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Some Uniqueness Results for Fractional Differential Equation of Arbitrary Order with Nagumo Like Conditions

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Abstract In this work, we generalize the Krasnoselskii-Krein type of uniqueness theorem to $q > 1$ arbitrary along with Kooi and Rogers ones. The initial value problem is of the Riemann-Liouville type fractional differential equation, where the nonlinearity is depending on $D^{q-1}x$. Further, we establish the convergence of successive approximations of the Picard iterations of the equivalent Volterra integral equation. Finally, we give a numerical example illustrating the convergence of the successive approximations to the unique solution.

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1. Introduction

In the last several years, fractional differential equations attracted more and more researchers and have proven to be very useful tools for modeling phenomena in physics, finance, and many other areas.

There are several ways to define the differential and integral operators of arbitrary order. For instance, we can name the Caputo, the Grunwald-Letnikov and the Riemann-Liouville definitions. The last one has nice mathematical properties [1, 2], however it is not always the most convenient one to describe directly real physical problems and frequently leads to difficulties when attempting to handle initial conditions in a meaningful way. The reason is that $D^q c \neq 0$, for a constant c. Nevertheless, the Riemann-Liouville formulation arises in a natural way for problems such as transport problems from the continuum random walk scheme or generalized Chapman-Kolmogorov models [3, 4]. It was

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also applied for modeling of the behavior of viscoelastic and viscoplastic materials under external influences [5] and the continuum and statistical mechanics for visco-elasticity problems [6].

On the other hand, one of the most important theorems in ordinary differential equations is Picard's existence and uniqueness theorem for first-order differential equations. One reason is that it can be generalized to establish existence and uniqueness results for higher-order ordinary differential equations. For a monograph on different types of uniqueness theorems for differential equations, we refer to [7]. But what about fractional order differential equations (FDE)?

Recently, the Krasnoselskii-Krein, Nagumo type uniqueness results and successive approximations have been extended to differential equations of fractional order. In [8, 9], the author considered the problem $D^q x = f(t, x)$, $x(t_0) = x_0$ for $0 < q < 1$. Later, in [10], uniqueness results were obtained for $D^q x = f(t, x(t), D^{q-1}x(t))$, $x(0) = 0$, $D^{q-1}x(0) = 0$, with $1 < q < 2$. And several other paper were interested by Nagumo like uniqueness conditions, see for instance, [11, 12]. Motivated by the increasing interest in these theorems, in this paper we generalize the previous uniqueness results of Krasnoselkii-Krein, Kooi and Rogers to initial value problems (IVP) of arbitrary $q > 1$. We study the following IVP:

$$\begin{cases} D^q x(t) = f(t, x(t), D^{q-1}x(t)) \\ x(0) = 0,\ D^{(q-i)}x(0) = 0,\ i = 1, \ldots, [q]. \end{cases} \tag{1.1}$$

The organization of this paper is as follows. Section 2 will contain some basic definitions and concerning of fractional order differential equations after exposing some known uniqueness results for ordinary and fractional differential equations and the results that we wish to generalize. In Section 3, we prove the equivalence between the problem (1.1) and the associated Volterra integral equation. Section 4 is devoted to the main result ; a Krasnoselskii-Krein uniqueness theorem along with the convergence of successive approximations of (1.1). In Section 5, we give an example illustrating numerically our result.

2. Preliminaries

Consider the following IVP:

$$x'(t) = f(t, x(t)),\ x(t_0) = x_0. \tag{2.1}$$

It is well-known that if the function $f(t, x)$ is continuous and if it satisfies a Lipschitz condition in x, then the problem (2.1) has a unique solution that is the limit of Picard's iterates. Over the past several decades, uniqueness conditions less restrictive than the Lipschitz continuity were obtained. A different type of uniqueness result involving two conditions instead of one Lipschitz type condition was first given by Krasnoselskii and Krein [7], which we state below:

Let $S = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b,\ a, b \in \mathbb{R}^+\}$.

Theorem 2.1. *Let $f(t, x)$ be a continuous function on S, which satisfies for all $(t, x), (t, \overline{x}) \in S$:*

(i) $|f(t, x) - f(t, \overline{x})| \leq k|t - t_0|^{-1}|x - \overline{x}|$, $t \neq t_0$,
(ii) $|f(t, x) - f(t, \overline{x})| \leq c|x - \overline{x}|^{\alpha}$, *where c and k are positive constants, the real number α is such that $0 < \alpha < 1$, and $k(1 - \alpha) < 1$.*

Then the IVP (2.1) has at most one solution in $|t - t_0| \leq a$.

Another version for the Riemann-Liouville type fractional differential equation was obtained in [10].

Let $R_0 = \{(t, x, y) : 0 \leq t \leq 1, |x| \leq b, |y| \leq d,\ b, d \in \mathbb{R}^+\}$ and $\mathcal{C}(R_0, \mathbb{R})$ the Banach space of all continuous functions endowed with the norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$.

