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On the Solutions of a Third Order Rational Difference Equation

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Abstract In this paper, we discuss the global behavior of all admissible solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B - Cx_nx_{n-1}x_{n-2}}, \quad n = 0, 1, \ldots,$$

where A, B, C are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

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1. Introduction

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [1–9] and the references therein. Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic processes, etc. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.
In [10], M. Aloqeili has discussed the stability properties and semicycle behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_nx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

with real initial conditions and positive real number a.

In this paper, we discuss the global behavior of all admissible solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B - Cx_nx_{n-1}x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

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where A, B, C are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers. The difference equation (1.2) is a more extended case of the difference equation (1.1).

2. Linearized Stability and Solutions of Equation (1.2)

In this section we introduce an explicit formula for the solutions of the difference equation (1.2) and study its linearized stability.

It is convenient to reduce the parameters on which equation (1.2) depends.

The change of variables $\sqrt[3]{\frac{C}{A}}x_n = y_n$ reduces equation (1.2) to the equation

$$y_{n+1} = \frac{y_{n-2}}{r - y_n y_{n-1} y_{n-2}}, \quad n = 0, 1, \dots, \tag{2.1}$$

where $r = \frac{B}{A}$.

We will investigate the global behavior of the difference equation (2.1) rather than equation (1.2).

If $y_{-i} = 0$ for some but not all $i \in \{0, 1, 2\}$, then

$$y_n = \begin{cases} (\frac{1}{r})^{\frac{n-1}{3}+1} y_{-2} & , n = 1, 4, 7, ..., \\ (\frac{1}{r})^{\frac{n-2}{3}+1} y_{-1} & , n = 2, 5, 8, ..., \\ (\frac{1}{r})^{\frac{n-k}{3}+1} y_0 & , n = 3, 6, 9, ... \end{cases} \tag{2.2}$$

Now suppose that $y_{-i} \neq 0$, for all $i \in \{0, 1, 2\}$ and let $\alpha = y_{-2}y_{-1}y_0$.

From equation (2.1) and using the substitution $l_n = \frac{1}{y_n y_{n-1} y_{n-2}}$, we can obtain the linear nonhomogeneous difference equation

$$l_{n+1} = r l_n - 1, \quad l_0 = \frac{1}{y_{-2}y_{-1}y_0} = \frac{1}{\alpha}, \quad n = 0, 1, ... \tag{2.3}$$

We can deduce the forbidden set and the solution of equation (2.1) from the solution of equation (2.3).

Proposition 2.1. *The forbidden set F of equation (2.1) is*

$$F = \bigcup_{n=0}^{\infty} \{(u_0, u_1, u_2) : u_0 u_1 u_2 = \frac{r}{\sum_{l=0}^{n} (\frac{1}{r})^l}\}.$$

Proof. The solution of equation (2.3) is

$$l_n = l_0 r^n - \sum_{i=0}^{n-1} r^i = \frac{r^n - \alpha \sum_{i=0}^{n-1} r^i}{\alpha}.$$

Then

$$y_n y_{n-1} y_{n-2} = \frac{\alpha}{r^n - \alpha \sum_{i=0}^{n-1} r^i}, \quad n = 0, 1, ... \tag{2.4}$$

When $n = n_0$ for some $n_0 \in \mathbb{N}$, if we set $\alpha = \frac{r}{\sum_{l=0}^{n_0} (\frac{1}{r})^l}$ in equation (2.4), we obtain

$$y_{n_0} y_{n_0-1} y_{n_0-2} = r.$$

Therefore, y_{n_0+1} is undefined.

