Controllability Analysis of One- and Two-Dimensional Additive Real-Valued Cellular Automata

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Abstract In this paper, we study the regional controllability of one- and two-dimensional additive real-valued cellular automata with periodic, fixed, and reflective boundary conditions. The global transition functions of cellular automata are formulated in the matrix form to update the cofiguration. These results are applied to derive and prove the sufficient conditions of regional controllability for these additive cellular automata. In addition, the formulae of the control vectors and control vectors with least-norm for which the additive cellular automata are controllable have been obtained. The illustrations of simulation examples are provided to support the theoretical results.

1. INTRODUCTION

Mathematical models are useful tools to understand, predict, and control of real world systems by using mathematical language. The models can take many different forms depending on the type of the system and purpose of the model. Differential equation and partial differential equation are most often used to represent continuous and deterministic systems such as biological [1], physical [2], and chemical processes [3]. Stochastic models, which take into account random variables and probability, represent the random or probabilistic properties of many systems. Cellular automata (CA), agent-based models, and lattice gas models are stochastic discrete models, which are a popular technique for describing the states of individual elements of a system over discrete intervals.

Cellular automata are discrete models that consist of a regular lattice of cells with a finite number of states. The states of cells are updated synchronously according to identical transition functions (or transition rules) that relying on the previous states of cells in their neighbourhood [4]. The CA were first proposed by John Von Neumann

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in 1950 as a self-reproduction model in biological systems [5] and were systematically studied by Wolfram in the 1980s [6]. "The Game of Life" was constructed by Conway, which became a famous example of CA with two states, "alive" and "dead". The CA have been widely used to simulate complex biological, environmental, power systems such as mealybugs spreading [7], calcium signaling [8], forest fire spreading [9], transmission of disease [10], forest dynamics [11], atmospheric dispersion [12], dispersion of pollutants [13], and smart energy grids design [14].

The main components of CA are the lattice, a collection of states, neighbourhood, and the transition function. Many various types of CA have been proposed with different components [15]. Among them, the additive CA are the simplest kind of CA whose update rule is an additive transfer function. The additive CA were introduced by Itô et al. [16] to present criteria for surjectivity and injectivity of the global transition map of additive CA. Several important properties of additive CA have been studied. In 1994, analytical studies were carried out by Nandi in which the CA with EXNOR rules can generate an alternating group [17]. In addition, Choudhuri et al. studied an algebraic structure of additive real-valued CA in 1997 [18]. Additive CA have been used to describe real phenomenon such as forest fire spreading [19, 20] and oil slick spreading [21].

In control theory, controllability is one of major concepts and has been applied in many fields. In particular, this conceptual framework has been extensively utilised to CA. For example, Baros et al. [22] applied CA to the case of fire spreading as well as morphogenesis and tumor growth. However, it appeared that CA may not be controllable within the whole domain but only in the subregion of the domain [23]. This leads to an introduction of regional controllability of additve CA proposed by Zerrik El Jai and Bourray [23]. Bel Fekih and El Jai [24] further postulated the conditions for regional controllability and observability for real-valued additive CA. Whilst El Yacoubi [25] presented the regional controllability of one- and two-dimensional additive CA with discrete state sets. Although this study derived the conditions for regional controllability of the system, the controlled system has only one excited cell with periodic boundaries. In 2019, Dridi, Bangnoil, and El Yacoubi [26] proved the regional controllability of Boolean cellular automata by using Markov chains approach. They also presented some necessary and sufficient conditions for the regional controllability of Boolean cellular automata base on graph theory notations [27].

Motivated by studies of Bel Fekih and El Jai [24] and El Yacoubi [25], in this paper, we present conditions for regional controllability of one- and two-dimensional additive real-valued CA with periodic, fixed, and reflective boundary conditions based on theory of linear equations. The paper is organized as follows: In Section 2, we present definitions of classical CA and the most common types of boundary conditions. In addition, a definition of additive CA and regional controllability of the additive CA is introduced. In Section 3, we formulate the global transition functions of one- and two-dimensional additive real-valued CA with the three boundary conditions. Furthermore, the sufficient conditions for regional controllability of the additive CA are derived and proved. We obtain the formulae of control vectors and those with least-norm vector for which the additive CA are controllable. Some simulation examples are later illustrated to verify the theoretical results of the regional controllability the additive real-valued CA. A brief conclusion is given in Section 4.

2. Preliminaries

Cellular automata models are dynamic models in which space and time are discrete entities. The models consist of lattice, state, neighbourhood, and a transition function which are defined as follows.

Definition 2.1. [28] A cellular automaton is defined by a 4-tuple $\mathcal{A} = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$, where

- $\mathcal{L}$ is a lattice or cellular space which is a finite or infinite discrete regular grid of cells on $\mathbb{R}^d$, where d is a dimension of the lattice. Each cell in the lattice is described by its position $c \in \mathcal{L}$.
- $\mathcal{S}$ is a state set which indicates the possible states of each cell at each time step. The state of cell c is written as $s_t(c)$.
- $\mathcal{N}$ represents a neighbourhood which is a mapping of cell c into the cells neighbourhood. The neighbourhood $\mathcal{N}$ is defined by

$$\mathcal{N} : \mathcal{L} \longrightarrow \mathcal{L}^k$$
$$c \longrightarrow \mathcal{N}(c) = \{c_1^*, c_2^*, ..., c_k^*\},$$

where c_i^* is a cell for $i = 1, 2, ..., k$, k is a neibourhood size and $r \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ is a set of positive integers) is the radius of neighbourhood. The most common types of neighbourhoods are the Von Neumann neighbourhood and the Moore neighbourhood. The state of the neighbourhood of cell c is written as

$$s_t(\mathcal{N}(c)) = \{s_t(c_1^*), ..., s_t(c_k^*) \mid c_i^* \in \mathcal{N}(c), 1 \leq i \leq k\}.$$

- f is a transition function (transition rule) which assigns the state $s_{t+1}(c)$ of a cell c at time step $t + 1$ depending on the state of its neighbourhood $s_t(\mathcal{N}(c))$ at time step t. The transition function f is given by

$$f : \mathcal{S}^k \longrightarrow \mathcal{S}$$
$$s_t(\mathcal{N}(c)) \longrightarrow s_{t+1}(c) = f(s_t(\mathcal{N}(c))).$$

Definition 2.2. [28] A cellular automaton state or configuration at time step t is the mapping s_t defined by

$$s_t : \mathcal{L} \longrightarrow \mathcal{S},$$

which associates to every cell of the lattice $\mathcal{L}$ and element of the state set $\mathcal{S}$.

Definition 2.3. [29] The global transition function (global rule) is a mapping $\mathcal{F}$ of the configuration at time t into the configuration at time $t + 1$, which is given by

$$\mathcal{F} : \mathcal{S}^{N_\mathcal{L}} \longrightarrow \mathcal{S}^{N_\mathcal{L}}$$
$$(s_1, s_2, ..., s_{N_\mathcal{L}})_t \longrightarrow (s_1, s_2, ..., s_{N_\mathcal{L}})_{t+1},$$
$$\forall s_j \in \mathcal{S}^{N_\mathcal{L}}, \ j = 1, 2, ..., N_\mathcal{L}, \ s_j = s_j(c), \ \forall c \in \mathcal{L}, \tag{2.1}$$

where $N_\mathcal{L}$ is a number of cells in the lattice.

Boundary Conditions

Boundary conditions are an important part of CA models, which assist in updating the state set. There are three common types of boundary conditions:

(i) Periodic boundary conditions are obtained by connecting the points on one boundary of the lattice to points on the opposite boundary, which lead to a torus-like shape. For the two-dimensional case, the top and bottom edges of the lattice are connected, and then the left and right edges are connected [4].

(ii) Reflective boundary conditions are induced by reflecting the lattice at the boundary, in which the cells at the boundary have the same states as the cells adjacent to them [30].

