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Constant Generalized Riesz Potential Functions and Polarization Optimality Problems

Nattapong Bosuwan

Department of Mathematics, Faculty of Science, Mahidol University, Rama VI Road, Ratchathewi District, Bangkok 10400, Thailand and Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand
e-mail : nattapong.bos@mahidol.ac.th

Abstract An extension of a conjecture of Nikolov and Rafailov [N. Nikolov, R. Rafailov, On extremums of sums of powered distances to a finite set of points, Geom. Dedicata 167(1) (2013) 69–89] by considering the following potential function defined on $\mathbb{R}^2$:

$$f_s(x) = \sum_{j=1}^{N} \left(|x - x_j|^2 + h\right)^{-s/2}, \qquad h \geq 0,$$

for $s = 2 - 2N$ is given. We obtain a characterization of sets of N distinct points $\{x_1, x_2, \ldots, x_N\}$ such that f_{2-2N} is constant on some circle in $\mathbb{R}^2$. Using this characterization, we prove some special cases of this new conjecture. The other problems considered in this paper are polarization optimality problems. We find all maximal and minimal polarization constants and configurations of two concentric circles in $\mathbb{R}^2$ using the above potential function for certain values of s.

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1. Introduction

Riesz potential functions play important role in the subject of discrete energy problems and point configurations (see the recent book by Borodachov, Hardin, and Saff [1], and papers [2–5]). In this paper, we investigate some analytic properties of the generalized Riesz potential functions on $\mathbb{R}^2$ stated below. Moreover, using these analytic properties of these potential functions, we further solve some polarization optimality (Chebyshev) problems which were initiated by Fekete, Pólya, and Szegő [6, 7]. For a fixed multiset of N points $\omega_N := \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$, a given constant $s \in \mathbb{R}$, and a given constant $h \geq 0$, we define the potential function $U^{s,h}(\cdot;\omega_N) : \mathbb{R}^2 \to [0, \infty]$ as the following:

$$U^{s,h}(x;\omega_N) := \sum_{j=1}^{N} \left(|x - x_j|^2 + h\right)^{-s/2}, \tag{1.1}$$

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where $x \in \mathbb{R}^2$ and $|\cdot|$ is the 2-dimensional Euclidean norm in $\mathbb{R}^2$. In this paper, we call $U^{s,h}(\cdot, \omega_N)$ a *Riesz* (s,h)*-potential function of* ω_N. The geometric interpretation of the function $U^{s,h}(\cdot;\omega_N)$ is as follows. Let us consider two parallel planes in $\mathbb{R}^3$: one is $\mathbb{R}^2 \times \{0\}$ and the other is $\mathbb{R}^2 \times \{\sqrt{h}\}$. Basically, the potential function $U^{s,h}(x, \omega_N)$ is the Riesz s-potential function in the 3-dimensional Euclidean space $\mathbb{R}^3$ of $\omega_N' \subset \mathbb{R}^2 \times \{0\}$ at $x' \in \mathbb{R}^2 \times \{\sqrt{h}\}$, where the projection from $\mathbb{R}^2 \times \{0\}$ to $\mathbb{R}^2$ of ω_N' is ω_N and the projection from $\mathbb{R}^2 \times \{\sqrt{h}\}$ to $\mathbb{R}^2$ of x' is x. Moreover, if $h = 0$, then $U^{s,h}(\cdot;\omega_N)$ is the Riesz s-potential function in the 2-dimensional Euclidean space $\mathbb{R}^2$ of ω_N. We refer the reader to [2, 8–10] for more information on Riesz s-potential functions in a d-dimensional Euclidean space $\mathbb{R}^d$.

Now, let ω_N be a fixed set of distinct equally spaced points on a circle $T \subset \mathbb{R}^2$, Γ be a circle concentric to T, and $h \geq 0$ be fixed. In [11, Theorem 1], Nikolov and Rafailov showed that $f_s(x) := U^{s,h}(x;\omega_N)$ is constant as a function of x on Γ if and only if $s = 0, -2, -4, \ldots, 4-2N$, or $2-2N$. Furthermore, for $s \in \mathbb{R} \setminus \{0, -2, -4, \ldots, 2-2N\}$, they located extremum points of $U^{s,h}(\cdot;\omega_N)$ on Γ in [11, Theorem 1].

In the same paper, they also proved the following inverse type result (see [11, Theorem 2]) of what proceeds.

Theorem A. Given a set of N distinct points $\omega_N := \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$ and a circle $\Gamma \subset \mathbb{R}^2$ such that for each $s = -2, -4, \ldots, 2-2N$,

$$U^{s,0}(x;\omega_N) = \sum_{j=1}^{N} |x - x_j|^{-s}$$

is independent of the position of $x \in \Gamma$. Then, ω_N forms a set of distinct equally spaced points on a circle concentric to Γ.

Moreover, they proposed the following conjecture (see [11, Conjecture 1]):

Conjecture B. Given a set of N distinct points $\omega_N := \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$ and a circle $\Gamma \subset \mathbb{R}^2$ such that

$$U^{2-2N,0}(x;\omega_N) = \sum_{j=1}^{N} |x - x_j|^{2N-2}$$

is constant as a function of x on Γ. Then, ω_N forms a set of distinct equally spaced points on a circle concentric to Γ.

Translating and scaling the circle Γ in the above conjecture, it is easy to check that Conjecture B is equivalent to the following conjecture.

Conjecture C. Given a set of N distinct points $\omega_N := \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$ such that

$$U^{2-2N,0}(x;\omega_N) = \sum_{j=1}^{N} |x - x_j|^{2N-2}$$

is constant as a function of x on the unit circle. Then, ω_N forms a set of distinct equally spaced points on a circle centered at 0.