On the other hand, from equation (2.3) we have that

$$y_{n-1}y_{n-2}y_{n-3} = \frac{ry_ny_{n-1}y_{n-2}}{1+y_ny_{n-1}y_{n-2}}. \tag{2.5}$$

For a fixed $n_0 \in \mathbb{N}$, suppose that we have y_{n_0+1} is undefined. This implies that

$$y_{n_0}y_{n_0-1}y_{n_0-2} = r.$$

Hence we have the following:

$$y_{n_0-1}y_{n_0-2}y_{n_0-3} = \frac{r^2}{1+r} = \frac{r}{\sum_{i=0}^{1}(\frac{1}{r})^i},$$

$$y_{n_0-2}y_{n_0-3}y_{n_0-4} = \frac{r^3}{1+r+r^2} = \frac{r}{\sum_{i=0}^{2}(\frac{1}{r})^i},$$

$$\vdots$$

$$\alpha = y_0y_{-1}y_{-2} = \frac{r^{n_0+1}}{1+r+r^2+...+r^{n_0}} = \frac{r}{\sum_{i=0}^{n_0}(\frac{1}{r})^i}. \tag{2.6}$$

This completes the proof. ■

Theorem 2.2. *Let y_{-2}, y_{-1} and y_0 be real numbers such that $\alpha = y_{-2}y_{-1}y_0 \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$. Then the solution of equation (2.1) is*

$$y_n = \begin{cases} y_{-2}\prod_{j=0}^{\frac{n-1}{3}} \frac{r^{3j}-\alpha\sum_{l=0}^{3j-1} r^l}{r^{3j+1}-\alpha\sum_{l=0}^{3j} r^l} & ,n=1,4,7,...,\\ y_{-1}\prod_{j=0}^{\frac{n-2}{3}} \frac{r^{3j+1}-\alpha\sum_{l=0}^{3j} r^l}{r^{3j+2}-\alpha\sum_{l=0}^{3j+1} r^l} & ,n=2,5,8,...,\\ y_{0}\prod_{j=0}^{\frac{n-3}{3}} \frac{r^{3j+2}-\alpha\sum_{l=0}^{3j+1} p^l}{r^{3j+3}-\alpha\sum_{l=0}^{3j+2} r^l} & ,n=3,6,9,.... \end{cases} \tag{2.7}$$

Corollary 2.3. *Assume that $r = 1$ and $\alpha = y_{-2}y_{-1}y_0 \neq \frac{1}{n+1}$ for any $n \in N$. Then the solution of equation (2.1) is*

$$y_n = \begin{cases} y_{-2}\prod_{j=0}^{\frac{n-1}{2}} \frac{1-3j\alpha}{1-(3j+1)\alpha} & ,n=1,4,7,...,\\ y_{-1}\prod_{j=0}^{\frac{n-2}{3}} \frac{1-(3j+1)\alpha}{1-(3j+2)\alpha} & ,n=2,5,8,...,\\ y_{0}\prod_{j=0}^{\frac{n-3}{3}} \frac{1-(3j+2)\alpha}{1-(3j+3)\alpha} & ,n=3,6,9,.... \end{cases} \tag{2.8}$$

Corollary 2.4. *Assume that $r < 1$ and let $\{y_n\}_{n=-2}^{\infty}$ be a nontrivial solution of equation (2.1). If $\alpha = y_{-2}y_{-1}y_0 = 0$, then the solution $\{y_n\}_{n=-2}^{\infty}$ is unbounded.*

Example (1) Figure 1 shows that if $\{y_n\}_{n=-2}^{\infty}$ is a nontrivial solution of the equation

$$y_{n+1} = \frac{y_{n-2}}{0.3 - y_ny_{n-1}y_{n-2}}, \quad n = 0, 1, \ldots,$$

with initial conditions $y_{-2} = 0$, $y_{-1} = 2$, $y_0 = 1.5$, then the solution $\{y_n\}_{n=-2}^{\infty}$ is unbounded.

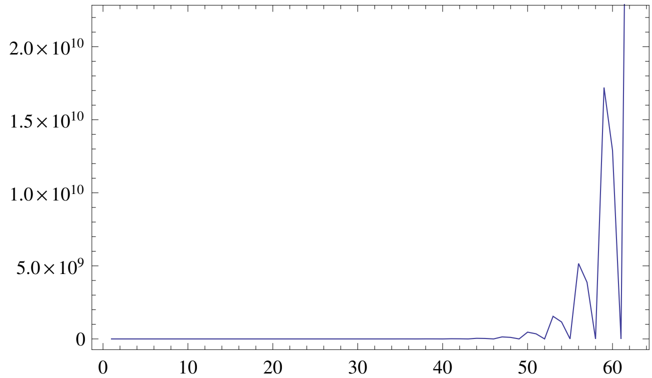

FIGURE 1. The difference equation $y_{n+1} = \frac{y_{n-2}}{0.3 - y_n y_{n-1} y_{n-2}}$

We end this section with the discussion of the local stability of the equilibrium points of equation (2.1).