(iii) Fixed boundary conditions are imposed by assigning a fixed value for the states of cells on the boundary [31].

Additive Cellular Automata

Additive cellular automata are a class of cellular automata, whose transition function is additive, which is defined in the next definition.

Definition 2.4. [29] A global transition function $\mathcal{F}$ is additive if

$$\mathcal{F}(s_i + s_j) = \mathcal{F}(s_i) + \mathcal{F}(s_j), \quad \text{for all} \quad s_i, s_j \in \mathcal{S}^{N_\mathcal{L}}.$$

Consequently, if $S = \mathbb{R}$, a local transition function f of an additive CA can be rewritten in the form

$$s_{t+1}(c_i) = f(s_t(\mathcal{N}(c_i))) = \sum_{1 \leq i \leq k} a_i s_t(c_i),$$

where $a_0, a_1, ..., a_k$ are real scalars, t is the time step, and k is the neighbourhood size.

Regional Controllability of Additive CA

Let $\mathcal{L} = \{c_1, ..., c_{N_\mathcal{L}}\}$. Consider the cellular automaton $\mathcal{A} = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$, where $S = \mathbb{R}$ and f is the additive transition function. Provide that $\mathcal{L}_p = \{c_1^*, c_2^*, ..., c_p^*\} \subseteq \mathcal{L}$ and $\omega = \{\omega_1, ..., \omega_n\} \subseteq \mathcal{L}$. Let $\mathcal{S}^\omega = \{s_t|_\omega = [s_t(\omega_1), ..., s_t(\omega_n)] \mid 0 \leqslant t \leqslant T\}$ be a configuration of $\mathcal{A}$ on ω, where $t = 0$ is an initial time step and T is a final time step.

In order to control the cellular automaton $\mathcal{A}$, it is excited at time step t in a subregion $\mathcal{L}_p$ by $u_t \in \mathcal{U}$, where $u_t = \left[u_t^{c_1^*}, u_t^{c_2^*}, ..., u_t^{c_p^*}\right]$ for $t = 0, ..., T-1$ and $\mathcal{U}$ is a control space which is a set of all the bounded controls [25]. The criterion for regional controllability of the CA can be defined as follows.

Definition 2.5. [25] The cellular automaton $\mathcal{A} = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$ is said to be regionally controllable if for a given $s_d \in \mathcal{S}^\omega$, there exists a control vector sequence $u = \{u_0, u_1, ..., u_{T-1}\}$ with $u_t \in \mathcal{U}$ for $t = 0, ..., T-1$ such that

$$s_T = s_d \text{ on } \omega,$$

where s_T is the final configuration at the final time step T.

The regionally controllable cellular automaton $\mathcal{A}$ means that the subregion $\mathcal{L}_p$ is controllable at the final time step T, which results in a given desired configuration on subregion ω at the final time step T.

The global transition function of additive CA with the control vector u_t is expressed by

$$s_{t+1} = \mathcal{F}(s_t) + \mathcal{G}(u_t) \tag{2.2}$$

$$= \mathcal{F}^{t+1}(s_0) + \sum_{\tau=0}^{t} \mathcal{F}^{t-\tau}\mathcal{G}(u_\tau),$$

where $\mathcal{F}$ is the global additive transition function, $\mathcal{G}$ is the global control function, and $s_0 \in \mathcal{S}^{N_\mathcal{L}}$ is the initial configuration (see the details in [25]).

In the next section, we investigate the conditions for regionally controllable of one- and two-dimensional additive real-valued CA.

3. Main Results

One-Dimensional Additive Real-Valued CA

Let $\mathcal{L} = \{c_1, c_2, ..., c_{N_\mathcal{L}}\}$, where $c_i = i$ for $i = 1, ..., N_\mathcal{L}$. Consider an additive cellular automaton $\mathcal{A}_1 = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$, having the state space $\mathcal{S} = \mathbb{R}$, the neighbourhood $\mathcal{N}$ of radius $r = 1$ such that $\mathcal{N}(c_i) = \{c_{i-1}, c_i, c_{i+1}\}$, and the additive transition function f. The transition function f is given by

$$s_{t+1}(c_i) = f(s_t(\mathcal{N}(c_i))) = \sum_{-1 \leq j \leq 1} a_j s_t(c_{i+j}), \quad c_i \in \mathcal{L},$$

where a_j are real coefficients for $-1 \leq j \leq 1$.

The boundary cells of the lattice $\mathcal{L}$ are c_{-1} and $c_{N_\mathcal{L}+1}$, where $c_{-1} = -1$ and $c_{N_\mathcal{L}+1} = N_\mathcal{L} + 1$.

In this study, we are interested in the three boundary conditions which have most often been often used in CA, including periodic, reflective, and fixed boundary conditions. The global transition function $\mathcal{F}$ with these boundary conditions for $\mathcal{A}_1$ can be represented as follows.

(i) Periodic boundary conditions: the global transition function at time step $t + 1$ is expressed in term of an $N_\mathcal{L} \times N_\mathcal{L}$ matrix M_1 as

$$s_{t+1} = \mathcal{F}(s_t) = M_1 s_t$$

and

$$s_t = M_1^t s_0,$$

where M_1 is an $N_\mathcal{L} \times N_\mathcal{L}$ matrix, which is given as follows.

$$(M_1)_{i,j} = \begin{cases} a_{-1} & \text{if } i = 1 \text{ and } j = N_\mathcal{L}, \\ a_1 & \text{if } i = N_\mathcal{L} \text{ and } j = 1, \\ a_0 & \text{if } j \in \{i-1, i, i+1\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

that is,

$$
M_1 = \begin{bmatrix}
a_0 & a_1 & 0 & 0 & 0 & \ldots & 0 & a_{-1} \\
a_{-1} & a_0 & a_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & a_{-1} & a_0 & a_1 & 0 & \ldots & 0 & 0 \\
0 & 0 & a_{-1} & a_0 & a_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & a_{-1} & a_0 & a_1 & 0 \\
0 & 0 & \ldots & 0 & 0 & a_{-1} & a_0 & a_1 \\
a_1 & 0 & \ldots & 0 & 0 & 0 & a_{-1} & a_0
\end{bmatrix}. \tag{3.2}
$$

(ii) Reflective boundary conditions: the global transition function at time step $t+1$ is updated by using configuration at time step t and an $N_{\mathcal{L}} \times N_{\mathcal{L}}$ matrix M_2.

$$
s_{t+1} = \mathcal{F}(s_t) = M_2 s_t \quad \text{and}
$$
$$
s_t = M_2^t s_0,
$$

where M_2 is defined as follows.

$$
(M_2)_{i,j} = \begin{cases}
a_1 + a_0 & \text{if } i = 1 \text{ and } j = 1, \\
a_0 + a_{-1} & \text{if } i = N_{\mathcal{L}} \text{ and } j = N_{\mathcal{L}}, \\
a_{-1} & \text{if } j = i - 1, \\
a_1 & \text{if } j = i + 1, \\
a_0 & \text{if } j \in \{2, 3, ..., N_{\mathcal{L}} - 1\} \text{ and } j = i, \\
0 & \text{otherwise,}
\end{cases} \tag{3.3}
$$

that is,

$$
M_2 = \begin{bmatrix}
a_1 + a_0 & a_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
a_{-1} & a_0 & a_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & a_{-1} & a_0 & a_1 & 0 & \ldots & 0 & 0 \\
0 & 0 & a_{-1} & a_0 & a_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & a_{-1} & a_0 & a_1 & 0 \\
0 & 0 & \ldots & 0 & 0 & a_{-1} & a_0 & a_1 \\
0 & 0 & \ldots & 0 & 0 & 0 & a_{-1} & a_0 + a_{-1}
\end{bmatrix}. \tag{3.4}
$$

(iii) Fixed boundary conditions: assume that the cell at the boundaries of $\mathcal{A}_1$ are fixed at the state s. Then the global transition function can be defined by

$$
s_{t+1} = \mathcal{F}(s_t) = M_3 s_t + [a_{-1}s, 0, ..., 0, a_1 s]' \quad \text{and}
$$
$$
s_t = M_3^t s_0 + (M_3^{t-1} + M_3^{t-2} + ... + M_3 + I)[a_{-1}s, 0, ..., 0, a_1 s]',
$$

where I is the $N_{\mathcal{L}} \times N_{\mathcal{L}}$ identity matrix and M_3 is an $N_{\mathcal{L}} \times N_{\mathcal{L}}$ matrix, which is given by

$$
(M_3)_{i,j} = \begin{cases}
a_{-1} & \text{if } j = i - 1, \\
a_0 & \text{if } j = i, \\
a_1 & \text{if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases} \tag{3.5}
$$

That is

$$M_3 = \begin{bmatrix} a_0 & a_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{-1} & a_0 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{-1} & a_0 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{-1} & a_0 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{-1} & a_0 & a_1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_{-1} & a_0 & a_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & a_{-1} & a_0 \end{bmatrix}. \tag{3.6}$$

Remark 3.1. The notation $(\cdot)'$ represent the transpose.