Theorem 2.2. *Let $f \in \mathcal{C}(R_0, \mathbb{R})$ satisfy the following Krein-type conditions:*

(i) $|f(t, x, y) - f(t, \overline{x}, \overline{y})| \leq \Gamma(q) \dfrac{k + \alpha(q-1)}{2t^{1-\alpha(q-1)}} [|x - \overline{x}| + |y - \overline{y}|]$, $t \neq 0$ and $0 < \alpha < 1$,

(ii) $|f(t, x, y) - f(t, \overline{x}, \overline{y})| \leq c \left[|x - \overline{x}|^\alpha + t^{\alpha(q-1)}|y - \overline{y}|^\alpha\right]$,

where c and k are positive constants and $k(1-\alpha) < 1+\alpha(q-1)$, for every $(t, x, y), (t, \overline{x}, \overline{y}) \in R_0$.

Then, the following successive approximations

$$x_{n+1}(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, x_n(s), D^{q-1}x_n(s))ds,$$
$$x_0(t) = 0, n = 0, 1, \ldots$$

given by the Picard's iterates converge uniformly to the unique solution x of (1.1) with $1 < q < 2$ on $[0, \eta]$, where $\eta = \min\left\{1, \left(\frac{b\Gamma(1+q)}{M}\right)^{1/q}, \frac{d}{M}\right\}$, M is the bound for f on R_0.

For other uniqueness theorems, namely Kooi's and Rogers's ones, we refer to [7, 10].

Now, Let us recall some basic definitions about fractional derivatives. Here and in the rest of the article, we denote by Γ the Gamma function and $[\alpha]$ the integer part of α.

Definition 2.3. The fractional integral of the function $h : (0, \infty) \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^\alpha h(t) = I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s)ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.4. For a function $h \in \mathcal{C}((0, \infty), \mathbb{R})$, the Riemann-Liouville fractional derivative of h is defined by

$$D^\alpha h(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t - s)^{n-\alpha-1} h(s)ds,$$

where $n \in \mathbb{N}^*$ and $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

It's well known that if $q \in (0, 1)$ then I^q maps $L(0, 1)$ to $L(0, 1)$ see [13] and for $q > 0$ the operator I^q maps $L(0, 1)$ to $L(0, 1)$.

We denote by $AC(0, 1)$ the space of absolutely continuous functions defined on $[0, 1]$. In fact, $x \in AC(0, 1)$ if and only if there exist $\phi \in L(0, 1)$ and $c \in \mathbb{R}$ such that

$$x(t) = c + \int_0^t \phi(s)ds \text{ for } t \in (0, 1).$$

Also, we define $AC^{n-1}(0, 1)$ by

$$AC^{n-1}(0, 1) = \{x \in \mathcal{C}^{n-2}, x^{(n-1)} \in AC(0, 1)\}$$

A function x is called a solution of the IVP (1.1) on an $[0, 1]$, if $x \in \mathcal{C}([0, 1], \mathbb{R})$, $D^q x$ exists and is continuous on $[0, 1]$ and x satisfies (1.1).

3. Equivalence with the Associated Volterra Integral Equation

In order to prove the results, we first study the relation between the IVP and the associated Volterra integral equation given by

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), D^{q-1}x(s))ds. \tag{3.1}$$

The following theorem gives the result of composing Riemann-Liouville fractional right side derivative and integral of the same order.

Theorem 3.1 ([1])**.** *For any given $n-1 \leq q < n$ and for some $x \in AC^n[0,1]$ we have*

$$I^q D^q x(t) = D^{q-q}x(t) - \sum_{j=1}^{n} [D^{(q-j)}x(0)] \frac{t^{(q-j)}}{\Gamma(q-j+1)}. \tag{3.2}$$

For a detailed discussion and other properties and theorems about fractional derivatives and fractional differential equations, we refer to [1, 13–15].

Remark 3.2. If $x \in \mathcal{C}^{n-1}[0,1]$, $n-1 \leq q < n$, then Theorem 3.1 is still true due to the integrability of $D^q x(t)$ and consequently all the $D^{(q-j)}x$ are bounded at 0.

Lemma 3.3. *Let $F \in \mathcal{C}([0,1], \mathbb{R})$ such that $F(t) = f(t, x(t), D^{q-1}x(t))$ and x verifies (3.1), then the function $I^q F$ has the following properties*

(i) $I^q F \in \mathcal{C}^{n-1}([0,1], \mathbb{R})$,

(ii) $D^{q-i} I^q F(0) = 0, \quad \forall i \in \{1, 2, \ldots, n-1\}$,

Proof. Since F and $t^{q-1}F \in \mathcal{C}([0,1], \mathbb{R})$ and from the property of integration's operator see [1]

$$I^q F(t) = \underbrace{I^1.I^1 \ldots I^1}_{n-1}.I^{q-n+1}F(t),$$

we get the part one and the second part two is an immediate consequence. Indeed $D^{q-i} I^q F(t) = I^i F(t) = D^{q-i}x(t)$ so $\lim_{t \to 0} D^{q-i} I^q F(t) = 0$. ∎

Remark 3.4. The previous Lemma shows that the condition $f \in \mathcal{C}(R_0, \mathbb{R})$ is sufficient to make $I^q f$ satisfying the initial conditions of (1.1) for arbitrary order q.

By Theorem 3.1 and Lemma 3.3 the IVP (1.1) is equivalent to the following Volterra fractional integral equation

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), D^{q-1}x(s))ds.$$

4. Main Result

Now, we state the Krasnoselskii-Krein type conditions for the IVP (1.1) of the Riemann-Liouville type fractional differential equation which involves derivative term in the function f and the order $q > 1$.