In order to simplify further considerations, we shall study Conjecture C. This conjecture for the case when $N = 2$ is trivial. The proof of this conjecture when $N = 3$ is in [11, Proposition 2]. The one for the case when $x_1, x_2, \ldots x_N$ have the same norm is in [12, Proposition 1]. In the same paper, the authors also proved this conjecture for the case when N is prime and $x_1, x_2, \ldots, x_N$ have an equal angle distribution and rational norms (see [12, Proposition 2]).

In this paper, we extend Theorem A to more general potential functions defined in (1.1). Moreover, the extension of Conjecture C is proposed (see Conjecture 2.2 in Section 2). A characterization of sets of N distinct points ω_N that $U^{2-2N,h}(\cdot, \omega_N)$ is constant on some circle in $\mathbb{R}^2$ is given. Using this characterization, we prove some special cases of this new extended conjecture.

The next problems considered in this paper are polarization optimality problems corresponding to the potential functions defined in (1.1). Let $\omega_N = \{x_1, \ldots, x_N\}$ denote a configuration of N (not necessarily distinct) points in $\mathbb{R}^2$. Denote by

$$\mathbb{S}^1_R := \{x \in \mathbb{R}^2 : |x| = R\}$$

the circle centered at 0 of radius R in $\mathbb{R}^2$. When $R = 1$, we simply use the notation $\mathbb{S}^1$. Given $s \in \mathbb{R}, h \geq 0, R > 0$, and $r > 0$, we define polarization constants

$$M_N^{s,h}(\mathbb{S}^1_r; \mathbb{S}^1_R) := \max_{\substack{\omega_N \subset \mathbb{S}^1_r \\ \#\omega_N = N}} \min_{y \in \mathbb{S}^1_R} U^{s,h}(y; \omega_N), \qquad M_N^{0,h}(\mathbb{S}^1_r; \mathbb{S}^1_R) := N, \tag{1.2}$$

$$m_N^{s,h}(\mathbb{S}^1_r; \mathbb{S}^1_R) := \min_{\substack{\omega_N \subset \mathbb{S}^1_r \\ \#\omega_N = N}} \max_{y \in \mathbb{S}^1_R} U^{s,h}(y; \omega_N), \qquad m_N^{0,h}(\mathbb{S}^1_r; \mathbb{S}^1_R) := N, \tag{1.3}$$

where $\#\omega_N$ stands for the cardinality of the multiset ω_N. We will call ω_N a *maximal (minimal) N-point Riesz (s, h)-polarization configuration of* $(\mathbb{S}^1_r; \mathbb{S}^1_R)$ if ω_N attains the maximum in (1.2) (minimum in (1.3)). We give a brief history of such polarization optimality problems below.

The idea of two-plate polarization constants was introduced by Farkas and Révész [13] in general sense. However, almost all previous results on polarization optimality problems related to Riesz potentials [2, 8–10, 14, 15] were considered for the case when $R = r = 1$ and $h = 0$. The maximality of N distinct equally spaced points on the unit circle for the maximal Riesz $(s, 0)$-polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (1.2) when $s > 0$ was proved by Hardin, Kendall, and Saff [10] (see also [9] and [14] for the history of this problem). In [10], they also showed the minimality of N distinct equally spaced points on the unit circle for the minimal Riesz $(s, 0)$-polarization problem of $(\mathbb{S}^1; \mathbb{S}^1)$ in (1.3) for $-1 \leq s < 0$. Recently, a characterization of all maximal and minimal N-point Riesz $(s, 0)$-polarization configurations of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$ when $s = -2, -4, \ldots, 2 - 2N$ was given in [12, Theorem 2]. One of the aims of this paper is to provide a characterization analogous to Theorem 2 in [12] for the case when $h > 0$.

We would like call the reader's attention to papers [2, 8–10] that contain asymptotic results of polarization constants and configurations of subsets of $\mathbb{R}^d$ as $N \to \infty$ when $s > 0$ and $h = 0$.

An outline of this paper is as follows. In Section 2, we state the extension of Theorem A to more general potential functions in (1.1) and give an extension of Conjecture C. Some special cases of this new conjecture are considered. In Section 3, we state our results on polarization optimality problems. Section 4 and Section 5 are devoted to the proofs of all results in Section 2 and Section 3, respectively. Finally, we perform our auxiliary computations in Section 6.

2. Constant Riesz (s, h)-Potential Functions

The first theorem is a generalization of Theorem A.

Theorem 2.1. *Let $h \geq 0$. Given a set of N distinct points $\omega_N := \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$ such that for each $s = -2, -4, \ldots, 2-2N$,*

$$U^{s,h}(x;\omega_N) = \sum_{j=1}^{N} \left(|x - x_j|^2 + h\right)^{-s/2}$$

is independent of the position of $x \in \mathbb{S}^1$. Then, ω_N forms a set of distinct equally spaced points on a circle centered at 0. Moreover, if $|x_1| = |x_2| = \ldots = |x_N| = r$, then for each $p = 1, 2, \ldots, N-1$,

$$U^{-2p,h}(x;\omega_N) = \frac{N}{2^p} \sum_{q=0}^{p} \binom{p}{q}^2 (2r)^{2q} \left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)((r+1)^2 + h)}\right)^{p-2q}$$

for all $x \in \mathbb{S}^1$.

This theorem brings us to the following conjecture which generalizes Conjecture C.