For the regional controllability problem, $\mathcal{A}_1$ is excited at time step t in a subregion $\mathcal{L}_p$ with p cells by the vector u_t. In this study, we assume that the global control function is given by

$$\mathcal{G}(u_t) = BV_t,$$

where B is an $N_\mathcal{L} \times N_\mathcal{L}$ matrix,

$$V_t = \begin{bmatrix} V_t(1) \\ V_t(2) \\ \vdots \\ V_t(N_\mathcal{L}) \end{bmatrix} \text{ is an } N_\mathcal{L} \times 1 \text{ matrix such that}$$

$$V_t(i) = \begin{cases} u_t^i & \text{if } i \in \{c_1^*, c_2^*, ..., c_p^*\}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = 1, ..., N_\mathcal{L}.$$

By equation (2.2), we have

$$s_{t+1} = \mathcal{F}(s_t) + BV_t. \tag{3.7}$$

From equation (3.7), the configuration of $\mathcal{A}_1$ with the vector u_t for these boundary conditions can be calculated at the final time step T depending on the initial configuration using the following proposition.

Proposition 3.2. *The configuration at time step T of $\mathcal{A}_1$ can be determined as follows.*
(i) Periodic boundary conditions:

$$s_T = M_1^T s_0 + \sum_{i=0}^{T-1} M_1^{T-i-1} BV_i. \tag{3.8}$$

(ii) Reflective boundary conditions:

$$s_T = M_2^T s_0 + \sum_{i=0}^{T-1} M_2^{T-i-1} BV_i. \tag{3.9}$$

(iii) Fixed boundary conditions:

$$s_T = M_3^T s_0 + \left(\sum_{i=1}^{T} M_3^{T-i} \right) [a_{-1}s, 0, ..., 0, a_1 s]' + \sum_{i=0}^{T-1} M_3^{T-i-1} BV_i. \tag{3.10}$$

Proof. Let s_0 be the initial configuration of $\mathcal{A}_1$.

(i) Assume that the boundary conditions of the cellular automaton are periodic. By using equation (3.7), it follows that

$$
\begin{aligned}
s_1 &= M_1 s_0 + BV_0, \\
s_2 &= M_1 s_1 + BV_1 \\
&= M_1\left(M_1 s_0 + BV_0\right) + BV_1 \\
&= M_1^2 s_0 + M_1 BV_0 + BV_1, \\
s_3 &= M_1 s_2 + BV_2 \\
&= M_1\left(M_1^2 s_0 + M_1 BV_0 + BV_1\right) + BV_2 \\
&= M_1^3 s_0 + M_1^2 BV_0 + M_1 BV_1 + BV_2, \\
&\;\;\vdots \\
s_T &= M_1^T s_0 + M_1^{T-1} BV_0 + M_1^{T-2} BV_1 + ... + BV_T \\
&= M_1^T s_0 + \sum_{i=0}^{T-1} M_1^{T-i-1} BV_i.
\end{aligned}
$$

(ii) The idea of the proof is similar to the proof of (i).

(iii) Assume that the cell at the boundaries $\mathcal{A}_1$ are fixed by the state s. By using equation (3.7), we obtain that

$$
\begin{aligned}
s_1 &= M_3 s_0 + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_0, \\
s_2 &= M_3 s_1 + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_1 \\
&= M_3\left(M_3 s_0 + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_0\right) + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_1 \\
&= M_3^2 s_0 + M_3[a_{-1}s, 0, ..., 0, a_1 s]' + M_3 BV_0 + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_1 \\
&= M_3^2 s_0 + (M_3 + I)[a_{-1}s, 0, ..., 0, a_1 s]' + M_3 BV_0 + BV_1, \\
s_3 &= M_3 s_2 + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_2 \\
&= M_3\left(M_3^2 s_0 + (M_3 + I)[a_{-1}s, 0, ..., 0, a_1 s]' + M_3 BV_0 + BV_1\right) + BV_2 \\
&= M_3^3 s_0 + (M_3^2 + M_3)[a_{-1}s, 0, ..., 0, a_1 s]' + M_3^2 BV_0 + M_3 BV_1 + [a_{-1}s, 0, ..., 0, a_1 s]' + BV_2 \\
&= M_3^3 s_0 + (M_3^2 + M_3 + I)[a_{-1}s, 0, ..., 0, a_1 s]' + M_3^2 BV_0 + M_3 BV_1 + BV_2, \\
&\;\;\vdots \\
s_T &= M_3^T s_0 + (M^{T-1} + M^{T-2} + ... + I)[a_{-1}s, 0, ..., 0, a_1 s]' + M_3^{T-1} BV_0 + M_3^{T-2} BV_1 + ... + BV_T \\
&= M_3^T s_0 + \left(\sum_{i=1}^{T} M_3^{T-i}\right)[a_{-1}s, 0, ..., 0, a_1 s]' + \sum_{i=0}^{T-1} M_3^{T-i-1} BV_i. \qquad \blacksquare
\end{aligned}
$$

Let s_d be the desired state on ω. If there exist the vector V_t for $t = 0, ..., T-1$ such that $s_T|_\omega = s_d$, then $\mathcal{A}_1$ is regionally controllable.

From the above proposition, we can rewrite equation (3.8), (3.9) and (3.10) as follows.

(i) Periodic boundary conditions:

$$
\sum_{i=0}^{T-1} M_1^{T-i-1} BV_i = s_T - M_1^T s_0. \tag{3.11}
$$

(ii) Reflective boundary conditions:

$$
\sum_{i=0}^{T-1} M_2^{T-i-1} BV_i = s_T - M_2^T s_0. \tag{3.12}
$$

(iii) Fixed boundary conditions:

$$\sum_{i=0}^{T-1} M_3^{T-i-1} BV_i \quad = s_T - M_3^T s_0 - \left(\sum_{i=1}^{T} M_3^{T-i}\right)[a_{-1}s, 0, ..., 0, a_1 s]'. \qquad (3.13)$$

The left-hand side of equation (3.11), (3.12) and (3.13) can be represented in matrix form,

$$\sum_{i=0}^{T-1} M^{T-i-1} BV_i \quad = M^{T-1} BV_0 + M^{T-1} BV_1 + ... + BV_{T-1}$$

$$= \begin{bmatrix} M^{T-1}B & M^{T-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{T-1} \end{bmatrix}$$

$$= \begin{bmatrix} M^{T-1}B & M^{T-2}B & \cdots & B \end{bmatrix} V, \qquad (3.14)$$

where $M = \begin{cases} M_1 & \text{if periodic boundary conditions are stated,} \\ M_2 & \text{if reflective boundary conditions are stated,} \\ M_3 & \text{if the boundary cells are fixed at } s. \end{cases}$