Theorem 4.1. *Let $f \in \mathcal{C}(R_0, \mathbb{R})$ satisfy the following Krein-type conditions:*

(A) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \min\{\Gamma(q),1\}\dfrac{k+\alpha(q-[q])}{2t^{1-\alpha(q-[q])}}\left[|x-\overline{x}| + |y-\overline{y}|\right],$
$t \neq 0$ *and* $0 < \alpha < 1$,

(B) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq c\left[|x-\overline{x}|^{\alpha} + t^{\alpha(q-[q])}|y-\overline{y}|^{\alpha}\right],$

where c and k are positive constants and $k(1-\alpha) < 1+\alpha(q-[q])$, for $(t,x,y),(t,\overline{x},\overline{y}) \in R_0$. Then the successive approximations given by

$$x_{j+1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x_j(s), D^{q-1}x_j(s))ds$$
$$x_0(t) = 0, n = 0, 1, \ldots \tag{4.1}$$

converge uniformly to the unique solution x of (1.1) on $[0,\eta]$, where $\eta = \min\left\{1, \left(\frac{b\Gamma(1+q)}{M}\right)^{1/q}, \frac{d}{M}\right\}$, M is the bound for f on R_0.

Proof. First, we establish the uniqueness, we suppose x and y are any two solutions of (1.1) on $[0,\eta]$ and let $\phi(t) = |x(t) - y(t)|$ and $\theta(t) = |D^{q-1}x(t) - D^{q-1}y(t)|$. Note that $\phi(0) = \theta(0) = 0$. We define $R(t) = \int_0^t \left[\phi^{\alpha}(s) + s^{\alpha(q-[q])}\theta^{\alpha}(s)\right] ds$, clearly $R(0) = 0$. Further, we have for $t \in [0,\eta]$

$$x(t) = I^q f(t, x(t), D^{q-1}x(t)),$$
$$D^{q-1}x(t) = D^{q-1}I^q[f(t,x(t),D^{q-1}x(t))] = \int_0^t f(s,x(s),D^{q-1}x(s))ds.$$

From the above inequalities and using the condition (B), we get

$$\begin{aligned}\phi(t) \leq& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))|ds \\ \leq& \frac{c}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[\phi^{\alpha}(s) + s^{\alpha(q-[q])}\theta^{\alpha}(s)\right] ds \leq \frac{c}{\Gamma(q)}t^{q-1}R(t).\end{aligned}$$

And

$$\theta(t) \leq \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))|ds \leq cR(t).$$

Seeking simplicity, we use the same symbol C to denote all different constants arising in the rest of the proof.
We have

$$\begin{aligned}R'(t) =& \phi^{\alpha}(t) + t^{\alpha(q-[q])}\theta^{\alpha}(t) \\ \leq& C\left[t^{\alpha(q-1)} + t^{\alpha(q-[q])}\right]R^{\alpha}(t).\end{aligned} \tag{4.2}$$

Since $R(t) > 0$ for $t > 0$, multiplying both sides of (4.2) by $(1-\alpha)R^{-\alpha}(t)$, and then integrating the resulting inequality, we get for $t > 0$

$$R(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1\right)} + t^{\left(\frac{\alpha}{1-\alpha}q+\frac{1-\alpha[q]}{1-\alpha}\right)}\right),$$

where we used the fact that for every $a, b \in (0,1)$ $2^{-1}(a+b)^{1-\alpha} \leq 2^{-(1-\alpha)}(a^{1-\alpha} + b^{1-\alpha})$. This leads to the following estimates on $\phi(t)$ and $\theta(t)$, for $t \in [0,\eta]$,

$$\phi(t) \leq C\left(t^{\left(\frac{q}{1-\alpha}\right)} + t^{\left(\frac{q}{1-\alpha}+\frac{\alpha(1-[q])}{1-\alpha}\right)}\right),$$

$$\theta(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1\right)} + t^{\left(\frac{\alpha}{1-\alpha}q+\frac{1-\alpha[q]}{1-\alpha}\right)}\right).$$

Define the function $\psi(t) = t^{-k}\max\{\phi(t), \theta(t)\}$ for $t \in (0,1]$. Then either $t^{-k}\phi(t)$ or $t^{-k}\theta(t)$ is the maximum. It follows that either

$$0 \leq \psi(t) \leq C\left(t^{\left(\frac{q}{1-\alpha}-k\right)} + t^{\left(\frac{q}{1-\alpha}+\frac{\alpha(1-[q])}{1-\alpha}-k\right)}\right)$$

or

$$0 \leq \psi(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1-k\right)} + t^{\left(\frac{\alpha q}{1-\alpha}+\frac{1-\alpha[q]}{1-\alpha}-k\right)}\right).$$

Since by assumption $k(1-\alpha) < 1+\alpha(q-[q])$, we verify easily that the inequalities in (4.3) hold

$$\begin{aligned} k(1-\alpha) &< q, \\ (k-1)(1-\alpha) &< \alpha q, \\ k(1-\alpha) &< q+\alpha-\alpha[q]), \\ k(1-\alpha) &< \alpha q+1-\alpha[q]. \end{aligned} \tag{4.3}$$