Conjecture 2.2. Let $h \geq 0$. Given a set of N distinct points $\omega_N := \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$ such that

$$U^{2-2N,h}(x;\omega_N) = \sum_{j=1}^{N} \left(|x - x_j|^2 + h\right)^{N-1}$$

is constant as a function of x on $\mathbb{S}^1$. Then, $\{x_1, x_2, \ldots, x_N\}$ forms a set of distinct equally spaced points on a circle centered at 0.

A characterization of sets of N distinct points ω_N such that $U^{2-2N,h}(\cdot, \omega_N)$ is constant on $\mathbb{S}^1$ is the following:

Theorem 2.3. *Let $h \geq 0$ and $\omega_N = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^2$ be a set of N distinct points. Then, the function*

$$U^{2-2N,h}(x;\omega_N) = \sum_{j=1}^{N} \left(|x - x_j|^2 + h\right)^{N-1}$$

is constant on $\mathbb{S}^1$ if and only if

$$\sum_{j=1}^{N} B_{k,j} x_j^k = 0, \qquad k = 1, \ldots, N-1, \tag{2.1}$$

where

$$x^k := (r^k \cos(kt), r^k sin(kt))$$

if $x = (r\cos(t), r\sin(t)) \in \mathbb{R}^2$ and

$$B_{k,j} := \sum_{q=0}^{N-k-1} \left[\binom{N-1}{q} \binom{N-1}{k+q} (2|x_j|)^{2q} \right.$$

$$\left. \times \left(|x_j|^2 + 1 + h + \sqrt{((|x_j| - 1)^2 + h)((|x_j| + 1)^2 + h)}\right)^{N-2q-k-1} \right]. \tag{2.2}$$

As a consequence of this characterization, we obtain the following corollary.

Corollary 2.4. *Let $h \geq 0$ and let $\omega_N := \{x_1, x_2, \ldots, x_N\}$ be a set of N distinct points in $\mathbb{R}^2$, which belong to a circle $\mathbb{S}^1_r \subset \mathbb{R}^2$. Assume that*

$$U^{2-2N,h}(x;\omega_N) = \sum_{j=1}^{N}(|x - x_j|^2 + h)^{N-1}$$

is constant on $\mathbb{S}^1$. Then, $\{x_1, x_2, \ldots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}^1_r$.

Applying Theorem 2.3 and Corollary 2.4, we prove Conjecture 2.2 when $N = 3$.

Corollary 2.5. *Let $h \geq 0$ and $\{x_1, x_2, x_3\} \subset \mathbb{R}^2$ be a set of 3 distinct points. If the function $U^{-4,h}(x, \{x_1, x_2, x_3\})$ is constant on $\mathbb{S}^1$, then $\{x_1, x_2, x_3\}$ forms a set of distinct equally spaced points on a circle centered at 0.*

3. Polarization Optimality Oroblems

A complete characterization of all maximal and minimal N-point Riesz (s, h)-polarization configurations of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$ when $s = -2, -4, \ldots, 2 - 2N$ and $h \geq 0$ is the following:

Theorem 3.1. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \ldots, N-1\}$, $R > 0$, $r > 0$, $h \geq 0$, and $\{x_1, x_2, \ldots, x_N\} \subset \mathbb{S}^1_r$. The following statements are equivalent:*

- *(a) $\{x_1, x_2, \ldots, x_N\}$ is a maximal N-point Riesz $(-2p, h)$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$;*
- *(b) $\{x_1, x_2, \ldots, x_N\}$ is a minimal N-point Riesz $(-2p, h)$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$;*
- *(c) $\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \cdots = \sum_{j=1}^N x_j^p = 0$, where $x^k := (r^k \cos(kt), r^k sin(kt))$ if $x = (r\cos(t), r\sin(t)) \in \mathbb{R}^2$.*

Furthermore,

$$M_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1_R) = m_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1_R)$$

$$= \frac{N}{2^p}\sum_{j=0}^{p}\binom{p}{j}^2 (2rR)^{2j}\left(r^2 + R^2 + h + \sqrt{((r-R)^2 + h)(r+R)^2 + h)}\right)^{p-2j}. \quad (3.1)$$

4. Proof of Section 2

The Euclidean space $\mathbb{R}^2$ and the complex space $\mathbb{C}$ have the same dimension and the same norm. However, the complex space $\mathbb{C}$ has a richer algebraic structure, for example, $\mathbb{C}$ is a field. Therefore, when we prove all results in Sections 2 and 3, any element $x \in \mathbb{R}^2$ will be replaced by $x \in \mathbb{C}$, the 2-dimensional Euclidean norm $|\cdot|$ is replaced by the modulus in $\mathbb{C}$, and the notation xy is adopted from the multiplication in $\mathbb{C}$ and the notation x/y is adopted from the division in $\mathbb{C}$. We recall that the usual dot product in $\mathbb{C}$ is defined by

$$(a_1 + a_2 i) \cdot (b_1 + b_2 i) := a_1 b_1 + a_2 b_2.$$

Lemma 4.1. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \ldots, N-1\}$, and $h \geq 0$. If $x_j := |x_j|\cos t_j + i|x_j|\sin t_j$ for all $j = 1, 2, \ldots, N$, then for all $y := \cos t + i \sin t \in \mathbb{S}^1$,*

$$\sum_{j=1}^{N}(|y-x_j|^2+h)^p = E_0^{(p)} + \sum_{k=1}^{p}\sum_{j=1}^{N} E_{k,j}^{(p)} \cos(kt_j - kt), \tag{4.1}$$