Consequently, we can evaluate equation (3.14) in the subregion ω as

$$\begin{bmatrix} M_{\omega_1,c_1^*}^{T-1}B & M_{\omega_1,c_1^*}^{T-2}B & \cdots & B_{\omega_1,c_1^*} & \cdots & M_{\omega_1,c_p^*}^{T-1}B & M_{\omega_1,c_p^*}^{T-2}B & \cdots & B_{\omega_1,c_p^*} \\ M_{\omega_2,c_1^*}^{T-1}B & M_{\omega_2,c_1^*}^{T-2}B & \cdots & B_{\omega_2,c_1^*} & \cdots & M_{\omega_2,c_p^*}^{T-1}B & M_{\omega_2,c_p^*}^{T-2}B & \cdots & B_{\omega_2,c_p^*} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ M_{\omega_n,c_1^*}^{T-1}B & M_{\omega_n,c_1^*}^{T-2}B & \cdots & B_{\omega_n,c_1^*} & \cdots & M_{\omega_n,c_p^*}^{T-1}B & M_{\omega_n,c_p^*}^{T-2}B & \cdots & B_{\omega_n,c_p^*} \end{bmatrix} \begin{bmatrix} u_0^{c_1^*} \\ \vdots \\ u_{T-1}^{c_1^*} \\ u_0^{c_2^*} \\ \vdots \\ u_{T-1}^{c_2^*} \\ \vdots \\ u_0^{c_p^*} \\ \vdots \\ u_{T-1}^{c_p^*} \end{bmatrix}$$

$$= \left(\sum_{i=0}^{T-1} M^{T-i-1} BV_i\right)\Bigg|_{\omega}, \qquad (3.15)$$

where $M_{i,j}^t$ is an element of matrix M^t at row i and column j for $i, j = 1, 2, ..., N_{\mathcal{L}}$ and $t = 1, 2, ..., T - 1$. Equations (3.11), (3.12) and (3.13) indicate that $\mathcal{A}_1$ is regionally controllable if the equation is solvable for V. From equation (3.11), (3.12) and (3.13), the system (3.15) can be written as

$$AU = b, \qquad (3.16)$$

where

$$A = \begin{bmatrix} M_{\omega_1,c_1^*}^{T-1}B & M_{\omega_1,c_1^*}^{T-2}B & \ldots & B_{\omega_1,c_1^*} & \ldots & M_{\omega_1,c_p^*}^{T-1}B & M_{\omega_1,c_p^*}^{T-2}B & \ldots & B_{\omega_1,c_p^*} \\ M_{\omega_2,c_1^*}^{T-1}B & M_{\omega_2,c_1^*}^{T-2}B & \ldots & B_{\omega_2,c_1^*} & \ldots & M_{\omega_2,c_p^*}^{T-1}B & M_{\omega_2,c_p^*}^{T-2}B & \ldots & B_{\omega_2,c_p^*} \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\ M_{\omega_n,c_1^*}^{T-1}B & M_{\omega_n,c_1^*}^{T-2}B & \ldots & B_{\omega_n,c_1^*} & \ldots & M_{\omega_n,c_p^*}^{T-1}B & M_{\omega_n,c_p^*}^{T-2}B & \ldots & B_{\omega_n,c_p^*} \end{bmatrix},$$

$$U = \begin{bmatrix} u_0^{c_1^*} & \ldots & u_{T-1}^{c_1^*} & u_0^{c_2^*} & \ldots & u_{T-1}^{c_2^*} & \ldots & u_0^{c_p^*} & \ldots & u_{T-1}^{c_p^*} \end{bmatrix}', \qquad \text{and}$$

$$b = \begin{cases} \left(s_T - M_1^T s_0\right)\Big|_\omega & \text{if periodic boundary conditions are stated,} \\ \left(s_T - M_2^T s_0\right)\Big|_\omega & \text{if reflective boundary conditions are stated,} \\ \left(s_T - M_3^T s_0 - \left(\sum_{i=1}^T M_3^{T-i}\right)[a_{-1}s,0,...,0,a_1 s]'\right)\Big|_\omega & \text{if the boundary cells are fixed at } s. \end{cases}$$

Thus, we have the conditions for regional controllability of $\mathcal{A}_1$ as expressed in the following theorem.

Theorem 3.3. *Let A and b be the $n \times (pT)$ and $N_{\mathcal{L}} \times 1$ matrices, respectively, defined by equation* (3.16) *let $\tilde{A} = [A|b]$ be the augmented matrix of the system* (3.16). *Then cellular automata $\mathcal{A}_1$ is regionally controllable if and only if $\text{rank}A = \text{rank}\tilde{A}$.*

Proof. This theorem can be proved by using Theorem 2.6.3. in reference [32]. ∎

Remark 3.4. $\text{rank}A$ denotes the rank of a matrix A.

Theorem 3.5. *Let A be the matrix defined by equation* (3.16). *Then the cellular automaton $\mathcal{A}_1$ is regionally controllable if $\text{rank}A = n$.*

Proof. The proof of Theorem 3.5 is done by using Theorem 2.9.3 in reference [33]. ∎

Theorem 3.6. *Let A and b be the matrices defined by equation* (3.16). *Then the cellular automaton $\mathcal{A}_1$ is regionally controllable with the control vector*

$$U = A'(AA')^{-1}b + \left(I - A'(AA')^{-1}A\right)Y, \tag{3.17}$$

for any $pT \times 1$ matrix Y if and only if $\text{rank}A = \text{rank}\ \tilde{A}$ and $\text{rank}A < pT$.

Proof. By using the result of reference [34], this proof is complete. ∎

Theorem 3.7. *Let A and b be the matrices defined by equation* (3.16). *Then the cellular automaton $\mathcal{A}_1$ is regionally controllable with the control vector $U_m = A'(AA')^{-1}b$, which is the least-norm vector, if $\text{rank}A = \text{rank}\ \tilde{A}$ and $\text{rank}A < pT$.*

Proof. The norm $\|\cdot\|_2$ represents the Euclidean norm. We will show that U_m is the smallest norm.
Assume that U_n and U_m are the solutions of equation (3.16) such that $U_n \neq U_m$.
We obtain that $\quad A(U_n - U_m) = 0$
Then, $\quad (U_n - U_m)'U_m' = (U_n - U_m)'(A'(AA')^{-1}b)$
$$= (A(U_n - U_m))'(AA')^{-1}b.$$
This implies that $U_n - U_m$ and U_m are perpendicular.
Hence, $\|U_m + (U_n - U_m)\|_2^2 = \|U_m\|_2^2 + \|U_n - U_m\|_2^2$.
Thus, $\quad \|U_n\|_2^2 = \|U_m + (U_n - U_m)\|_2^2$
$$= \|U_m\|_2^2 + \|U_n - U_m\|_2^2$$

$$\geqslant \|U_m\|_2^2.$$

Therefore, U_m is the least-norm vector.   ∎

Remark 3.8. If $Y = \underline{0}$ then, by equation (3.17), $U = U_m$.

Theorem 3.9. *Let A and b be the matrices defined by equation* (3.16). *Then the cellular automaton $\mathcal{A}_1$ is regionally controllable with a unique control vector $U = A'(AA')^{-1}b$ if and only if* $\mathrm{rank}A = \mathrm{rank}\ \tilde{A}$ *and* $\mathrm{rank}A < pT$.

Proof. We can prove this theorem by using the result of reference [34].   ∎

Remark 3.10. By Theorem 3.3, Theorem 3.5, and Theorem 3.9, it follows that the cellular automaton $\mathcal{A}_1$ is regionally controllable at time T if $\mathrm{rank}A = n$ and $T > \frac{n}{p}$. This result is the same as Bel Fekih and El Jai [24].