It is clear that the point $\bar{y} = 0$ is always an equilibrium point of equation (2.1). Moreover, equation (2.1) has a positive equilibrium point $\bar{y} = \sqrt[3]{r-1}$ if $r > 1$ and a negative equilibrium point $\bar{y} = \sqrt[3]{r-1}$ if $r < 1$.

The following theorem describes the local behavior of the equilibrium points.

Theorem 2.5. *The following statements are true.*

(1) The equilibrium point $\bar{y} = 0$ is locally asymptotically stable if $r > 1$ and unstable equilibrium point if $r < 1$.
(2) If $r > 1$, then the equilibrium point $\bar{y} = \sqrt[3]{r-1}$ is unstable equilibrium point.
(3) If $r < 1$, then the equilibrium point $\bar{y} = \sqrt[3]{r-1}$ is nonhyperbolic equilibrium point.

Proof. The linearized equation associated with equation (2.1) about an equilibrium point $\bar{y}$ is

$$z_{n+1} - \frac{\bar{y}^3}{(r-\bar{y}^3)^2}(z_n + z_{n-1}) - \frac{r}{(r-\bar{y}^3)^2} z_{n-2} = 0, \quad n = 0, 1, \ldots. \tag{2.9}$$

Its characteristic equation associated with this equation is

$$\lambda^3 - \frac{\bar{y}^3}{(r-\bar{y}^3)^2}(\lambda^2 + \lambda) - \frac{r}{(r-\bar{y}^3)^2} = 0. \tag{2.10}$$

Therefore, (1) follows directly.
Equation (2.9) about a nonzero equilibrium point $\bar{y}$ is

$$z_{n+1} - (r-1)(z_n + z_{n-1}) - r z_{n-2} = 0, \quad n = 0, 1, \ldots. \tag{2.11}$$

Also equation (2.10) becomes

$$\lambda^3 - (r-1)(\lambda^2 + \lambda) - r = 0. \tag{2.12}$$

Let

$$f(\lambda) = \lambda^3 - (r-1)(\lambda^2 + \lambda) - r.$$

We can see that

$$f(\lambda) = (\lambda - r)(1 + \lambda + \lambda^2).$$

Then the roots of equation (2.12) are r and 2 other roots with modulus 1.
Therefore, (2) and (3) follow directly. ∎

3. Global Behavior of Equation (2.1)

The solution of equation (2.1) can be written as

$$y_{3m+i} = y_{-3+i} \prod_{j=0}^{m} \frac{r^{3j+i-1} - \alpha \sum_{l=0}^{3j+i-2} r^l}{r^{3j+i} - \alpha \sum_{l=0}^{3j+i-1} r^l}, \quad i = 1,2,3 \quad \text{and} \quad m = 0,1,.... \quad (3.1)$$

But as

$$\frac{r^{3j+i-1} - \alpha \sum_{l=0}^{3j+i-2} r^l}{r^{3j+i} - \alpha \sum_{l=0}^{3j+i-1} r^l} = \frac{r^{3j+i-1}\theta - \alpha}{r^{3j+i}\theta - \alpha}, \quad \text{where} \quad \theta = 1 - r + \alpha.$$

We can write

$$y_{3m+i} = y_{-3+i} \prod_{j=0}^{m} \beta_i(j), \quad i = 1,2,3 \quad \text{and} \quad m = 0,1,...,$$

where

$$\beta_i(j) = \frac{r^{3j+i-1}\theta - \alpha}{r^{3j+i}\theta - \alpha}, \quad i = 1,2,3.$$

Theorem 3.1. *Assume that $\{y_n\}_{n=-2}^{\infty}$ is a solution of equation (2.1) such that $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$. If $\alpha = r - 1$, then $\{y_n\}_{n=-2}^{\infty}$ is a periodic solution with period 3.*