$$\sum_{j=1}^{N}(|y-x_j|^2+h)^p = E_0^{(p)} + \sum_{k=1}^{p}\sum_{j=1}^{N} \frac{E_{k,j}^{(p)}}{|x_j|^k}\left(y^k \cdot x_j^k\right), \tag{4.2}$$

where

$$E_0^{(p)} := \frac{1}{2^p}\sum_{j=1}^{N}\sum_{q=0}^{p}\binom{p}{q}^2 (2|x_j|)^{2q}\left(|x_j|^2+1+h+\sqrt{((|x_j|-1)^2+h)(|x_j|+1)^2+h)}\right)^{p-2q} \tag{4.3}$$

and for all $k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, N$,

$$E_{k,j}^{(p)} := \frac{(-1)^k}{2^{p-1}}\sum_{q=0}^{p-k}\left[\binom{p}{q}\binom{p}{k+q}(2|x_j|)^{2q+k}\right.$$
$$\left.\times\left(|x_j|^2+1+h+\sqrt{((|x_j|-1)^2+h)((|x_j|+1)^2+h)}\right)^{p-k-2q}\right]. \tag{4.4}$$

Proof of Lemma 4.1. Let $y := \cos t + i \sin t$ and $x_j := |x_j|\cos t_j + i|x_j|\sin t_j$ for all $j = 1, 2, \ldots, N$. A simple calculation shows that

$$f_j(t) := (|y-x_j|^2+h)^p = (|x_j|^2+1+h-2|x_j|\cos(t-t_j))^p.$$

We know that

$$A := \{1, \cos(t-t_j), \ldots, \cos(p(t-t_j))\}$$

forms an orthogonal set with respect to the inner product

$$\langle f, g\rangle := \int_0^{2\pi} f(t)g(t)dt.$$

Moreover,

$$\begin{aligned} f_j &\in \operatorname{span}\{1, \cos(t-t_j), \cos^2(t-t_j), \ldots, \cos^p(t-t_j)\} \\ &= \operatorname{span}\{1, \cos(t-t_j), \ldots, \cos(p(t-t_j))\}. \end{aligned}$$

Therefore,

$$f_j(t) = \sum_{k=0}^{p} E_{k,j}^{(p)} \cos(kt_j - kt).$$

This implies that

$$\sum_{j=1}^{N}(|y-x_j|^2+h)^p = \sum_{j=1}^{N} f_j(t) = E_0^{(p)} + \sum_{k=1}^{p}\sum_{j=1}^{N} E_{k,j}^{(p)}\cos(kt_j - kt),$$

where $E_0^{(p)} := \sum_{j=1}^N E_{0,j}^{(p)}$. By the orthogonality of the set A and the calculation in Lemma 6.2 (see Appendix), we have

$$E_0^{(p)} := \sum_{j=1}^N \frac{\langle f_j, 1\rangle}{2\pi}$$

$$= \frac{1}{2^p}\sum_{j=1}^N\sum_{q=0}^p \binom{p}{q}^2 (2|x_j|)^{2q}\left(|x_j|^2+1+h+\sqrt{((|x_j|-1)^2+h)(|x_j|+1)^2+h)}\right)^{p-2q}$$

and

$$E_{k,j}^{(p)} = \frac{\langle f_j, \cos k(t-t_j)\rangle}{\pi} = \frac{(-1)^k}{2^{p-1}}\sum_{q=0}^{p-k}\left[\binom{p}{q}\binom{p}{k+q}(2|x_j|)^{2q+k}\right.$$
$$\left.\times\left(|x_j|^2+1+h+\sqrt{((|x_j|-1)^2+h)((|x_j|+1)^2+h)}\right)^{p-k-2q}\right],$$

for all $k \in \{0,1,\ldots,p\}$ and $j \in \{1,\ldots,N\}$. Moreover, it is clear that the equations (4.1) and (4.2). ∎

Proof of Theorem 2.1. Suppose that there exist constants C_p, $p = 1,2,\ldots,N-1$, such that

$$U^{-2p,h}(x;\omega_N) = C_p, \qquad x \in \mathbb{S}^1, \qquad p = 1,2,\ldots,N-1,$$

where $\omega_N = \{x_1, x_2, \ldots, x_N\}$. If $x = \cos t + i\sin t$ and $x_j := |x_j|\cos t_j + i|x_j|\sin t_j$, then by (4.1), for each $p = 1,2,\ldots,N-1$, we have for all $t \in [0,2\pi]$,

$$C_p = E_0^{(p)} + \sum_{k=1}^p\sum_{j=1}^N\left[E_{k,j}^{(p)}\cos(kt_j)\cos(kt) + E_{k,j}^{(p)}\sin(kt_j)\sin(kt)\right]$$

and

$$0 = (E_0^{(p)} - C_p) + \sum_{k=1}^p\left[\left(\sum_{j=1}^N E_{k,j}^{(p)}\cos(kt_j)\right)\cos(kt) + \left(\sum_{j=1}^N E_{k,j}^{(p)}\sin(kt_j)\right)\sin(kt)\right].$$

Since $\{1, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(pt), \sin(pt)\}$ is linearly independent over $\mathbb{R}$, for all $p = 1,2,\ldots,N-1$,

$$C_p = E_0^{(p)}, \tag{4.5}$$

$$\sum_{j=1}^N E_{p,j}^{(p)}\cos(kt_j) = 0 \quad \text{and} \quad \sum_{j=1}^N E_{p,j}^{(p)}\sin(kt_j) = 0. \tag{4.6}$$

