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Best Proximity Point Theorems for Cyclic Wardowski Type Contraction

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Abstract In this article, we give an extended version of fixed point result for Wardowski type contraction and define a new type of contraction namely, cyclic Wardowski type contraction or cyclic F-contraction in a complete metric space. Moreover, we prove the existence of best proximity point for cyclic F-contraction and also establish best proximity result in the setting of uniformly convex Banach space.

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1. Introduction

It is widely known that Banach contraction principle is the first Fixed point theorem and one of the most powerful and versatile results in the field of functional analysis. This result is a very popular and effective tool in solving existence problems in many branches of mathematical analysis and engineering. Over the years, several extensions of this remarkable result have been emerged in the literature (see, for example, [1–5] and references therein). On the other hand one such valuable extension was delivered by Eldred and Veeramani [6] by taking A and B, two non-empty closed subsets of a metric space X and a mapping $T : A \cup B \to A \cup B$ satisfying:

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$,
(2) $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A, y \in B,$ where $k \in (0, 1)$.

Then T has a fixed point in $A \cap B$. Moreover in [4], authors defined a generalized version of cyclic contraction.

It is enough to the authors to be motivated for the case $A \cap B = \phi$, where the question of existence of fixed points may be raised immediately. It is better to examine the case for $d(A, B) > 0$, where it is natural to ask the existence and uniqueness question of the smallest value of $d(x, Tx)$. This is the main motivation of a best proximity point. A best proximity point is a point x in $A \cup B$ such that $d(x, Tx) = d(A, B)$. A best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the existence of x that satisfies

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$d(x, Tx) = d(A, B)$ (see [7–9]). Eldred and Veeramani [6] delivered an algorithm to find a best proximity point for the similar kind of mappings in uniformly convex Banach space and subsequently Al-Thagafi and Shahzad [10] gave some related results in a reflexive Banach space.

In [5], Dariusz Wardowski has proved fixed point theorem on F-contraction, which is perceived to be one of the most general non-linear contraction in complete metric spaces. Motivated by this result our main aim is to resolve a more general problem on the existence of fixed point of T satisfying (1) for F-contraction. On the other hand, we also establish best proximity point result on a complete metric space and in a uniformly convex Banach space using new type cyclic F-contraction.

2. Preliminaries

In this section we recall some of the basic definitions and notations which will be essential throughout the article.

Definition 2.1. [5] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

- $(F1)$: F is strictly increasing i.e. for all $\alpha, \beta \in \mathbb{R}^+$ such that $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$,
- $(F2)$: For each sequence $\{\alpha_n\}, n \in \mathbb{N}$ of positive real numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
- $(F3)$: There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0+} \alpha^k F(\alpha) = 0$.

A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$, such that

$$\forall x, y \in X \qquad d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{2.1}$$

Depending on the types of mappings F, we get different types of contraction mappings, for detailed study one is referred to see the examples in [5, 11].

Remark 2.2. [12] From $(F1)$ and the equation (2.1) it is clear that every F-contraction T is a contractive mapping i.e.

$$d(Tx, Ty) < d(x, y), \qquad \forall x, y \in X, \ Tx \neq Ty.$$

Thus every F-contraction is a continuous mapping.

Theorem 2.3. [5] *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Definition 2.4. [6] Let A and B be two non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. We say that T is cyclic contraction if

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B),$$

for some $\alpha \in (0, 1)$ and for all $x \in A$ and $y \in B$ where $d(A, B) = inf\{d(x, y) : x \in A, y \in B\}$.

A point $x \in A \cup B$ is said to be a best proximity point for T if $d(x, Tx) = d(A, B)$.

Definition 2.5. A Banach space X is uniformly convex if there exists a strictly increasing function $\delta : (0,2] \to [0,1]$ such that the following condition holds for all $x, y, z \in X,\ R > 0$ and $r \in [0, 2R]$

$$\left.\begin{array}{l} ||x-z|| \leq R \\ ||y-z|| \leq R \\ ||x-y|| \geq r \end{array}\right\} \Rightarrow \left|\left|\frac{x+y}{2} - z\right|\right| \leq \Big(1 - \delta(\frac{r}{R})\Big)R.$$

Definition 2.6. [6] A subset K of a metric space X is boundedly compact if every bounded sequence in K has a sub-sequence converging to a point in K.