Lemma 3.11. *Let M_2 and M_3 be the matrices which defined in equation* (3.4) *and* (3.6), *respectively. Let $\mathcal{L}_p = \{c_1^*, c_2^*, ..., c_p^*\}$ and $\omega = \{\omega_1, ..., \omega_n\}$. Then $M_{\omega_i,c_j^*}^q = 0$ for $\omega_i \in \{1, ..., c_j^* - q - 1\} \cup \{c_j^* + q + 1, ..., N_{\mathcal{L}}\}$, $i = 1, ..., n$, $j = 1, ..., p$.*

Proof. Suppose that M_2 and M_3 be the matrices which defined in equation (3.4) and (3.6), respectively.
Assume that $M = M_2$.
Let $P(k)$ be the statement $M_{\omega_i,c_j^*}^k = 0$ for $\omega_i \in \{1, ..., c_j^* - k - 1\} \cup \{c_j^* + k + 1, ..., N_{\mathcal{L}}\}$, $i = 1, ..., n$, $j = 1, ..., p$ for $k \in \mathbb{N}$.
From equation (3.3), we have

$$M_{\omega_i,c_j^*} = 0 \text{ for } \omega_i \in \{1, ..., c_j^* - 2\} \cup \{c_j^* + 2, ..., N_{\mathcal{L}}\}, i = 1, ..., n, j = 1, ..., p.$$

This implies that $P(1)$ holds.
Suppose that $P(k)$ is true, we get

$$M_{\omega_i,c_j^*}^k = 0 \text{ for } \omega_i \in \{1, ..., c_j^* - k - 1\} \cup \{c_j^* + k + 1, ..., N_{\mathcal{L}}\}, i = 1, ..., n, j = 1, ..., p. \quad (3.18)$$

Assume that $\omega_i \in \{1, ..., c_j^* - q - 1\} \cup \{c_j^* + q + 1, ..., N_{\mathcal{L}}\}$.
From $M^{k+1} = M^k \times M$, we obtain

$$M_{\omega_i,c_j^*}^{k+1} = M_{\omega_i,1}^k M_{1,c_j^*} + M_{\omega_i,2}^k M_{2,c_j^*} + ... + M_{\omega_i,N_{\mathcal{L}}}^k M_{N_{\mathcal{L}},c_j^*}. \quad (3.19)$$

From equation (3.3), it follows that $M_{\omega_i,c_j^*} = 0$ for $\omega_i \in \{1, ..., c_j^* - 2\} \cup \{c_j^* + 2, ..., N_{\mathcal{L}}\}, i = 1, ..., n, j = 1, ..., p$.
Consequently, by equation (3.19),

$$M_{\omega_i,c_j^*}^{k+1} = M_{\omega_i,c_j^*-1}^k M_{c_j^*-1,c_j^*} + M_{\omega_i,c_j^*}^k M_{c_j^*,c_j^*} + M_{\omega_i,c_j^*+1}^k M_{c_j^*+1,c_j^*}. \quad (3.20)$$

From equation (3.18), we obtain
$M_{\omega_i,c_j^*}^k = 0$ for $\omega_i \in \{1, ..., c_j^* - k - 1\} \cup \{c_j^* + k + 1, ..., N_{\mathcal{L}}\}$, $i = 1, ..., n$, $j = 1, ..., p$,
$M_{\omega_i,c_j^*-1}^k = 0$ for $\omega_i \in \{1, ..., c_j^* - k - 2\} \cup \{c_j^* + k, ..., N_{\mathcal{L}}\}$, $i = 1, ..., n$, $j = 1, ..., p$, and
$M_{\omega_i,c_j^*+1}^k = 0$ for $\omega_i \in \{1, ..., c_j^* - k\} \cup \{c_j^* + k + 2, ..., N_{\mathcal{L}}\}$, $i = 1, ..., n$, $j = 1, ..., p$.

Hence, $M^k_{\omega_i,c^*_j}, M^k_{\omega_i,c^*_j-1}, M^k_{\omega_i,c^*_j+1} = 0$ for $\omega_i \in \{1,...,c^*_j-k\} \cup \{c^*_j+k+2,...,N_{\mathcal{L}}\}$.
From equation (3.20), we conclude that

$$M^{k+1}_{\omega_i,c^*_j} = 0 \text{ for } \omega_i \in \{1,...,c^*_j-k-1\} \cup \{c^*_j+k+3,...,N_{\mathcal{L}}\}, i=1,...,n, j=1,...,p.$$

By the induction rule, $P(k+1)$ holds.
Hence, in case of $M = M_3$, the proof is done by using idea of this proof.
Therefore, Lemma 3.11 have been proven. ∎

Theorem 3.12. *Let B be the matrix defined in equation* (3.7) *such that*
$$B = \begin{cases} M_2 & \text{if reflective boundary conditions are stated,} \\ M_3 & \text{if the boundary cells are fixed at } s, \end{cases}$$
let $\mathcal{L}_p = \{c^*_1, c^*_2, ..., c^*_n\}$ and $\omega = \{\omega_1, \omega_2, \omega_3, ..., \omega_n\}$ be the subregion of $\mathcal{L}$ and consider $\mathcal{A}_1$ to be excited on $\mathcal{L}_p$. Suppose that $\omega_i \notin \{c^*_1 - T, c^*_1 - T + 1, ..., c^*_j + T - 1, c^*_j + T\}$ for $i = 1, ..., n$, $j = 1, ..., p$.
Then, the cellular automaton $\mathcal{A}_1$ is regionally controllable if and only if

$$s_d = \begin{cases} \left(M_2^T s_0\right)\big|_\omega & \text{if reflective boundary conditions are stated,} \\ \left(M_3^T s_0 + \left(\sum_{i=0}^{T-1} M_3^{T-i}\right) [a_{-1}s, 0, ..., 0, a_1 s]'\right)\big|_\omega & \text{if the boundary cells are fixed at } s, \end{cases}$$

where s_d is the desired state on ω.

Proof. Let $\omega_i \notin \{c^*_1 - T, c^*_1 - T + 1, ..., c^*_j + T - 1, c^*_j + T\}, i = 1, ..., n$.
Then $\omega_i \in \{1, 2, ..., c^*_j - T - 1\} \cup \{c^*_j + T + 1, c^*_j + T + 2, ..., N_{\mathcal{L}}\}$.
By Lemma 3.11, we have

$$M^T_{\omega_i,c^*_j} = 0 \text{ for } \omega_i \in \{1,...,c^*_j-T-1\} \cup \{c^*_j+T+1,...,N_{\mathcal{L}}\}, \ i=1,...,n, \ j=1,...,p,$$
$$M^{T-1}_{\omega_i,c^*_j} = 0 \text{ for } \omega_i \in \{1,...,c^*_i-T\} \cup \{c^*_i+T,...,N_{\mathcal{L}}\}, \ i=1,...,n, \ j=1,...,p,$$
$$\vdots$$
$$M_{\omega_i,c^*_j} = 0 \text{ for } \omega_i \in \{1,...,c^*_j-1\} \cup \{c^*_j+1,...,N_{\mathcal{L}}\}, \ i=1,...,n, \ j=1,...,p.$$

From equation (3.16), we obtain $A = \underline{0}$, where $\underline{0}$ is the zero matrix.
Consequently,

$$b = AU = \underline{0} \text{ for any matrix } U. \tag{3.21}$$

Assume that $\mathcal{A}_1$ is regionally controllable. Then there exist U such that $s_T|_\omega = s_d$.

