance.

Theorem 2. Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. If $\xi_n \xrightarrow{a.u.} \xi$, then $\xi_n \xrightarrow{\text{Ch}} \xi$.

Proof. Suppose

$$\xi_n \xrightarrow{a.u.} \xi.$$

Then for any given $\delta > 0$, there exist $E \in \mathcal{A}$ and $F \in \Sigma$ with $\text{Cr}(E) < \delta$, $\text{Pr}(F) < \delta$ such that $\{\xi_n\}$ converges to ξ uniformly on $\Gamma \setminus E \times \Omega \setminus F$. For any given $\varepsilon > 0$, there exists some positive integer N such that for every $(\gamma, \omega) \in \Gamma \setminus E \times \Omega \setminus F$,

$$|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| < \varepsilon$$

whenever $n \geq N$. As a consequence, one has

$$\begin{aligned} & \text{Ch}\{|\xi_m - \xi| \geq \varepsilon\} \\ &= \int_0^1 \text{Cr}\{\gamma \in \Gamma \mid \text{Pr}\{|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq p\} dp \\ &\leq \int_0^1 \text{Cr}\{\gamma \in \Gamma \setminus E \mid \text{Pr}\{|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq p\} dp \\ &\quad + \int_0^1 \text{Cr}\{\gamma \in E \mid \text{Pr}\{|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq p\} dp \\ &\leq \int_0^\delta dp + \int_0^1 \delta dp = 2\delta \end{aligned}$$

whenever $n \geq N$. It follows from the arbitrary of δ that

$$\xi_n \xrightarrow{\text{Ch}} \xi,$$

which completes the proof. \square

The following theorem compares convergence in chance and convergence a.s..

Theorem 3. Suppose that γ is a discrete fuzzy variable assuming finite number of values, if $\xi_n \xrightarrow{\text{Ch}} \xi$, then there exists some subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that

$$\xi_{n_k} \xrightarrow{a.s.} \xi.$$

Proof. Assume that γ has the following possibility distribution

$$\gamma \sim \begin{pmatrix} \gamma_1, & \gamma_2, & \cdots, & \gamma_N \\ p_1, & p_2, & \cdots, & p_N \end{pmatrix}$$

with $p_i > 0$ and $\max_{i=1}^N p_i = 1$. Since

$$\xi_n \xrightarrow{\text{Ch}} \xi,$$

by Proposition 3, for every $\varepsilon > 0$,

$$\Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \xrightarrow{\text{Cr}} 0.$$

Noting that in credibility theory, convergence in credibility implies convergence almost sure [5], one has

$$\Pr\{\omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq \varepsilon\} \rightarrow 0, \quad i = 1, 2, \dots, N.$$

That is,

$$\xi_{n,\gamma_i} \xrightarrow{\text{Pr}} \xi_{\gamma_i}$$

for $i = 1, 2, \dots, N$. It follows from Riesz's theorem [14] that there exists some subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that

$$\xi_{n_k,\gamma_i}(\omega) \rightarrow \xi_{\gamma_i}(\omega)$$

for every $\omega \in \Omega$ and $i = 1, 2, \dots, N$. The proof of the theorem is complete. \square

6 Conclusion

The major new results of this paper include the following three aspects.

- (i) Several new modes of convergence for random fuzzy variables such as convergence almost uniform, convergence in chance and convergence in distribution were introduced.
- (ii) The convergence criteria for convergence almost sure, convergence almost uniform, and convergence in chance were provided.
- (iii) The interconnections between convergence almost uniform and convergence almost sure, convergence almost uniform and convergence in chance, and convergence in chance and convergence almost sure were established.

To conclude, we want to mention several potential applications of the obtained results in this paper. For instance, we will employ the convergent results obtained to design intelligent algorithms and approximation approaches to random fuzzy optimization problems, which result in finite-dimensional optimization problems that can be tackled easily. By using the proposed modes of convergence, we can deal with the convergence of the optimal solutions of the finite-dimensional approximating optimization problems to the optimal solutions of the original infinite-dimensional optimization problems.

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Yan-Kui Liu and Xiao-Dong Dai
College of Mathematics & Computer Science,
Hebei University, Baoding 071002, Hebei, China.
e-mail:liuyankui@tsinghua.org.cn