3. Main Results

Firstly we are going to deliver one fixed point result for cyclic type F-contraction as follows:

Theorem 3.1. *Suppose there exists two non-empty closed subsets A and B of a complete metric space (X,d) and the mapping $T : A \cup B \to A \cup B$ satisfies:*

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$,
(2) $\forall x \in A$ and $\forall y \in B$, there exists $\tau > 0$ such that

$$d(T(x),T(y)) > 0 \Rightarrow \tau + F(d(T(x),T(y))) \leq F(d(x,y)).$$

Then T has a unique fixed point in $A \cap B$.

Proof. First note that if x and y are two different fixed points of T, then $d(x,y) > 0$. Now by the definition of F-contraction we have

$$\tau + F(d(x,y)) = \tau + F(d(T(x),T(y))) \leq F(d(x,y)).$$

Since $\tau > 0$, we have $x = y$. Thus the uniqueness part of the theorem is done.

For the existence part, suppose $x_0 \in A \cup B$. Define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n, \forall n \geq 0$. Set $\gamma_n := d(x_{n+1}, x_n),\ \forall n \geq 0$. First note that if $\gamma_n = 0$ for some $n \geq 0$, then $x_n = x_{n+1} = Tx_n$. Therefore, $x_n \in A \cap B$ is a fixed point of T and the proof follows. We now assume that $\gamma_n \neq 0$ for all $n \geq 0$.

Since T is F-contraction, from the equation (2.1)

$$\begin{aligned} F(\gamma_n) &\leq F(\gamma_{n-1}) - \tau \\ &\leq F(\gamma_{n-2}) - 2\tau \\ &\leq \cdots \\ &\leq F(\gamma_0) - n\tau. \end{aligned}$$

Then as $n \to \infty$, $F(\gamma_n) \to -\infty$ together with $(F2)$ we have,

$$\lim_{n\to\infty} \gamma_n = 0.$$

From $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0.$$

Now,

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq \gamma_n^k (F(\gamma_0) - n\tau) - \gamma_n^k F(\gamma_0) = -\gamma_n^k n\tau \leq 0, \text{ for all } n \in \mathbb{N}.$$

Therefore as $n \to \infty$, we have

$$n\gamma_n^k = 0.$$

From above, as $\lim_{n\to\infty} n\gamma_n^k = 0$, there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. Therefore, $\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}$, for all $n \geq n_1$. Let $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &= \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n \\ &< \sum_{i=n}^{\infty} \gamma_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since $d(x_m, x_n)$ is less than a convergent series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ as $k \in (0,1)$, it implies $d(x_m, x_n) < \varepsilon$, for all $m > n \geq n_1$ and $\varepsilon > 0$. Therefore $\{x_n\}$ is a Cauchy sequence in A, let $\lim_{n\to\infty} x_n = z$. So there are infinitely many number of terms of $\{x_n\}$ in A as well as in B. Therefore $z \in A \cap B$, so $A \cap B \neq \phi$.

Now (1) implies $T : A \cap B \to A \cap B$ while (2) implies that T restricted to $A \cap B$ is an F-contraction. Thus Theorem 2.3, applies to T on $A \cap B$ and we have a unique fixed point in $A \cap B$. ∎

Remark 3.2. Theorem 3.1 gives an extension result of F-contraction [5] if we set $A = B$ in the statement.

Before going to deliver the convergence and existence results for best proximity points we first introduce a new concept of cyclic F-contraction. We also prove the existence result for best proximity point of cyclic F-contraction in the setting of uniformly convex Banach space.

Definition 3.3. Let A and B be two non-empty subsets of a complete metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called cyclic F-contraction if the following conditions hold:

(a) $T(A) \subseteq B$ and $T(B) \subseteq A$,
(b) $\forall x \in A,\ y \in B$ and if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq \alpha F(d(x, y)) + (1-\alpha) F(d(A, B)),$$

for some $\alpha \in (0, 1)$, provided $d(A, B) > 0$.

Note that (b) implies that T satisfies $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, for all $x \in A,\ y \in B$. On the other hand, (b) can be written as:

$$\tau + F(d(Tx, Ty)) - F(d(A, B)) \leq \alpha(F(d(x, y)) - F(d(A, B))), \text{ for all } x \in A,\ y \in B.$$

Example 3.4. Consider the complete metric space $X = \mathbb{R}$ with the usual metric d. If we consider the map $F(x) = -\frac{1}{\sqrt{x}}, x > 0$ and for any $k \in (\frac{1}{2}, 1)$. Then clearly F satisfies all the conditions from $(F1) - (F3)$. In this case, any $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ satisfying the condition

$$d(Tx, Ty) \leq \frac{1}{\left\{\alpha + (1-\alpha)\sqrt{\frac{d(x,y)}{d(A,B)}} + \tau\sqrt{d(x, y)}\right\}^2} d(x, y),$$

$\forall x \in A,\ \forall y \in B,\ d(Tx,Ty) > 0,\ \alpha \in (0,1)$ and $\tau > 0$ provided $d(A,B) > 0$, is a cyclic F-contraction.