Proof. It is sufficient to see that, if $\alpha = r - 1$, then $\theta = 0$. Therefore,

$$y_{3m+i} = y_{-3+i} \prod_{j=0}^{m} \frac{r^{3j+i-1}\theta - \alpha}{r^{3j+i}\theta - \alpha} = y_{-3+i}, \quad i = 1,2,3.$$ ∎

Proposition 3.2. *Assume that $r < 1$ and let $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$. Then there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.*

Proof. We have three situations:

(1) If $\alpha < r - 1 < 0$, then $0 < \theta - \alpha < -\alpha$. Hence for each $j \in \mathbb{N}$, $r^{3j+i-1}\theta - \alpha > \theta - \alpha > 0$, $i = 1,2,3$. Then

$$\beta_i(j) = \frac{r^{3j+i-1}\theta - \alpha}{r^{3j+i}\theta - \alpha} > 0 \quad \text{for all} \quad j \geq 0.$$

(2) If $r - 1 < \alpha < 0$, then $0 < -\alpha < \theta - \alpha$.
But

$$\lim_{j\to\infty} \beta_i(j) = \lim_{j\to\infty} \frac{r^{3j+i-1}\theta - \alpha}{p^{3j+i}\theta - \alpha} = 1.$$

Then there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.

(3) When $r - 1 < 0 < \alpha$, the situation is similar to that in (2).
In all cases there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.

∎

Theorem 3.3. *Assume that $\{y_n\}_{n=-2}^{\infty}$ is a solution of equation (2.1) such that $\alpha \neq r-1$ and $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$. Then the following statements are true.*

(1) If $r > 1$, then $\{y_n\}_{n=-2}^{\infty}$ converges to $\bar{y} = 0$.
(2) If $r < 1$ and $\alpha \neq 0$, then $\{y_n\}_{n=-2}^{\infty}$ is bounded.

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of equation (2.1) such that $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$.

The condition $\alpha \neq r-1$ ensures that the solution $\{y_n\}_{n=-2}^{\infty}$ is not a 3-periodic solution.

(1) Suppose that $r > 1$. It is clear that, as the equilibrium point $\frac{1}{r-1}$ of equation (2.3) is repelling, every non-constant solution of equation (2.3) approaches ∞ or $-\infty$ according to the value of $l_0 = \frac{1}{\alpha}$.

We shall consider the following situations:

(a) If $\alpha = \frac{1}{l_0} < 0$, then according to equation (2.3), we have $y_n y_{n-1} y_{n-2} = \frac{1}{l_n} < 0$, for each $n \in \mathbb{N}$. Therefore,

$$\mid y_{n+1} \mid = \frac{\mid y_{n-2} \mid}{\mid r - y_n y_{n-1} y_{n-2} \mid} < \frac{\mid y_{n-2} \mid}{r}, \quad n = 0, 1, \ldots .$$

(b) If $0 < \alpha = \frac{1}{l_0} < r-1$, then according to equation (2.3), $y_n y_{n-1} y_{n-2} = \frac{1}{l_n} \to 0$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that $0 < y_n y_{n-1} y_{n-2} < r-1$ for each $n > n_0$. Therefore,

$$\mid y_{n+1} \mid = \frac{\mid y_{n-2} \mid}{\mid r - y_n y_{n-1} y_{n-2} \mid} < \mid y_{n-2} \mid, \quad n \geq n_0.$$

(c) If $r-1 < \alpha = \frac{1}{l_0} < r$, then according to equation (2.3), there exists $n_0 \in \mathbb{N}$ such that $y_n y_{n-1} y_{n-2} = \frac{1}{l_n} < 0$ for each $n \geq n_0$. Therefore,

$$\mid y_{n+1} \mid = \frac{\mid y_{n-2} \mid}{\mid r - y_n y_{n-1} y_{n-2} \mid} < \frac{\mid y_{n-2} \mid}{r}, \quad n \geq n_0.$$

(d) If $\alpha = \frac{1}{l_0} > r > 0$, then according to equation (2.3), $y_n y_{n-1} y_{n-2} = \frac{1}{l_n} < 0$ for each $n > 0$. Therefore,

$$\mid y_{n+1} \mid = \frac{\mid y_{n-2} \mid}{\mid p - y_n y_{n-1} y_{n-2} \mid} < \frac{\mid y_{n-2} \mid}{r}, \quad n = 0, 1, \ldots .$$

In all cases, $y_n \to 0$ as $n \to \infty$.