Using (4.4), we can compute

$$E_{p,j}^{(p)} = \frac{(-1)^p}{2^{p-1}}(2|x_j|)^p. \tag{4.7}$$

Combining (4.6) and (4.7), we have for all $p = 1,2,\ldots,N-1$,

$$0 = \sum_{j=1}^N \frac{(-1)^p}{2^{p-1}}(2|x_j|)^p(\cos(pt_j) + i\sin(pt_j)) = (-1)^p 2\sum_{j=1}^N x_j^p$$

which implies that $\sum_{j=1}^{N} x_j^p = 0$ for all $p = 1, 2, \ldots, N-1$. Using Newton's identities, we have

$$e_p(x_1, x_2, \ldots, x_N) = 0, \qquad p = 1, 2, \ldots, N-1.$$

Then,

$$\prod_{j=1}^{N}(x - x_j) = x^N + (-1)^N \prod_{j=1}^{N} x_j.$$

Hence, $|x_1| = |x_2| = \ldots = |x_N| = r$ for some $r > 0$ and $\{x_1, x_2, \ldots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}^1_r$. In turn, the equality (4.5) implies that for all $x \in \mathbb{S}^1$ and for all $p = 1, 2, \ldots, N-1$,

$$U^{-2p,h}(x; \omega_N) = C_p = E_0^{(p)}$$

$$= \frac{N}{2^p} \sum_{q=0}^{p} \binom{p}{q}^2 (2r)^{2q} \left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)((r+1)^2 + h)}\right)^{p-2q}.$$

■

Proof of Theorem 2.3. Set

$$x_j := |x_j| \cos(t_j) + i|x_j| \sin(t_j)$$

for all $j = 1, 2, \ldots, N$.

($\Rightarrow$) By our assumption, we assume that $f(y) := \sum_{j=1}^{N}(|y - x_j|^2 + h)^{N-1}$ is constant on $\mathbb{S}^1$, say $f(y) = C$ on $\mathbb{S}^1$. Set $y = \cos t + i \sin t \in \mathbb{S}^1$. By (4.1), for all $t \in [0, 2\pi]$,

$$C = f(y) = \sum_{j=1}^{N}(|y - x_j|^2 + h)^{N-1}$$

$$= E_0^{(N-1)} + \sum_{k=1}^{N-1} \sum_{j=1}^{N} \left[E_{k,j}^{(N-1)} \cos(kt_j) \cos(kt) + E_{k,j}^{(N-1)} \sin(kt_j) \sin(kt)\right]$$

$$= E_0^{(N-1)} + \sum_{k=1}^{N-1} \left[\left(\sum_{j=1}^{N} E_{k,j}^{(N-1)} \cos(kt_j)\right) \cos(kt) + \left(\sum_{j=1}^{N} E_{k,j}^{(N-1)} \sin(kt_j)\right) \sin(kt)\right]. \tag{4.8}$$

Because $\{1, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos((N-1)t), \sin((N-1)t)\}$ is linearly independent over $\mathbb{R}$,

$$C - E_0^{(N-1)} = 0$$

and for all $k = 1, 2, \ldots, N-1$,

$$\sum_{j=1}^{N} E_{k,j}^{(N-1)} \cos(kt_j) = 0 \quad \text{and} \quad \sum_{j=1}^{N} E_{k,j}^{(N-1)} \sin(kt_j) = 0. \tag{4.9}$$

Then, for all $k = 1, 2, \ldots, N-1$,

$$0 = \sum_{j=1}^{N} E_{k,j}^{(N-1)} (\cos(kt_j) + i \sin(kt_j)) = \sum_{j=1}^{N} \frac{E_{k,j}^{(N-1)}}{|x_j|^k} x_j^k. \tag{4.10}$$

Using the calculation in (4.4), it is not difficult to check that the equations (4.10) imply the equations (2.1).

($\Leftarrow$) Assume that the equations (2.1) hold true. Then,

$$\sum_{j=1}^{N} \frac{E_{k,j}^{(N-1)}}{|x_j|^k} x_j^k = 0, \qquad k = 1, \dots, N-1.$$

From (4.10), we have (4.9). Combining the relations (4.9) and the identity (4.8), we have for all $y \in \mathbb{S}^1$,

$$\sum_{j=1}^{N} (|y - x_j|^2 + h)^{N-1} = E_0^{(N-1)},$$

which implies that $U^{2-2N,h}(\cdot;\omega_N)$ is constant on $\mathbb{S}^1$. This completes the proof. ■

Proof of Corollary 2.4. Assume that $\{x_1, x_2, \dots, x_N\} \subset \mathbb{S}_r^1$. It is easy to check that the constants $B_{k,j} \neq 0$ do not depend on j. Therefore, by the system of equations (2.1), $\sum_{j=1}^{N} x_j^k = 0$ for all $k = 1, 2, \dots, N-1$. Using Newton's identities, we have

$$e_k(x_1, x_2, \dots, x_N) = 0, \qquad k = 1, 2, \dots, N-1.$$

Then,

$$\prod_{j=1}^{N} (x - x_j) = x^N + (-1)^N \prod_{j=1}^{N} x_j.$$

Hence, $|x_1| = |x_2| = \dots = |x_N| = r$ and $\{x_1, x_2, \dots, x_N\}$ forms a set of distinct equally spaced points on $\mathbb{S}_r^1$. ■

Proof of Corollary 2.5. Using Theorem 2.3, we have

$$x_1^2 + x_2^2 + x_3^2 = 0, \tag{4.11}$$

$$E(|x_1|)x_1 + E(|x_2|)x_2 + E(|x_3|)x_3 = 0, \tag{4.12}$$

where

$$E(x) := \frac{\left(x^2 + 1 + h + \sqrt{((x-1)^2 + h)((x+1)^2 + h)}\right)^2 + 4x^2}{\left(x^2 + 1 + h + \sqrt{((x-1)^2 + h)((x+1)^2 + h)}\right)}.$$