From equation (3.16), we have

$$b = \underline{0} = \begin{cases} s_d - \left(M_2^T s_0\right)\big|_\omega & \text{if reflective boundary conditions are stated,} \\ s_d - \left(M_3^T s_0 + \left(\sum_{i=1}^{T} M_3^{T-i}\right)[a_{-1}s, 0, ..., 0, a_1 s]'\right)\big|_\omega & \text{if the boundary cells are fixed at } s. \end{cases}$$

Thus,

$$s_d = \begin{cases} \left(M_2^T s_0\right)\big|_\omega & \text{if reflective boundary conditions are stated,} \\ \left(M_3^T s_0 + \left(\sum_{i=1}^{T} M_3^{T-i}\right)[a_{-1}s, 0, ..., 0, a_1 s]'\right)\big|_\omega & \text{if the boundary cells are fixed at } s. \end{cases}$$

Conversely, assume that

$$s_d = \begin{cases} \left(M_2^T s_0\right)\big|_\omega & \text{if reflective boundary conditions are stated,} \\ \left(M_3^T s_0 + \left(\sum_{i=1}^T M_3^{T-i}\right)[a_{-1}s, 0, ..., 0, a_1 s]'\right)\big|_\omega & \text{if the boundary cells are fixed at } s, \end{cases}$$

and $s_d = s_T|_\omega$. From equation (3.21), there are multiple solutions of the system $AU = \underline{0}$. Therefore, $\mathcal{A}_1$ is regionally controllable.   ∎

Simulation Examples for One-Dimensional Additive Real-Valued CA

We consider the additive cellular automaton $\mathcal{A}_1$ with lattice $\mathcal{L} = \{c_1, c_2, ..., c_{200}\}$. Each state of cell c_i can take a value in the state set $\mathcal{S} = \mathbb{R}$. The local transition function f with neighbourhood $\mathcal{N}$, where $\mathcal{N}(c_i) = \{c_{i-1}, c_i, c_{i+1}\}$, under the reflective boundary conditions is given by

$$s_{t+1}(c_i) = f(s_t(\mathcal{N}(c_i))) = 0.9s_t(c_{i-1}) + 0.2s_t(c_i) - 0.5s_t(c_{i+1}).$$

Let $\mathcal{L}_p = \{c_{55}, c_{100}\}$, $T = 6$ and $B = M_2$. Suppose that s_0 is the initial configuration, which the initial states are randomly generated between -20 to 20.

We then demonstrate the simulation examples of the regional controllability problem in order to represent the results of various excited subregion and of different cells which the controls are active. The examination of these effects is divided into the three following cases.

Case(i) Let $\omega = \{c_{25}, ..., c_{30}, c_{151}, ..., c_{155}\}$ be a given subregion of $\mathcal{L}$, $u = \{u_0, u_1, ..., u_5\}$, where $u_i = \left[u_i^{55}, u_i^{100}\right]$, $i = 0, 1, ..., 5$, and $s_d = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]'$. From equation (3.16), we obtain $AU = b$, where

$$A = \begin{bmatrix} M_{25,55}^6 & \cdots & M_{25,55} & M_{25,100}^6 & \cdots & M_{25,100} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{30,55}^6 & \cdots & M_{30,55} & M_{30,100}^6 & \cdots & M_{30,100} \\ M_{151,55}^6 & \cdots & M_{151,55} & M_{151,100}^6 & \cdots & M_{151,100} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{155,55}^6 & \cdots & M_{155,55} & M_{155,100}^6 & \cdots & M_{155,100} \end{bmatrix} = \underline{0}$$

$$U = \left[u_0^{55}, \ldots, u_5^{55}, u_0^{100}, \ldots, u_5^{100}\right]',$$

$$b = s_d - \left(M^6 s_0\right)\big|_\omega = [-1.98, 0.86, 11.94, -1.62, -6.03, 13.25, -12.01, 1.01, 3.93, 1.07, 12.37]'.$$

This result indicates that this system has no solution. Thus, $\mathcal{A}_1$ is not regionally controllable. This conclusion corresponds with the result of Theorem 3.12 which show that subregion $\omega \nsubseteq \{49, 50, ..., 106\}$ and $s_d \neq (M^5 s_0)|_\omega$. Consequently, $\mathcal{A}_1$ is not regionally controllable.

Case(ii) Let $\omega = \{c_{55}, ..., c_{60}, c_{100}, ..., c_{105}\}$ be a given subregion of $\mathcal{L}$, $u = \{u_0, u_1, ..., u_5\}$, where $u_i = \left[u_i^{55}, u_i^{100}\right]$, $i = 0, 1, ..., 5$, and $s_d = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]'$. From equation (3.16), we get $AU = b$, where

$$A = \begin{bmatrix} M_{55,55}^6 & M_{55,55}^5 & \cdots & M_{55,55} & M_{55,100}^6 & M_{55,100}^5 & \cdots & M_{55,100} \\ M_{56,55}^6 & M_{56,55}^5 & \cdots & M_{56,55} & M_{56,100}^6 & M_{56,100}^5 & \cdots & M_{56,100} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{60,55}^6 & M_{60,55}^5 & \cdots & M_{60,55} & M_{60,100}^6 & M_{60,100}^5 & \cdots & M_{60,100} \\ M_{100,55}^6 & M_{100,55}^5 & \cdots & M_{100,55} & M_{100,100}^6 & M_{100,100}^5 & \cdots & M_{100,100} \\ M_{101,55}^6 & M_{101,55}^5 & \cdots & M_{101,55} & M_{101,100}^6 & M_{101,100}^5 & \cdots & M_{101,100} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{105,55}^6 & M_{105,55}^5 & \cdots & M_{105,55} & M_{105,100}^6 & M_{105,100}^5 & \cdots & M_{105,100} \end{bmatrix}$$

$$= \begin{bmatrix} -1.12 & 1.14 & 1.00 & -0.53 & -0.86 & 0.20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.99 & 1.34 & -0.94 & -1.11 & 0.36 & 0.90 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.61 & -1.39 & -1.26 & 0.49 & 0.81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.85 & -1.35 & 0.58 & 0.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.38 & 0.66 & 0.66 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.71 & 0.59 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.12 & 1.14 & 1.00 & -0.53 & -0.86 & 0.20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.99 & 1.34 & -0.94 & -1.11 & 0.36 & 0.90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1.61 & -1.39 & -1.26 & 0.49 & 0.81 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.85 & -1.35 & 0.58 & 0.73 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.38 & 0.66 & 0.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0.59 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$U = \left[u_0^{55}, \ldots, u_5^{55}, u_0^{100}, \ldots, u_5^{100}\right]'$,

$b = s_d - \left(M^6 s_0\right)\Big|_\omega$

$= [35.28, 92.31, 15.11, -66.91, -23.85, 30.12, 0.98, -1.09, 2.39, -1.17, -2.14, 1.83]'.$

According to Theorem 3.3 and 3.5, rank$\tilde{A} = $ rank$A = 12 = n$, and $\mathcal{A}_1$ is regionally controllable. Moreover, by Theorem 3.9, we have the unique control vector

$U = \left[u_0^{55}, ..., u_5^{55}, u_0^{100}, ..., u_5^{100}\right]'$

$= [-112.31, 185.78, -458.00, 333.03, -353.55, 145.14, -8.47, 13.26, -34.30, 28.87, -28.29, 8.63]'.$

Consequently, $u = \{u_0, u_1, u_2, u_3, u_4, u_5\}$, where $u_0 = [-112.31, -8.47]$, $u_1 = [185.78, 13.26]$, $u_2 = [-458.00, -34.30]$, $u_3 = [333.03, 28.87]$, $u_4 = [-353.55, -28.29]$, $u_5 = [145.14, 8.63]$. The evolution of $\mathcal{A}_1$ is illustrated in Figure 1. The first row and last row in the diagram represent the initial state and the final state, respectively. In the last row $s_d = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]'$.