So all of the exponents of t in the above inequalities are positive. Hence, $\lim_{t\to 0^+}\psi(t) = 0$. Therefore, if we define $\psi(0) = 0$, the function $\psi(t)$ is continuous in $[0,\eta]$. We want to prove that $\psi \equiv 0$. Since he function ψ is continuous, if ψ doesn't vanish at some points t that is $\psi(t) > 0$ on $]0,\eta]$, then there exists a maximum $m > 0$ reached when t is equal to some t_1: $0 < t_1 \leq \eta \leq 1$ such that $\psi(s) < m = \psi(t_1)$, for $s \in]0, t_1)$. Therefor from condition (A) we get for either cases

$$m = \psi(t_1) = t_1^{-k}\phi(t_1) \leq \min(\Gamma(q),1)mt_1^{q-1+\alpha(q-[q])} < m$$

or

$$m = \psi(t_1) = t_1^{-k}\theta(t_1) \leq \min(\Gamma(q),1)mt_1^{\alpha(q-[q])} < m,$$

which is a contradiction. Thus, the uniqueness of the solution is established.

For the second part, we use Arzela-Ascoli Theorem. First, we show that the successive approximations $x_{j+1}(t), j = 0,1,\ldots$ given by (4.1) are well-defined and continuous on $[0,\eta]$. In fact,

$$\begin{aligned} |x_{j+1}(t)| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left|f(s,x_j(s),D^{q-1}x_j(s))\right| ds \\ |D^{q-1}x_{j+1}(t)| &\leq \int_0^t \left|f(s,x_j(s),D^{q-1}x_j(s))\right| ds. \end{aligned}$$

For $j = 0$ and $t \in [0,\eta]$, we have

$$|x_1(t)| \leq \frac{Mt^q}{\Gamma(q+1)} \leq b \quad \text{and} \quad |D^{q-1}x_1(t)| \leq Mt \leq d.$$

Moreover, for every $i \in \{0,\ldots,n-1\}$ we have

$$\begin{aligned} |x_1^{(i)}(t)| &= |D^i I^q f(t,x_0(t),D^{q-1}x_0(t))| \\ &= |I^{q-i} f(t,x_0(t),D^{q-1}x_0(t))| \\ &= \left|\frac{1}{\Gamma(q-i)}\int_0^t (t-s)^{q-i-1} f(s,x_0(s),D^{q-1}x_0(s))ds\right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{M}{\Gamma(q-i)} \int_0^t (t-s)^{q-i-1} ds \\
&\leq \frac{Mt^{q-i}}{(q-i)\Gamma(q-i)} \\
&\leq \frac{Mt^{q-i}}{\Gamma(q-i+1)}.
\end{aligned}$$

By induction, the sequences $\{x_{j+1}(t)\}$ and $\{D^{q-1}x_{j+1}(t)\}$ are well-defined and uniformly bounded on $[0,\eta]$. We verify that the family $\{D^{q-1}j_{n+1}(t)\}$ is equicontinuous in $\mathcal{C}[0,1]$ and that the family $\{x_{j+1}(t)\}$ is equicontinuous in $\mathcal{C}^{n-1}[0,1]$. We may prove that y and z are continuous functions in $[0,\eta]$, where y and z are defined by

$$\begin{aligned}
y(t) &= \limsup_{j\to\infty} |x_j(t) - x_{j-1}(t)|, \\
z(t) &= \limsup_{j\to\infty} |D^{q-1}x_j(t) - D^{q-1}x_{j-1}(t)|.
\end{aligned}$$

Let us note

$$m(t) = \sup_{i\leq n-1} \limsup_{j\to\infty} |x_j^{(i)}(t) - x_{j-1}^{(i)}(t)|.$$

For $t_1, t_2 \in [0,\eta]$ we have

$$|x_{j+1}(t_1) - x_j(t_1)| \leq |x_j(t_2) - x_{j-1}(t_2)| + \frac{4M}{\Gamma(q+1)}(t_2-t_1)^q,$$

and we get for every $i \in \{0,\ldots,n-1\}$

$$|x_{j+1}^{(i)}(t_1) - x_j^{(i)}(t_1)| \leq |x_j^{(i)}(t_2) - x_{j-1}^{(i)}(t_2)| + \frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i}.$$

In fact, for $0 \leq t_1 \leq t_2$ and for every $i \in \{0,\ldots,n-1\}$, consider the difference

$$\begin{aligned}
&|x_{j+1}^{(i)}(t_1) - x_j^{(i)}(t_1)| - |x_{j+1}^{(i)}(t_2) - x_j^{(i)}(t_2)| \\
&\qquad \leq |x_{j+1}^{(i)}(t_1) - x_j^{(i)}(t_1) - x_{j+1}^{(i)}(t_2) + x_j^{(i)}(t_2)| \\
&\qquad \leq \frac{1}{\Gamma(q-i)} \left[\left| \int_0^{t_1} (t_1-s)^{q-1-i} D(s) ds \right.\right. \\
&\qquad\quad \left.\left. - \int_0^{t_2} (t_2-s)^{q-1-i} D(s) ds \right| \right] \\
&\qquad \leq \frac{2M}{\Gamma(q-i)} \left[\left| \int_0^{t_1} ((t_1-s)^{q-1-i} - (t_2-s)^{q-1-i}) ds \right.\right. \\
&\qquad\quad \left.\left. - \int_{t_1}^{t_2} (t_2-s)^{q-1-i} ds \right| \right] \\
&\qquad \leq \frac{2M}{(q-i)\Gamma(q-i)} \left[t_1^{q-i} - t_2^{q-i} + 2(t_2-t_1)^{q-i} \right] \\
&\qquad \leq \frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i},
\end{aligned}$$