Without loss of generality, we can assume that $|x_1| \geq |x_2| \geq |x_3|$. Moreover, it is easy to check that $E(x)$ is a positive increasing function on $[0, \infty)$. Therefore, $E(|x_1|) \geq E(|x_2|) \geq E(|x_3|) > 0$. From (4.12), we have

$$E(|x_3|)x_3 = -E(|x_1|)x_1 - E(|x_2|)x_2$$

and

$$E(|x_3|)\overline{x_3} = -E(|x_1|)\overline{x_1} - E(|x_2|)\overline{x_2},$$

which imply that

$$|x_3|^2 E(|x_3|)^2 = |x_1|^2 E(|x_1|)^2 + |x_2|^2 E(|x_2|)^2 + E(|x_1|)E(|x_2|)(x_1\overline{x_2} + x_2\overline{x_1}). \tag{4.13}$$

Note that since $|x_1| \geq |x_2| \geq |x_3|$ and x_1, x_2, x_3 are distinct,

$$x_1\overline{x_2} + x_2\overline{x_1} \in (-\infty, 0). \tag{4.14}$$

From (4.11), we have

$$x_3^2 = -x_1^2 - x_2^2 \qquad \text{and} \qquad \overline{x_3}^2 = -\overline{x_1}^2 - \overline{x_2}^2,$$

which imply that

$$|x_3|^4 = |x_1|^4 + |x_2|^4 + x_1^2\overline{x_2}^2 + x_2^2\overline{x_1}^2 = |x_1|^4 + |x_2|^4 - 2|x_1|^2|x_2|^2 + (x_1\overline{x_2} + x_2\overline{x_1})^2.$$

Therefore,

$$(x_1\overline{x_2} + x_2\overline{x_1})^2 = |x_3|^4 - (|x_1|^2 - |x_2|^2)^2.$$

By (4.14),

$$(x_1\overline{x_2} + x_2\overline{x_1}) = -\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2}$$

From (4.13), we obtain

$$|x_3|^2E(|x_3|)^2 + E(|x_1|)E(|x_2|)\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2} = |x_1|^2E(|x_1|)^2 + |x_2|^2E(|x_2|)^2.$$

Since

$$E(|x_1|)E(|x_2|)\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2} \leq |x_1|^2E(|x_1|)^2$$

and

$$|x_3|^2E(|x_3|)^2 \leq |x_2|^2E(|x_2|)^2,$$
$$\sqrt{|x_3|^4 - (|x_1|^2 - |x_2|^2)^2} = |x_1|^2,$$

which implies $|x_1| = |x_2| = |x_3|$. Applying Corollary 2.4, $\{x_1, x_2, x_3\}$ forms a set of distinct equally spaced points on a circle cantered at 0. ■

5. Proof of Section 3

Recall that for the proofs in this section, we also consider our problems in the complex plane (see our discussion at the beginning of Section 4).

Lemma 5.1. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \ldots, N-1\}$, $R > 0$, $r > 0$, and $h \geq 0$. Then, any configuration of N distinct equally spaced points on $\mathbb{S}_r^1$ is both maximal and minimal N-point Riesz $(-2p, h)$-polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}_R^1)$.*

Proof of Lemma 5.1. Let $\omega_N := \{x_1, \ldots, x_N\}$ be a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$, $p \in \{1, 2, \ldots, N-1\}$ be fixed, and $h \geq 0$ be fixed. By [11, Theorem 1], we know that $f(x) := \sum_{j=1}^{N}(|x - x_j|^2 + h)^p$ is constant as a function of x on $\mathbb{S}_R^1$, say $f(x) \equiv C$ for all $x \in \mathbb{S}_R^1$.

Let $\{y_1, \ldots, y_N\}$ be any N-point configuration on $\mathbb{S}_r^1$. Clearly, $y_j/r, x_j/r \in \mathbb{S}^1$ for all $j = 1, 2, \ldots, N$. Then,

$$NC = \sum_{i=1}^{N} f\left(\frac{R}{y_i/r}\right) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\left|x_j - \frac{R}{y_i/r}\right|^2 + h\right)^p$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\left|\frac{x_j/r}{y_i/r}\right|^2\left|y_i - \frac{R}{x_j/r}\right|^2 + h\right)^p = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\left|y_i - \frac{R}{x_j/r}\right|^2 + h\right)^p$$
$$= \sum_{j=1}^{N}\sum_{i=1}^{N}\left(\left|y_i - \frac{R}{x_j/r}\right|^2 + h\right)^p.$$

Therefore, there exist $j_0, j_0' \in \{1,2,\dots,N\}$ such that

$$\sum_{i=1}^{N}\left(\left|y_i - \frac{R}{x_{j_0}/r}\right|^2 + h\right)^p \geq C \quad \text{and} \quad \sum_{i=1}^{N}\left(\left|y_i - \frac{R}{x_{j_0'}/r}\right|^2 + h\right)^p \leq C.$$