Case(iii) Let $\omega = \{c_{56}, ..., c_{60}, c_{100}, ..., c_{105}\}$, $u = \{u_0, u_1, ..., u_5\}$, where $u_i = \left[u_i^{55}, u_i^{100}\right]$, $i = 0, 1, ..., 5$ and $s_d = [5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5]'$. From equation (3.16), it follows that
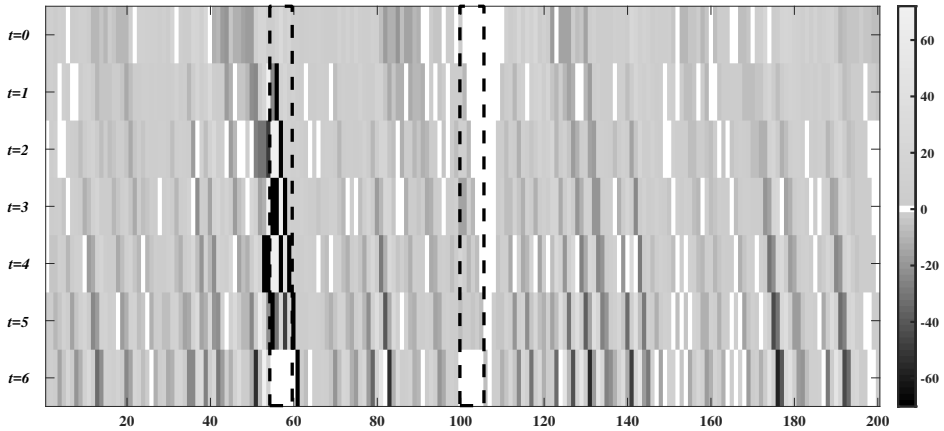
FIGURE 1. A space-time diagram of the evolution of the cellular automaton $\mathcal{A}_1$ for case(ii) with control vector sequence $u = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ for $t = 0, 1, ..., 6$ on a lattice of 1×200 cells.

$AU = b$, where

$$A = \begin{bmatrix} M^6_{56,55} & M^5_{56,55} & \cdots & M_{56,55} & M^6_{56,100} & M^5_{56,100} & \cdots & M_{56,100} \\ M^6_{57,55} & M^5_{57,55} & \cdots & M_{57,55} & M^6_{57,100} & M^5_{57,100} & \cdots & M_{57,100} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M^6_{60,55} & M^5_{60,55} & \cdots & M_{60,55} & M^6_{60,100} & M^5_{60,100} & \cdots & M_{60,100} \\ M^6_{100,55} & M^5_{100,55} & \cdots & M_{100,55} & M^6_{100,100} & M^5_{100,100} & \cdots & M_{100,100} \\ M^6_{101,55} & M^5_{101,55} & \cdots & M_{101,55} & M^6_{101,100} & M^5_{101,100} & \cdots & M_{101,100} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M^6_{105,55} & M^5_{105,55} & \cdots & M_{105,55} & M^6_{105,100} & M^5_{105,100} & \cdots & M_{105,100} \end{bmatrix}$$

$$= \begin{bmatrix} 1.99 & 1.34 & -0.94 & -1.11 & 0.36 & 0.90 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.61 & -1.39 & -1.26 & 0.49 & 0.81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.85 & -1.35 & 0.58 & 0.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.38 & 0.66 & 0.66 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.71 & 0.59 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.12 & 1.14 & 1.00 & -0.53 & -0.86 & 0.20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.99 & 1.34 & -0.94 & -1.11 & 0.36 & 0.90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.61 & -1.39 & -1.26 & 0.49 & 0.81 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.85 & -1.35 & 0.58 & 0.73 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.38 & 0.66 & 0.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0.59 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$U = \left[u_0^{55}, \ldots, u_5^{55}, u_0^{100}, \ldots, u_5^{100} \right]'$,

$b = s_d - \left(M^6 s_0 \right)\Big|_\omega = [97.31, 20.11, -61.91, -18.85, 35.12, 5.98, 3.91, 7.39, 3.83, 2.86, 6.83]'$.

Since rank$A = $ rank$\tilde{A} = 11 < 12 = pT$, then by Theorem 3.6, $\mathcal{A}_1$ is regionally controllable with $U = A'(AA')^{-1}b + \left(I - A'(AA')^{-1}A \right) Y$. In this case, there are many values of the control vector sequence u, depending on Y. In order to illustrate how to compute the

control vector sequence u with different Y, two examples are given below.

I. If $Y = [1210, 1130, 1150, 1200, 130, 1180, 120, 190, 170, 1120, 150, 1200]'$, then

$$U = \left[u_0^{55}, ..., u_5^{55}, u_0^{100}, ..., u_5^{100} \right]'$$

$$= [40.16, 11.28, 44.32, 2.49, 32.29, 38.88, -45.31, 65.94, -156.73, 137.54, -114.70, 57.11]'.$$

Thus, we obtain $u = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ where $u_0 = [40.16, -45.31]$, $u_1 = [11.28, 65.94]$, $u_2 = [44.32, -156.73]$, $u_3 = [2.49, 137.54]$, $u_4 = [32.29, -114.70]$, $u_5 = [38.88, 57.11]$, and $\|U\|_2 = 269.38$.

II. If $Y = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]'$, then

$$U = \left[u_0^{55}, ..., u_5^{55}, u_0^{100}, ..., u_5^{100} \right]'$$

$$= [31.73, 21.40, 16.49, 22.05, 11.26, 45.77, -45.31, 65.94, -156.73, 137.54 - 114.70, 57.11]'.$$

This result indicates that $u = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ where $u_0 = [31.73, -45.31]$, $u_1 = [21.40, 65.94]$, $u_2 = [16.49, -156.73]$, $u_3 = [22.05, 137.54]$, $u_4 = [11.26, -114.70]$, $u_5 = [45.77, 57.11]$, and $\|U\|_2 = 265.98$. Thus, the space-time configurations of $\mathcal{A}_1$ with u can be shown in Figure 2.
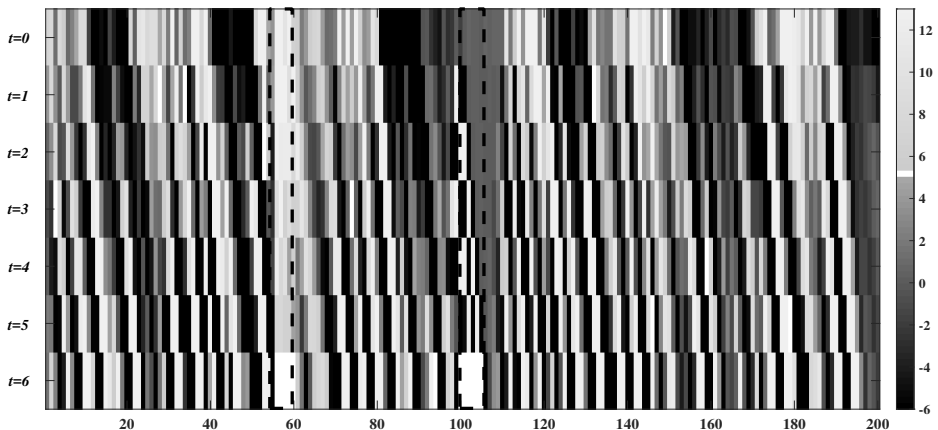

FIGURE 2. A space-time diagram of the evolution of the cellular automaton $\mathcal{A}_1$ for case (iii) with control vector sequence $u = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ for $t = 0, 1, ..., 6$ on a lattice of 1×200 cells.

Remark 3.13. The traditional methods to find rank of a matrix are Gaussian elimination and singular value decomposition(SVD). However, SVD approach is more reliable and

effective than Gaussian elimination [35]. There are many algorithms of the SVD method that compute a SVD of an $m \times n$ $(m \geq n)$ matrix with computational time being $O(mn^2 + m^2 n)$ [36, 37]. In this study, we use the rank function based on SVD method in MATLAB software to evaluate rank of matrices.

Two-Dimensional Additive Real-Valued CA

Let $\mathcal{L} = \{(i,j) | i, j = 1, ..., N_\mathcal{L}\} \subseteq \mathbb{Z}^2$. Assume that $\mathcal{N}$ is the Von Neumann neighbourhood of radius $r = 1$ such that $\mathcal{N}(i,j) = \{(i-1,j), (i,j), (i+1,j), (i,j-1), (i,j+1)\}$. We consider the additive cellular automaton $\mathcal{A}_2 = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$, where $\mathcal{S} = \mathbb{R}$, and the additive transition function f.