Remark 3.5. It is clear form above example that we can get a special case of non-linear contraction of the type $d(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$ where $\alpha(d(x,y)) < 1$, for details see [1, 9].

Now we present our main results regarding the convergence and existence of best proximity point as below.

Proposition 3.6. *Let A and B be non-empty closed subsets of a complete metric space (X,d). Suppose the mapping $T : A\cup B \to A\cup B$ is a cyclic F-contraction. Let $x_0 \in A\cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then we have*

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A,B).$$

Proof. Suppose $x_0 \in A\cup B$. Define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \geq 0$. Now,

$$\begin{aligned} F(d(x_2,x_1)) &= F(d(Tx_1,Tx_0)) \\ &\leq \alpha F(d(x_1,x_0)) + (1-\alpha)F(d(A,B)) - \tau \\ &< \alpha F(d(x_1,x_0)) + (1-\alpha)F(d(A,B)) \end{aligned}$$

which implies that

$$F(d(x_2,x_1)) - F(d(A,B)) \leq \alpha\{F(d(x_1,x_0)) - F(d(A,B))\}.$$

Again

$$\begin{aligned} F(d(x_3,x_2)) &= F(d(Tx_2,Tx_1)) \\ &\leq \alpha F(d(x_2,x_1)) + (1-\alpha)F(d(A,B)) - \tau \\ &< \alpha F(d(x_2,x_1)) + (1-\alpha)F(d(A,B)) \end{aligned}$$

which implies that

$$\begin{aligned} F(d(x_3,x_2)) - F(d(A,B)) &< \alpha\{F(d(x_2,x_1)) - F(d(A,B))\} \\ &< \alpha^2\{F(d(x_1,x_0)) - F(d(A,B))\}. \end{aligned}$$

Recursively we say that as $\alpha \in (0,1)$,

$$F(d(x_{n+1},x_n)) - F(d(A,B)) < \alpha^n\{F(d(x_1,x_0)) - F(d(A,B))\}.$$

This implies that for $n \to \infty$

$$F(d(x_{n+1},x_n)) = F(d(A,B)).$$

Since F is increasing

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

That is

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A,B).$$

∎

Theorem 3.7. *Let A and B be two non-empty closed subsets of a complete metric space X and $T : A \cup B \to A \cup B$ be a cyclic F-contraction mapping. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.*

Proof. Let $\{x_{2n_k}\}$ be a sub-sequence of $\{x_{2n}\}$ with $x_{2n_k} \to x \in A$. Since

$$d(A, B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})$$

for each $k \geq 1$, then by Proposition 3.6 we have $d(x, x_{2n_k-1}) \to d(A, B)$ as $k \to \infty$.

Again

$$d(A, B) \leq d(x_{2n_k}, Tx) = d(Tx_{2n_k-1}, Tx) \leq d(x_{2n_k-1}, x)$$

for each $k \geq 1$, hence

$$d(x, Tx) = d(A, B).$$

■

The following Theorem easily prove from Theorem 3.7.

Theorem 3.8. *Let A and B be two non-empty closed subsets of a complete metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic F-contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$ such that*

(1) the sequence $\{x_n\}$ is bounded and
(2) if either A or B is boundedly compact.

Then there exists $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$.

Now we present the following lemmas from [6] in order to prove our best proximity result of cyclic F-contraction in uniformly convex Banach space.

Lemma 3.9. [6] *Let A be a non-empty closed and convex subset and B be a non-empty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequence in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $||z_n - y_n|| \to d(A, B)$,
(ii) For every $\varepsilon > 0$ there exists N_0 such that for all $m > n > N_0, ||x_m - y_n|| \leq d(A, B) + \varepsilon$.

Then for every $\varepsilon > 0$ there exists N_1 such that for all $m > n > N_1, ||x_m - z_n|| \leq \varepsilon$.

Lemma 3.10. [6] *Let A be a non-empty closed and convex subset and B be a non-empty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequence in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $||x_n - y_n|| \to d(A, B)$,
(ii) $||z_n - y_n|| \to d(A, B)$.

Then $||x_n - z_n||$ converges to 0.