(2) Suppose that $r < 1$. Using proposition (3.2), there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$. Hence for each $i \in \{1, 2, 3\}$, we have for large values of m

$$\begin{aligned} y_{3m+i} &= y_{-3+i} \prod_{j=0}^{m} \beta_i(j) = y_{-3+i} \prod_{j=0}^{j_0-1} \beta_i(j) \prod_{j=j_0}^{m} \beta_i(j) \\ &= y_{-3+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp(\ln \prod_{j=j_0}^{m} \beta_i(j)) \\ &= y_{-3+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp\big(\sum_{j=j_0}^{m} \ln \beta_i(j)\big). \end{aligned}$$

It is sufficient to test the convergence of the series $\sum_{j=j_0}^{\infty} \mid \ln \beta_i(j) \mid$.
But

$$\lim_{j\to\infty} \frac{\ln \beta_i(j+1)}{\ln \beta_i(j)} = \frac{0}{0}.$$

Then

$$\begin{aligned}\lim_{j\to\infty} \frac{\ln \beta_i(j+1)}{\ln \beta_i(j)} &= \lim_{j\to\infty} \frac{\frac{d}{dj}(\ln \beta_i(j+1))}{\frac{d}{dj}(\ln \beta_i(j))} \\ &= \lim_{j\to\infty} \frac{\frac{3(r-1)(\ln r)\theta r^{3(j+1)+i-1}}{(r^{3(j+1)+i-1}\theta-\alpha)(r^{3(j+1)+i}\theta-\alpha)}}{\frac{3(r-1)(\ln r)\theta r^{3j+i-1}}{(r^{3j+i-1}\theta-\alpha)(r^{3j+i}\theta-\alpha)}} \\ &= r^3 < 1.\end{aligned}$$

It follows from D' Alemberts' test that the series $\sum_{j=j_0}^{\infty} \mid \ln \beta_i(j) \mid$ is convergent. This ensures that the solution $\{y_n\}_{n=-2}^{\infty}$ is bounded.

■

Example (2) Figure 2 shows that if $\{y_n\}_{n=-2}^{\infty}$ is a solution of the equation

$$y_{n+1} = \frac{y_{n-2}}{2 - y_n y_{n-1} y_{n-2}}, \quad n = 0, 1, \ldots,$$

with initial conditions $y_{-2} = 2$, $y_{-1} = 1$, $y_0 = 2$ ($\alpha \neq r - 1$ and $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$) where ($r = 2$), then the solution $\{y_n\}_{n=-2}^{\infty}$ converges to zero.

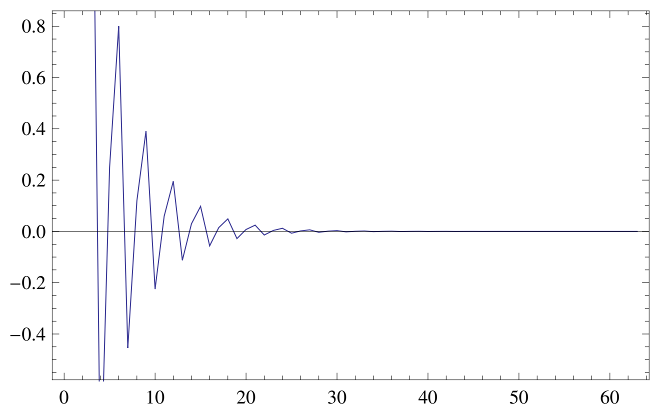

FIGURE 2. The difference equation $y_{n+1} = \frac{y_{n-2}}{2-y_n y_{n-1} y_{n-2}}$

Example (3) Figure 3 shows that if $\{y_n\}_{n=-2}^{\infty}$ is a solution of the equation

$$y_{n+1} = \frac{y_{n-2}}{0.8 - y_n y_{n-1} y_{n-2}}, \quad n = 0, 1, \ldots,$$

with initial conditions $y_{-2} = 0.2$, $y_{-1} = -1$, $y_0 = 0.3$ ($\alpha \neq r - 1$, $\alpha \neq 0$ and $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$) where ($r = 0.8$), then the solution $\{y_n\}_{n=-2}^{\infty}$ is bounded.
Moreover, the solution $\{y_n\}_{n=-2}^{\infty}$ converges to 3-periodic solution.