where $D(s) = |f(s, x_j(s), D^{q-1}x_j(s)) - f(s, x_{j-1}(s), D^{q-1}x_{j-1}(s))| \leq 2M$. Let us note

$$\sigma = \max_{i \leq n-1} \left\{ \frac{4M}{\Gamma(q-i+1)} (t_2 - t_1)^{q-i} \right\}.$$

The right-hand side in the above inequalities is at most $m(t_2) + \epsilon + \sigma(t)$ for large n if $\epsilon > 0$ provided that

$$|t_2 - t_1| \leq \eta \leq \frac{Mt^q}{\Gamma(q+1)} \leq \sigma.$$

Since ϵ is arbitrary and t_1, t_2 can be interchangeable, we get $|m(t_1) - m(t_2)| \leq \sigma$. The same goes for $z(t)$: that is, $|z(t_1) - z(t_2)| \leq 2M|t_2 - t_1|$. These imply that $y(t)$ and $z(t)$ are continuous on $[0, \eta]$. Also, using condition (B) and the definition of successive approximations we obtain

$$\begin{aligned} |x_{j+1}(t) - x_j(t)| \leq & c \int_0^t (t-s)^{q-1} \left[|x_j(s) - x_{j-1}(s)|^\alpha \right. \\ & \left. + s^{\alpha(q-[q])} |D^{q-1}x_j(s) - D^{q-1}x_{j-1}(s)|^\alpha \right] ds \end{aligned}$$

and

$$\begin{aligned} |x^{(i)}_{j+1}(t) - x^{(i)}_j(t)| \leq & C \int_0^t (t-s)^{q-i-1} \left[|x_j(s) - x_{j-1}(s)|^\alpha \right. \\ & \left. + s^{\alpha(q-[q])} |D^{q-1}x_j(s) - D^{q-1}x_{j-1}(s)|^\alpha \right] ds. \end{aligned}$$

As a consequence, we obtain the following estimation for a certain $i = i_0$

$$\begin{aligned} \|x_{j+1} - x_j\| \leq & C \int_0^1 (1-s)^{q-i_0-1} \left[|x_j(s) - x_{j-1}(s)|^\alpha \right. \\ & \left. + s^{\alpha(q-[q])} |D^{q-1}x_j(s) - D^{q-1}x_{j-1}(s)|^\alpha \ \right] ds. \end{aligned}$$

All of the Arzela-Ascoli Theorem conditions are fulfilled for the family $\{x_j\}$ in $\mathcal{C}^{n-1}[0,1]$, respectively $\{D^{q-1}x_j\}$ in $\mathcal{C}[0,1]$. Hence, there exists a subsequence $\{x_{j_k}\}$, respectively $\{D^{q-1}x_{j_k}\}$ converging uniformly on $[0, \eta]$ as $j_k \to \infty$. Let us define for every $t \in [0, \eta]$

$$\begin{aligned} m^*(t) = & \limsup_{k \to \infty} |x_{j_k}(t) - x_{j_{k-1}}(t)|, \\ z^*(t) = & \limsup_{k \to \infty} |D^{q-1}x_{j_k}(t) - D^{q-1}x_{j_{k-1}}(t)|. \end{aligned}$$

Further, if $\{|x_j - x_{j-1}|\} \to 0$ and $\{|D^{q-1}x_j - D^{q-1}x_{j-1}|\} \to 0$ as $j \to \infty$, then (4.1) implies that the limit of any such subsequence is the unique solution x of (1.1). It follows that a selection of subsequences is unnecessary and that the entire sequence $\{x_j\}$ converges uniformly to x. For that, it suffices that $y \equiv 0$ and $z \equiv 0$ which lead to $m^*(t)$ and $z^*(t)$ being null. Setting

$$R(t) = \int_0^t \left[y(s)^\alpha + s^{\alpha(q-[q])} z(s)^\alpha \right] ds,$$

and by defining $\psi^*(t) = t^{-k} \max\{y(t), z(t)\}$, we show that $\lim_{t \to 0^+} \psi^*(t) = 0$. We shall now show that $\psi^*(t) \equiv 0$. If $\psi^*(t) > 0$ at any point in $[0, \eta]$, then there exists t_1 such that $0 < \overline{m} = \psi^*(t_1) = \max_{0 \leq t \leq \eta} \psi^*(t)$. Hence, from condition (A), we obtain

$$\overline{m} = \psi(t_1) = t_1^{-k} y(t_1) \leq \min(\Gamma(q), 1) \overline{m} t_1^{q-1+\alpha(q-[q])} < \overline{m}$$

or

$$\overline{m} = \psi(t_1) = t_1^{-k} z(t_1) \le \min(\Gamma(q), 1)\overline{m} t_1^{\alpha(q-[q])} < \overline{m}.$$

In both cases, we end up with a contradiction. So $\psi^* \equiv 0$. Therefore, the Picard iterates converge uniformly to the unique solution x of (1.1) on $[0, \eta]$. ■

Remark 4.2. Some may ask about the involving of $q - [q]$ and what if it's changed with $0 < p < 1$. The answer is that it's possible. All of $q - [q]$ in the above Theorem and proof can be changed by p but $k(1-\alpha) < 1+\alpha(q-[q])$ will be $k(1-\alpha) < 1+\alpha \min(q-[q], p)$. The choice of $q - [q]$ was made so to simplify and make our Theorem similar and generalizing the previous results, for more details we give the following Corollary.