Theorem 3.11. *Let A and B be two non-empty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is cyclic F-contraction map and F is a continuous map. Then there exists a unique best proximity point $x \in A$ (that is with $||x - Tx|| = d(A, B)$). Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose $d(A,B) \neq 0$. Then by Proposition 3.6,

$$||x_{2n} - x_{2n+1}|| = ||x_{2n} - Tx_{2n}|| \to d(A,B)$$

and

$$||x_{2n+2} - x_{2n+1}|| = ||T^2 x_{2n} - Tx_{2n}|| \to d(A,B).$$

Then by Lemma 3.10

$$||x_{2n} - x_{2n+2}|| \to 0.$$

Similarly we can show that

$$||Tx_{2n} - Tx_{2n+2}|| \to 0.$$

We now show that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n > N_0$, $||x_{2m} - Tx_{2n}|| \leq d(A,B)+\varepsilon$. Let $\varepsilon > 0$. If possible, suppose for all $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ such that,

$$||x_{2m_k} - Tx_{2n_k}|| \geq d(A,B) + \varepsilon$$

and

$$||x_{2(m_k-1)} - Tx_{2n_k}|| \leq d(A,B) + \varepsilon.$$

Now,

$$\begin{aligned} d(A,B) + \varepsilon &\leq ||x_{2m_k} - Tx_{2n_k}|| \\ &\leq ||x_{2m_k} - x_{2(m_k-1)}|| + ||x_{2(m_k-1)} - Tx_{2n_k}|| \\ &\leq ||x_{2m_k} - x_{2(m_k-1)}|| + d(A,B) + \varepsilon. \end{aligned}$$

Thus,

$$\lim_{k\to\infty} ||x_{2m_k} - Tx_{2n_k}|| = d(A,B) + \varepsilon.$$

Consequently,

$$\begin{aligned} ||x_{2m_k} - Tx_{2n_k}|| \leq\ & ||x_{2m_k} - x_{2(m_k+1)}|| + ||x_{2(m_k+1)} - Tx_{2(n_k+1)}|| \\ & + ||Tx_{2(n_k+1)} - Tx_{2n_k}||. \end{aligned}$$

Therefore, as F is continuous

$$\begin{aligned} \lim_{k\to\infty} F(||x_{2m_k} - Tx_{2n_k}||) &\leq \lim_{k\to\infty} F(||Tx_{2m_k+1} - Tx_{2n_k+2})|| \\ &\leq \lim_{k\to\infty} \{\alpha F(||x_{2m_k+1} - x_{2m_k+2}||) + (1-\alpha)F(d(A,B)) - \tau\} \\ &\leq \lim_{k\to\infty} \{\alpha F(||Tx_{2m_k} - T^2 x_{2m_k}||) + (1-\alpha)F(d(A,B)) - \tau\} \\ &\leq \lim_{k\to\infty} \{\alpha^2 F(||x_{2m_k} - Tx_{2m_k}||) + (1-\alpha^2)F(d(A,B)) - (1+\alpha)\tau\} \\ F(d(A,B)+\varepsilon) &\leq \alpha^2 F(d(A,B)+\varepsilon) + (1-\alpha^2)F(d(A,B)) - (1+\alpha)\tau \\ &< \alpha^2 F(d(A,B)+\varepsilon) + (1-\alpha^2)F(d(A,B)). \end{aligned}$$

Hence

$$\begin{aligned} (1-\alpha^2)F(d(A,B)+\varepsilon) &< (1-\alpha^2)F(d(A,B)) \\ F(d(A,B)+\varepsilon) &< F(d(A,B)), \end{aligned}$$

Now, we prove the uniqueness of x. Suppose $x, y \in A$ and $x \neq y$ with

$$||x - Tx|| = d(A,B) \text{ and } ||y - Ty|| = d(A,B)$$

where necessarily,

$$T^2x = x \text{ and } T^2y = y.$$

Therefore,

$$||Tx - y|| = ||Tx - T^2y|| < ||x - Ty||,$$
$$||Ty - x|| = ||Ty - T^2x|| < ||y - Tx||.$$

Therefore we arrive at a contradiction. Hence $x = y$. ■

Remark 3.12. If we drop the convexity assumption from Theorem 3.11, then the convergence and uniqueness is not guaranteed even in finite dimensional case. For example, consider $X = R^6, A = \{e_1, e_3, e_5\}$ and $B = \{e_2, e_4, e_6\}$. Define $T(e_i) = e_{i+1}$, where $e_{6+i} = e_i$.

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