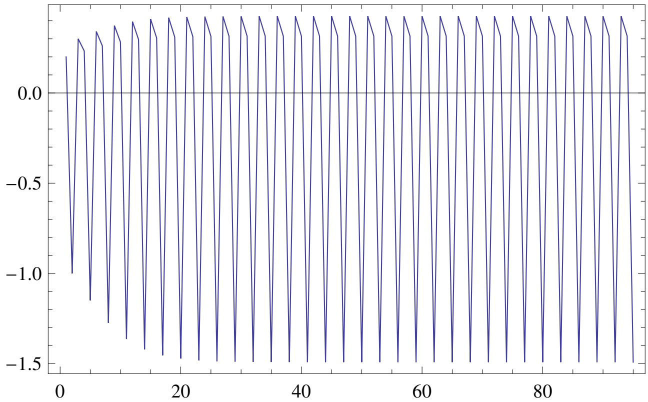

FIGURE 3. The difference equation $y_{n+1} = \frac{y_{n-2}}{0.8 - y_n y_{n-1} y_{n-2}}$

We can observe in case $r < 1$ that, the behavior of the solution $\{y_n\}_{n=-2}^{\infty}$ is totally different according to whether $\alpha = 0$ or $\alpha \neq 0$. This is obvious in corollary (2.4) and theorem (3.3).

Theorem 3.4. *Assume that $r < 1$ and let $\{y_n\}_{n=-2}^{\infty}$ be a solution of equation (2.1) such that $\alpha \neq \frac{r}{\sum_{l=0}^{n}(\frac{1}{r})^l}$ for any $n \in N$. Then $\{y_n\}_{n=-2}^{\infty}$ converges to a 3-periodic solution $\{\rho_0, \rho_1, \rho_2\}$ of equation (2.1) with $\rho_0\rho_1\rho_2 = r - 1$.*

Proof. By theorem (3.3), there exist 3 real numbers $\rho_i \in \mathbb{R}$ such that

$$\lim_{j\to\infty} y_{3m+i} = \rho_i, \quad i \in \{0, 1, 2\}.$$

If we set $n = 3m + i - 1$, $i = 0, 1, 2$ in equation (2.1), we get

$$y_{3m+i} = \frac{y_{3(m-1)+i}}{r - y_{3(m-1)+i+2}\, y_{3(m-1)+i+1}\, y_{3(m-1)+i}}, \quad i = 0, 1, 2 \quad \text{and} \quad m = 0, 1, \ldots.$$

By taking the limit as $m \to \infty$, we obtain

$$\rho_i = \frac{\rho_i}{r - \rho_{i+2}\rho_{i+1}\rho_i}, \quad i = 0, 1, 2.$$

But from equation (2.3) we have $y_n y_{n-1} y_{n-2} = \frac{1}{l_n} \to r - 1$ as $n \to \infty$.
This implies that $y_{3m} y_{3m+1} y_{3m+2} \to \rho_0\rho_1\rho_2 = r - 1$ as $m \to \infty$.
Therefore, $\{y_n\}_{n=-2}^{\infty}$ converges to the 3-periodic solution

$$\{\ldots, \rho_0, \rho_1, \frac{r-1}{\rho_0\rho_1}, \rho_0, \rho_1, \frac{r-1}{\rho_0\rho_1}, \ldots\}.$$

■

4. CASE $r = 1$

We end this work with the discussion of the case $r = 1$.
If we set $r = 1$ in equation (3.1), we get

$$y_{3m+i} = y_{-3+i} \prod_{j=0}^{m} \zeta_i(j), \quad i = 1, 2, 3 \quad \text{and} \quad m = 0, 1, \ldots. \tag{4.1}$$

where

$$\zeta_i(j) = \frac{1-\alpha(3j+i-1)}{1-\alpha(3j+i)}, \quad i = 1,2,3.$$

Proposition 4.1. *Assume that $r = 1$ and let $\alpha \neq \frac{1}{n+1}$ for any $n \in N$. Then there exists $j_0 \in \mathbb{N}$ such that $\zeta_i(j) > 0$ for all $j \geq j_0$.*