Corollary 4.3. *Let $f \in \mathcal{C}(R_0, \mathbb{R})$ satisfy the following Krein-type conditions:*

(A) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \le \min\{\Gamma(q), 1\} \frac{k+\alpha p}{2t^{1-\alpha p}} \left[|x-\overline{x}| + |y-\overline{y}|\right]$, $t \ne 0$ *and* $0 < \alpha < 1$.

(B) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \le c\left[|x-\overline{x}|^\alpha + t^{\alpha p}|y-\overline{y}|^\alpha\right]$,

where c and k are positive constants, $0 < p < 1$ and $k(1-\alpha) < 1 + \alpha \min(q-[q], p)$, for $(t,x,y), (t,\overline{x},\overline{y}) \in R_0$. Then the successive approximations given by (4.1) converge uniformly to the unique solution x of (1.1) on $[0,\eta]$,

where $\eta = \min\left\{1, \left(\frac{b\Gamma(1+q)}{M}\right)^{1/q}, \frac{d}{M}\right\}$, *$M$ is the bound for f on R_0.*

Remark 4.4. For the case $1 < q < 2$, Theorem 4.1 is reduced to the uniqueness result provided in [10].

Theorem 4.5 (Kooi Type Uniqueness Theorem)**.** *Let f satisfy the following conditions:*

(i) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \le \min\{\Gamma(q), 1\} \frac{k+\alpha(q-[q])}{2t^{1-\alpha(q-[q])}} \left[|x-\overline{x}| + |y-\overline{y}|\right]$, $t \ne 0$ *and* $0 < \alpha < 1$,

(ii) $t^\beta |f(t,x,y) - f(t,\overline{x},\overline{y})| \le c\left[|x-\overline{x}|^\alpha + t^{\alpha(q-[q])}|y-\overline{y}|^\alpha\right]$,

where c and k are positive constants, the order $q > 1$ and $k(1-\alpha) < 1 + \alpha(q-[q]) - \beta$, for $(t,x,y), (t,\overline{x},\overline{y}) \in R_0$. Then the successive approximations given by (4.1) converge to the unique solution x on $[0,\eta]$.

Proof. It's similar to that of Theorem 4.1, thus we omit it. ■

Lemma 4.6. *Let ϕ and θ be nonnegative continuous functions in $[0,a]$ for a real number $a > 0$. Let $\psi(t) = \int_0^t \frac{\phi(s)+s^{q-[q]}\theta(s)}{2s^{q-[q]+2}} ds$. Assume the following:*

(i) $\phi(t) \le t^{q-[q]}\psi(t)$,

(ii) $\theta(t) \le \psi(t)$,

(iii) $\phi(t) = o(t^{q-[q]} e^{-1/t})$,

(iv) $\theta(t) = o(e^{-1/t})$.

Then $\phi \equiv \theta \equiv 0$.

Proof. Let $\psi(t) = \int_0^t \frac{\phi(s)+s^{q-[q]}\theta(s)}{2s^{q-[q]+2}} ds$. After differentiating ψ and using (ii), we obtain for $t > 0$, $\psi'(t) \le \frac{1}{t^2}\psi(t)$, so that $e^{1/t}\psi(t)$ is decreasing. Now, from (iii) and (iv), if $\epsilon > 0$ then for a small t, we have

$$e^{1/t}\psi(t) \le e^{1/t} \int_0^t \frac{1}{2s^2} 2\epsilon e^{-1/s} ds = \epsilon.$$

Hence, $\lim_{t\to 0} e^{1/t}\psi(t) = 0$ which implies that $\psi(t) \leq 0$. Finally, ψ is nonnegative due to (i), and thus $\psi \equiv 0$. ∎

Theorem 4.7 (Rogers' uniqueness theorem)**.** *Let the function f be such that the following conditions hold:*

(i) $f(t,x,y) \leq \min\{\Gamma(q),1\} o\left(\frac{e^{-1/t}}{t^2}\right)$, *uniformly for positive and bounded x and y*

(ii) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \min\{\Gamma(q),1\}\frac{1}{2t^{q-[q]+2}}\left[|x-\overline{x}| + t^{(q-[q])}|y-\overline{y}|\right]$.

Then the problem has at most one solution.

The proof of this theorem is essentially based on the Lemma 4.6.