Then, we have

$$\max_{x\in\mathbb{S}^1_R}\sum_{i=1}^{N}\left(|y_i - x|^2 + h\right)^p \geq C = \max_{x\in\mathbb{S}^1_R}\sum_{i=1}^{N}\left(|x_i - x|^2 + h\right)^p$$

and

$$\min_{x\in\mathbb{S}^1_R}\sum_{i=1}^{N}\left(|y_i - x|^2 + h\right)^p \leq C = \min_{x\in\mathbb{S}^1_R}\sum_{i=1}^{N}\left(|x_i - x|^2 + h\right)^p$$

which imply

$$\max_{x\in\mathbb{S}^1_R}\sum_{i=1}^{N}\left(|x_i - x|^2 + h\right)^p = m_N^{-2p,h}(\mathbb{S}^1_r;\mathbb{S}^1_R)$$

and

$$\min_{x\in\mathbb{S}^1_R}\sum_{i=1}^{N}\left(|x_i - x|^2 + h\right)^p = M_N^{-2p,h}(\mathbb{S}^1_r;\mathbb{S}^1_R),$$

respectively. Therefore, ω_N is both maximal and minimal N-point Riesz $(-2p,h)$-polarization configuration of $(\mathbb{S}^1_r;\mathbb{S}^1_R)$. ■

Proof of Theorem 3.1. Because the proof of (a) ⇔ (c) is similar to the proof of (b) ⇔ (c), we will show only (b) ⇔ (c) and skip the proof of (a) ⇔ (c). Moreover, without loss of generality, we can assume that $R = 1$.

Let $N \in \mathbb{N}$, $p \in \{1,2,\dots,N-1\}$, $r > 0$, and $h \geq 0$ be fixed. Notice that for all configurations $\{x_1,x_2,\dots,x_N\} \subset \mathbb{S}^1_r$, the constants $E_0^{(p)}$ and $E_{k,j}^{(p)}$ in (4.3) and (4.4) depend only on k. For convenience, for all configurations $\{x_1,x_2,\dots,x_N\} \subset \mathbb{S}^1_r$, we set

$$E := E_0^{(p)} \qquad \text{and} \qquad E_k := \frac{E_{k,j}^{(p)}}{r^k}, \qquad k = 1,2,\dots,p.$$

First of all, we show that

$$m_N^{-2p,h}(\mathbb{S}^1_r;\mathbb{S}^1) = E. \tag{5.1}$$

Let $\omega_N' := \{x_1',x_2',\dots,x_N'\}$ be a configuration of distinct equally spaced points on $\mathbb{S}^1_r$. Using (4.2), we have for all $y \in \mathbb{S}^1$,

$$\sum_{j=1}^{N}(|y - x_j'|^2 + h)^p = E + \sum_{k=1}^{p}\sum_{j=1}^{N}E_k(y^k\cdot(x_j')^k) = E + \sum_{k=1}^{p}E_k\Big(y^k\cdot\sum_{j=1}^{N}(x_j')^k\Big) = E \tag{5.2}$$

where the last equality follows from the fact that $\sum_{j=1}^{N}(x_j')^k = 0$ for all $k = 1,2,\dots,N-1$. By Lemma 5.1, since ω_N' is a minimal N-point Riesz $(-2p,h)$-polarization configuration of $(\mathbb{S}^1_r;\mathbb{S}^1)$,

$$m_N^{-2p,h}(\mathbb{S}^1_r;\mathbb{S}^1) = \max_{y\in\mathbb{S}^1}U^{-2p,h}(y;\omega_N') = E \tag{5.3}$$

as we wanted.

Now, we prove (c)⇒(b). Assume that $\omega_N = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{S}^1_r$ and $\sum_{j=1}^N x_j^k = 0$ for all $k = 1, 2, \ldots, p$. Applying the same argument as in (5.2), we have for all $y \in \mathbb{S}^1$,

$$U^{-2p,h}(y;\omega_N) = E + \sum_{k=1}^{p} E_k\Big(y^k \cdot \sum_{j=1}^{N} x_j^k\Big) = E,$$

which implies that ω_N is a minimal N-point Riesz $(-2p, h)$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1)$.

Next, we show (b)⇒(c). Assume that $\omega_N := \{x_1, x_2, \ldots, x_N\}$ is a minimal N-point Riesz $(-2p, h)$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1)$. Then, for all $y \in \mathbb{S}^1$,

$$U^{-2p,h}(y;\omega_N) = \sum_{j=1}^{N}(|y - x_j|^2 + h)^p \le m_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1) = E.$$

Set $y = \cos(t) + i\sin(t) \in \mathbb{S}^1$ and $x_j = r\cos(t_j) + ir\sin(t_j) \in \mathbb{S}^1_r$ for all $j = 1, 2, \ldots, N$. Hence, by (4.1), for all $t \in [0, 2\pi]$,

$$E \ge U^{-2p,h}(y;\omega_N) = E + \sum_{k=1}^{p}\left[\left(\sum_{j=1}^{N}\frac{E_k}{r^k}\cos(kt_j)\right)\cos(kt) + \left(\sum_{j=1}^{N}\frac{E_k}{r^k}\sin(kt_j)\right)\sin(kt)\right].$$

Then, for all $t \in [0, 2\pi]$,

$$0 \ge \sum_{k=1}^{p}\left[\left(\sum_{j=1}^{N}\frac{E_k}{r^k}\cos(kt_j)\right)\cos(kt) + \left(\sum_{j=1}^{N}\frac{E_k}{r^k}\sin(kt_j)\right)\sin(kt)\right].$$

It is not difficult to check that for all $t \in [0, 2\pi]$,

$$\sum_{k=1}^{p}\left[\left(\sum_{j=1}^{N}\frac{E_k}{r^k}\cos(kt_j)\right)\cos(kt) + \left(\sum_{j=1}^{N}\frac{E_k}{r^k}\sin(kt_j)\right)\sin(kt)\right] = 0.$$