Proof. When $\alpha < 0$, the result is obvious where $\zeta_i(j) > 0$ for each $j \in \mathbb{N}$.
When $\alpha > 0$, It is sufficient to see that,

$$\lim_{j\to\infty} \zeta_i(j) = \lim_{j\to\infty} \frac{1-\alpha(3j+i-1)}{1-\alpha(3j+i)} = 1.$$

This implies that, there exists $j_0 \in \mathbb{N}$ such that $\zeta_i(j) > 0$ for all $j \geq j_0$. ∎

Theorem 4.2. *Assume that $r = 1$. Then any solution $\{y_n\}_{n=-2}^{\infty}$ of equation (2.1) with $\alpha \neq 0$ and $\alpha \neq \frac{1}{n+1}$ for any $n \in N$ converges to zero.*

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of equation (2.1) such that $\alpha \neq \frac{1}{n+1}$ for any $n \in N$. The condition $\alpha \neq 0$ ensures that the solution $\{y_n\}_{n=-2}^{\infty}$ is not a 3-periodic solution. Using proposition (4.1), there exists $j_0 \in \mathbb{N}$ such that $\zeta_i(j) > 0$ for all $j \geq j_0$. Hence for each $i \in \{1,2,3\}$, we have for large m

$$\begin{aligned} y_{3m+i} &= y_{-3+i} \prod_{j=0}^{m} \zeta_i(j) = y_{-3+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \prod_{j=j_0}^{m} \zeta_i(j) \\ &= y_{-3+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(\ln \prod_{j=j_0}^{m} \zeta_i(j)) \\ &= y_{-3+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(\sum_{j=j_0}^{m} \ln \zeta_i(j)) \\ &= y_{-3+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(-\sum_{j=j_0}^{m} \ln \frac{1}{\zeta_i(j)}). \end{aligned}$$

We shall show that $\sum_{j=j_0}^{\infty} \ln \frac{1}{\zeta_i(j)} = \sum_{j=j_0}^{\infty} \ln \frac{1-\alpha(3j+i)}{1-\alpha(3j+i-1)} = \infty$, by considering the series $\sum_{j=j_0}^{\infty} \frac{\alpha}{-1+\alpha(3j+i)}$. But as

$$\lim_{j\to\infty} \frac{\ln(1-\alpha(3j+i))/(1-\alpha(3j+i-1))}{\alpha/(-1+\alpha(3j+i))} = 1,$$

using the limit comparison test, we get $\sum_{j=j_0}^{\infty} \ln \frac{1}{\zeta_i(j)} = \infty$.
Therefore,

$$y_{3m+i} = y_{-3+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(-\sum_{j=j_0}^{m} \ln \frac{1}{\zeta_i(j)})$$

converges to zero as $m \to \infty$. ∎

References

[1] R. Abo-Zeid, Global behavior of a fourth order difference equation, Acta Commentaiones Univ. Tartuensis Math. 18 (2) (2014) 211–220.

[2] R. Abo-Zeid, On the solutions of two third order recursive sequences, Armenian J. Math. 6 (2) (2014) 64–66.

[3] R. Abo-Zeid, Global behavior of a third order rational difference equation, Math. Bohem. 139 (1) (2014) 25–37.

[4] R. Abo-Zeid, Global attractivity of a higher-order difference equation, Discrete Dyn. Nat. Soc. 2012 (2012) Article ID 930410.

[5] R.P. Agarwal, Difference Equations and Inequalities, First Edition, Marcel Dekker, 1992.

[6] E. Camouzis, G. Ladas, Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/HRC Boca Raton, 2008.

[7] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman and Hall/CRC, 2005.

[8] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic, Dordrecht, 1993.

[9] M.R.S. Kulenović, G. Ladas, Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures, Chapman and Hall/CRC Boca Raton, 2002.

[10] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comput. 176 (2006) 768–774.