Proof. Suppose x, y are two solutions of (1.1), we get for $t \in [0,a] \subset [0,1]$

$$\begin{aligned}
|x(t)-y(t)| \leq & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{2s^{q-[q]+2}}[|x(s)-y(s)| + s^{q-[q]}|D^{q-1}x(s) - D^{q-1}y(s)|]ds \\
\leq & t^{q-1}\int_0^t \frac{1}{2s^{q-[q]+2}}[|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]ds \\
\leq & t^{q-[q]}\int_0^t \frac{1}{2s^{q-[q]+2}}[|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]ds
\end{aligned}$$

$$\begin{aligned}
|D^{q-1}x(s) - D^{q-1}y(s)| \leq & \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
\leq & \int_0^t \frac{\min\{\Gamma(q),1\}}{2s^{q-[q]+2}}[|x(s)-y(s)| \\
& + s^{q-[q]}|D^{q-1}x(s) - D^{q-1}y(s)|]ds \\
\leq & \int_0^t \frac{1}{2s^{q-[q]+2}}[|x(s)-y(s)| \\
& + s^{q-[q]}|D^{q-1}x(s) - D^{q-1}y(s)|]ds.
\end{aligned}$$

Also, if $\epsilon > 0$, then from the condition (C) for small t, we have

$$\begin{aligned}
|x(t)-y(t)| \leq & \frac{t^{q-1}}{\Gamma(q)}\int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
< & t^{q-1}2\epsilon\int_0^t \frac{e^{-1/s}}{s^2}ds \leq t^{q-1}e^{-1/t}2\epsilon \\
< & \epsilon t^{q-[q]}e^{-1/t}2
\end{aligned}$$

and

$$\begin{aligned}
|D^{q-1}x(t) - D^{q-1}y(t)| \leq & \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
< & 2\epsilon\min\{1,\Gamma(q)\}\int_0^t \frac{e^{-1/s}}{s^2}ds \leq 2\epsilon e^{-1/t}.
\end{aligned}$$

According to Lemma 4.6, we obtain $|x-y| \equiv 0$, which ends the proof. ∎

5. Numerical Example

In this section, we apply our result obtained in Theorem 4.1 to a general version of the nonlinear fractional differential equation proposed in [10] for arbitrary $q > 1$.

Example 5.1. Let us consider the following equation

$$D^q x(t) = f(t, x(t)),$$

where

$$f(t,x) = \begin{cases} At^{q\alpha/(1-\alpha)}, & 0 \le t \le 1, -\infty < x < 0, \\ At^{q\alpha/(1-\alpha)} - Axt^{-q}, & 0 \le t \le 1, 0 \le x \le t^{q/(1-\alpha)}, \\ 0, & 0 \le t \le 1, t^{q/(1-\alpha)} < x < +\infty, \end{cases}$$

$$x(0) = 0,$$
$$D^{q-i}x(0) = 0 \, i = 1, \ldots, [q].$$

where $0 < \alpha < 1$, $A = \min(1, \Gamma(q))(q(k-1)+1)$, $k(1-\alpha) < 1 + \alpha(q - [q])$ and $c = 2^{(1-\alpha)}(q(k-1)+1)$.

The function $f(\cdot,\cdot)$ is continuous. Suppose $0 < x, \overline{x} < t^{q/(1-\alpha)}$, then

$$\begin{aligned} |f(t,x) - f(t,\overline{x})| \le& \left| -A\frac{x}{t^q} + A\frac{\overline{x}}{t^q} \right| \\ \le& \frac{A}{t^q}|x - \overline{x}| \end{aligned}$$

and

$$\begin{aligned} |f(t,x) - f(t,\overline{x})| \le& \frac{A}{t^q}|x - \overline{x}|^{1-\alpha}|x - \overline{x}|^{\alpha} \\ \le& \frac{A}{t^q}(|x| + |\overline{x}|)^{1-\alpha}|x - \overline{x}|^{\alpha} \\ \le& \frac{A}{t^q}2^{1-\alpha}t^q|x - \overline{x}|^{\alpha} \\ \le& A2^{1-\alpha}|x - \overline{x}|^{\alpha}. \end{aligned}$$

Suppose $t^{q/(1-\alpha)} < x < +\infty, -\infty < \overline{x} < 0$, then

$$\begin{aligned} |f(t,x) - f(t,\overline{x})| \le& \left| -At^{q\alpha/(1-\alpha)} \right| \le \frac{A}{t^q}x \\ \le& \frac{A}{t^q}|x - \overline{x}| \end{aligned}$$

and

$$\begin{aligned} |f(t,x) - f(t,\overline{x})| \le& At^{q\alpha/(1-\alpha)} \\ \le& A(|x| + |\overline{x}|)^{\alpha} \\ \le& A2^{1-\alpha}|x - \overline{x}|^{\alpha}. \end{aligned}$$

Suppose $t^{q/(1-\alpha)} < x < +\infty, 0 < \overline{x} < t^{q/(1-\alpha)}$, then

$$|f(t,x) - f(t,\overline{x})| \le \left| -At^{q\alpha/(1-\alpha)} + A\frac{\overline{x}}{t^q} \right| \le \frac{A}{t^q}\left| t^{q/(1-\alpha)} - \overline{x} \right|$$

$$\leq \frac{A}{t^q}|x - \overline{x}|$$

and

$$\begin{aligned}|f(t,x) - f(t,\overline{x})| \leq & A\left[\frac{t^{q/(1-\alpha)} - \overline{x}}{t^{(1/(1-\alpha))(1-\alpha)}}\right] \\ \leq & A\left[t^{q/(1-\alpha)} - \overline{x}\right]^{\alpha} \leq A(x - \overline{x})^{\alpha} \\ \leq & A2^{1-\alpha}|x - \overline{x}|^{\alpha}.\end{aligned}$$