Because $\{\cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(pt), \sin(pt)\}$ is linearly independent over $\mathbb{R}$, for all $k = 1, 2, \ldots, p$,

$$\sum_{j=1}^{N}\frac{E_k}{r^k}\cos(kt_j) = \sum_{j=1}^{N}\frac{E_k}{r^k}\sin(kt_j) = 0.$$

Since for all $k = 1, 2, \ldots, p$, $E_k \neq 0$ (see the formula in (4.4)),

$$\sum_{j=1}^{N}\cos(kt_j) = \sum_{j=1}^{N}\sin(kt_j) = 0, \qquad k = 1, 2, \ldots, p,$$

which imply that $\sum_{j=1}^N x_j^k = 0$ for all $k = 1, 2, \ldots, p$. Moreover, from (4.3), we have

$$M_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1) = m_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1)$$
$$= E = \frac{N}{2^p}\sum_{j=0}^{p}\binom{p}{j}^2(2r)^{2j}\left(r^2 + 1 + h + \sqrt{((r-1)^2 + h)(r+1)^2 + h)}\right)^{p-2j}.$$

To compute $M_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1_R) = m_N^{-2p,h}(\mathbb{S}^1_r; \mathbb{S}^1_R)$ in (9), we can use a similar argument as in the proof of Lemma 4.1 by replacing $y = R\cos t + iR\sin t$ and $f_j(t) := (|y - x_j|^2 + h)^p = (R_j^2 + R^2 + h - 2R_jR\cos(t - t_j))^p$. Applying the calculations as in Lemma 6.2, it is not difficult to check that if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}^1_r$, then for all $y \in \mathbb{S}^1_R$,

$$U^{-2p,h}(y;\omega_N)$$
$$= \frac{N}{2^p} \sum_{j=0}^{p} \binom{p}{j}^2 (2rR)^{2j} \left(r^2 + R^2 + h + \sqrt{((r-R)^2+h)(r+R)^2+h)}\right)^{p-2j}.$$

■

6. Appendix

We collect our computations of all integrals in this section.

Lemma 6.1. *Let $p \in \mathbb{N}$, $k \in \{0,1,\dots,p\}$, and $z \in \mathbb{C}$. Then,*

$$\int_0^{2\pi} (z^2+1-2z\cos(t))^p \cos(kt)dt = (-1)^k 2\pi \sum_{q=0}^{p-k} \binom{p}{q}\binom{p}{k+q} z^{2p-k-2q}. \tag{6.1}$$

Proof of Lemma 6.1. Let $p \in \mathbb{N}$ and $k \in \{1,\dots,p\}$. First, we prove the equality (6.1) for $z \in \mathbb{R}$. Let $z \in \mathbb{R}$. Then, for $\zeta = e^{it}$,

$$\int_0^{2\pi} (z^2+1-2z\cos(t))^p \cos(kt)dt = \int_0^{2\pi} (z^2+1-z(e^{it}+e^{-it}))^p e^{ikt}dt$$
$$= \int_0^{2\pi} (z-e^{it})^p (z-e^{-it})^p e^{ikt}dt = \frac{1}{i}\int_{\mathbb{S}^1} (z-\zeta)^p (z-1/\zeta)^p \zeta^{k-1} d\zeta$$
$$= 2\pi \cdot \operatorname{res}\left(\frac{(z-\zeta)^p(z\zeta-1)^p}{\zeta^{p-k+1}};0\right) = (-1)^k 2\pi \sum_{q=0}^{p-k}\binom{p}{q}\binom{p}{k+q} z^{2p-k-2q},$$

where the first equality follows from the fact that the last expression is a real number. Notice that the left-hand side and the right-hand side of the equation (6.1) are polynomials as functions of z. Then, both functions are analytic on $\mathbb{C}$ and we have the equation (6.1) for all $z \in \mathbb{C}$. ■

Lemma 6.2. *Let $p \in \mathbb{N}$ and $k \in \{0,1,\dots,p\}$. For $a, b \in \mathbb{C}$,*

$$\int_0^{2\pi} (a-b\cos(t))^p \cos(kt)dt = \frac{(-1)^k\pi}{2^{p-1}} \sum_{q=0}^{p-k}\binom{p}{q}\binom{p}{k+q} b^{2q+k}\left(a \pm \sqrt{a^2-b^2}\right)^{p-k-2q}, \tag{6.2}$$

where the square root function in (6.2) can be selected to be both branches of the complex square root function.

Proof of Lemma 6.2. Clearly, if $b = 0$, then the equation in (6.2) is $0 = 0$. Assume that $b \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C}$. To reduce the equation (6.2) to the equation (6.1), we consider

$$(\lambda a - \lambda b \cos(t))^p,$$

where λ is chosen to satisfy the equations

$$2z = b\lambda \qquad \text{and} \qquad z^2 + 1 = a\lambda,$$

for some $z \in \mathbb{C}$. From above equations,

$$z = \frac{a \pm \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \lambda = \frac{2a \pm 2\sqrt{a^2 - b^2}}{b^2}.$$

Moreover, $\lambda \neq 0$ because if $\lambda = 0$, then $z = 0$ which implies that $b = 0$. Therefore, by Lemma 6.1,

$$\begin{aligned}\int_0^{2\pi} (a - b\cos(t))^p \cos(kt)dt &= \frac{1}{\lambda^p} \int_0^{2\pi} (\lambda a - \lambda b\cos(t))^p \cos(kt)dt \\ &= \frac{1}{\lambda^p} \int_0^{2\pi} (z^2 + 1 - 2z\cos(t))^p \cos(kt)dt \\ &= \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} b^{2q+k} \left(a \pm \sqrt{a^2 - b^2}\right)^{p-k-2q}.\end{aligned}$$

■

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