Suppose $0 < x < t^{q/(1-\alpha)}, -\infty < \overline{x} < 0$, then

$$\begin{aligned}|f(t,x) - f(t,\overline{x})| \leq & \left|At^{q/(1-\alpha)} - A\frac{x}{t^q} - At^{q/(1-\alpha)}\right| \leq A\frac{x}{t^q} \\ \leq & \frac{A}{t^q}|x - \overline{x}|\end{aligned}$$

and

$$\begin{aligned}|f(t,x) - f(t,\overline{x})| \leq & A\frac{x}{t^q} < Ax^{\alpha}) \\ \leq & A2^{1-\alpha}|x - \overline{x}|^{\alpha}.\end{aligned}$$

From the below cases, it yields that

$$\begin{aligned}|f(t,x) - f(t,\overline{x})| \leq & \frac{A}{t^q}|x - \overline{x}|, \\ |f(t,x) - f(t,\overline{x})| \leq & A2^{1-\alpha}|x - \overline{x}|^{\alpha}.\end{aligned}$$

Thus all the condition of Theorem 4.1 are satisfied; the (IVP) has a unique solution on $[0, 1]$ limit of the successive approximations

$$x_{n+1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x_n(s))ds.$$

The works presented in [16–19] give several methods to approximate the solution of FDE. They have been of lower order, but the FracPECE [16] attracted our interest because of the relative ease of application and its reliability as far as convergence and stability are concerned.

The Figure 1 shows the approximate solution x_f given by the FracPECE with a stepsize $h = 0.01$, $q = 1.5$, $k = 1.7$ and $\alpha = 0.5$.

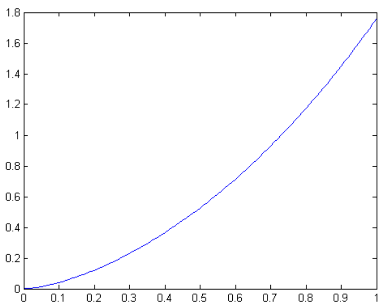

FIGURE 1. approximate solution x_f given by the FracPECE.

Next, we try to solve the problem using the successive approximations of Picard (4.1). First with different initial vectors x_0 and later by changing the values of q. We calculate the integral with the composite Simpson's rule with a stepsize $h = 0,01$ and we stop at the nth iteration whenever $err_1 = \|x_n - x_{n-1}\| \leq 0,01$. We then calculate $err_2 = \|x_n - x_f\|$ to see if x_n is converging to x_f or to an other function.

Our main Theorem stipulates that $x_n \to x$ independently of the chosen x_0. Moreover, the following two tables show that after discrediting our example the Picard's iterations continue to converge to the same solution.

In the first table we give the error between x_k and x_{k-1} and the error between x_k and the solution x_f for different values of x_0 and the number of iteration k needed to get $err_1 \leq 0,01$ continuing the iterations further won't give a divergence.

	err_1	err_2	err_1	err_2
x_0	0		$rand(0,1)$	
$k=1$	4.02e-1	1.35e-0	1.22e-0	4.61e-1
$k=2$	1.32e-0	3.02e-2	1.90e-1	2.27e-1
$k=3$	1.32e-0	3.02e-2	2.73e-1	3.02e-2
$k=4$	0	3.02e-2	0	3.02e-2
x_0	x_f		$-x_f$	
$k=1$	6.97e-3	6.97e-3	2.16e-0	1.35e-0
$k=2$	0	6.97e-3	1.32e-0	3.02e-2
$k=3$	0	6.97e-3	0	3.02e-2

TABLE 1. err_1 and err_2 for some values of x_0.

The number of iterations n needed to get an error bounded by $0,01$, is shown in the following Table for different values of the fractional order q. Although, one may note that for $x_0 = x_f$ the number of iterations to get to $\hat{x}$ is big but keep in mind that we didn't do any stability analysis and that we only used the Picard's iterates with no ajustements.

$x_0 \diagdown q$	1.01	1.44	1.5	2	4	4.001	4.98	5
0	3	3	3	3	3	3	5	5
rand(0,1)	4	4	4	4	4	4	4	4
x_f	1	1	1	1	2	2	2	2
$-x_f$	3	3	3	3	3	3	3	3

TABLE 2. Number of iteration to get $err_1 \leq 0.01$.

6. Conclusion

To summarize, the fundamental goal of this work has been to generalize the previous uniqueness results of F. Yoruk et al. [10] to arbitrary order using the Krasnoselskii-Krein, Rogers and Kooi conditions. Obviously, the numerical examples showed the convergence of the Picard's iterations even if it was somewhat slow for some values of q. The error, the stability analysis and the use of algorithms to solve this type of equations are going to be the subject of a future research. We finally hope that this work is a step in the study of the analytical and numerical aspect of fractional differential equations used in applied mathematics' fields.

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