Each cell's state at time step t is updated by the following local transition function,

$$s_{t+1}(i,j) = f(s_t(\mathcal{N}(i,j))) = \sum_{c^* \in \mathcal{N}(i,j)} a_{c^*} s_t(c^*).$$

Let $a_{-1} = a_{(i-1,j)}$, $a_0 = a_{(i,j)}$, $a_1 = a_{(i+1,j)}$, $a^*_{-1} = a_{(i,j-1)}$, and $a^*_1 = a_{(i,j+1)}$. In order to formulate the global transition function of $\mathcal{A}_2$, the columns of the configuration s_t are stacked to form a vector

$$z_t = Vec(s_t),$$

where $Vec(s_t) \in S^{(N_\mathcal{L})^2}$. We consequently proceed in the same manner as the one-dimensional case. The global transition function of $\mathcal{A}_2$ with periodic, reflective and fixed boundary conditions are given as follows.

(i) Periodic boundary conditions: the global transition function can be expressed in terms of a matrix $\mathcal{M}_1$ by the following equation

$$z_{t+1} = \mathcal{F}(z_t) = \mathcal{M}_1 z_t \quad \text{and}$$
$$z_t = \mathcal{M}_1^t z_0,$$

where $\mathcal{M}_1 = (I \otimes M_1) + (\mathbf{P_1} \otimes I)$ (the notation $\otimes$ represent the Kronecker product),
      I is the $N_\mathcal{L}^2 \times N_\mathcal{L}^2$ identity matrix,
      M_1 is defined in equation (3.2),
and $\mathbf{P_1}$ is constructed as follows.

$$\mathbf{P_1} = \begin{bmatrix} 0 & a^*_{-1} & 0 & 0 & 0 & \dots & 0 & a^*_1 \\ a^*_1 & 0 & a^*_{-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & a^*_1 & 0 & a^*_{-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & a^*_1 & 0 & a^*_{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a^*_1 & 0 & a^*_{-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & a^*_1 & 0 & a^*_{-1} \\ a^*_{-1} & 0 & \dots & 0 & 0 & 0 & a^*_1 & 0 \end{bmatrix}. \quad (3.22)$$

(ii) Reflexive boundary conditions: given $\mathbf{P_2}$ is a matrix,

$$
\mathbf{P_2} = \begin{bmatrix}
a^*_{-1} & a^*_{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\
a^*_1 & 0 & a^*_{-1} & 0 & 0 & \dots & 0 & 0 \\
0 & a^*_1 & 0 & a^*_{-1} & 0 & \dots & 0 & 0 \\
0 & 0 & a^*_1 & 0 & a^*_{-1} & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & a^*_1 & 0 & a^*_{-1} & 0 \\
0 & 0 & \dots & 0 & 0 & a^*_1 & 0 & a^*_{-1} \\
0 & 0 & \dots & 0 & 0 & 0 & a^*_1 & a^*_1
\end{bmatrix} . \tag{3.23}
$$

The matrix $\mathcal{M}_2$ is expressed in association with Kronecker product structures as $\mathcal{M}_2 = (I \otimes M_2) + (\mathbf{P_2} \otimes I)$ with M_2 is defined in equation (3.4). In order to express the global transition function, the vector z_{t+1} is in this formula,

$$
z_{t+1} = \mathcal{F}(z_t) = \mathcal{M}_2 z_t \quad \text{and}
$$
$$
z_t = \mathcal{M}_2^t z_0.
$$

(iii) Fixed boundary conditions: let the cell of the boundary of $\mathcal{A}_2$ be fixed at the state s. Then, the global transition function can be written as

$$
z_{t+1} = \mathcal{F}(z_t) = \mathcal{M}_3 z_t + K, \quad \text{and}
$$
$$
z_t = \mathcal{M}_3^t z_0 + (\mathcal{M}_3^{t-1} + \mathcal{M}_3^{t-2} + \dots + \mathcal{M}_3 + I)K,
$$

where I is the identity matrix,
$K = [a^*_1 s + a_{-1} s, a^*_1 s, \dots, a^*_1 s, a^*_1 s + a_1 s, a^{-1} s, 0, \dots, 0, a_1 s, \dots, a^{-1} s, 0, \dots, 0, a_1 s, a^*_{-1} s + a_{-1} s, a^*_1 s, \dots, a^*_1 s, a_1 s + a^*_{-1} s]'$,
$\mathcal{M}_3 = (I \otimes M_3) + (\mathbf{P_3} \otimes I)$,
M_3 is defined in equation (3.6),
and $\mathbf{P_3}$ has the following expression

$$
\mathbf{P_3} = \begin{bmatrix}
0 & a^*_{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\
a^*_1 & 0 & a^*_{-1} & 0 & 0 & \dots & 0 & 0 \\
0 & a^*_1 & 0 & a^*_{-1} & 0 & \dots & 0 & 0 \\
0 & 0 & a^*_1 & 0 & a^*_{-1} & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & a^*_1 & 0 & a^*_{-1} & 0 \\
0 & 0 & \dots & 0 & 0 & a^*_1 & 0 & a^*_{-1} \\
0 & 0 & \dots & 0 & 0 & 0 & a^*_1 & 0
\end{bmatrix} . \tag{3.24}
$$

Let $\mathcal{L}_p = \{c^*_1, c^*_2, \dots, c^*_p\} \subseteq \mathcal{L}$ and $\omega = \{\omega_1, \dots, \omega_n\} \subseteq \mathcal{L}$ such that $c^*_1 \prec c^*_2 \prec \dots \prec c^*_p$ and $\omega_1 \prec \omega_2 \prec \dots \prec \omega_n$, where $\prec$ is row-major order. The regional controllability problem is to find u_t in which a subregion $\mathcal{L}_p$ is excited by u_t to obtain the desired state s_d on ω at the final time step T, that is $s_d = s_T|_\omega$. In this study, the global control function is

$$
\mathcal{G}(u_t) = \mathcal{B}W_t,
$$

where $W_t = \begin{bmatrix} W_{t\ 1,1} & W_{t\ 1,2} & \cdots & W_{t\ 1,N_{\mathcal{L}}} \\ W_{t\ 2,1} & W_{t\ 2,2} & \cdots & W_{t\ 2,N_{\mathcal{L}}} \\ \vdots & \vdots & \ddots & \vdots \\ W_{t\ N_{\mathcal{L}},1} & W_{t\ N_{\mathcal{L}},2} & \cdots & W_{t\ N_{\mathcal{L}},N_{\mathcal{L}}} \end{bmatrix}$ is a $(N_{\mathcal{L}} \times N_{\mathcal{L}})$ matrix,

such that

$$W_{t\ i,j} = \begin{cases} u_t^{(i,j)} & \text{if } (i,j) \in \mathcal{L}_p, \\ 0 & \text{otherwise.} \end{cases}$$

for $i, j = 1, ..., N_{\mathcal{L}}$

Thus, according to equation (2.2), the transition rule of the regional controllability problem is given as

$$z_{t+1} = \mathcal{F}(z_t) + \mathcal{B}W_t. \tag{3.25}$$

for $i, j = 1, ..., N_{\mathcal{L}}$,

Similar to the one dimensional case, we can find the control vector sequence u by using following system,

$$\mathbf{AW} = \mathbf{B}, \tag{3.26}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathcal{M}_{\omega_1,c_1^*}^{T-1}\mathcal{B} & \mathcal{M}_{\omega_1,c_1^*}^{T-2}\mathcal{B} & \cdots & \mathcal{B}_{\omega_1,c_1^*} & \mathcal{M}_{\omega_1,c_2^*}^{T-1}\mathcal{B} & \mathcal{M}_{\omega_1,c_2^*}^{T-2}\mathcal{B} & \cdots & \mathcal{B}_{\omega_1,c_2^*} & \cdots & \mathcal{B}_{\omega_1,c_p^*} \\ \mathcal{M}_{\omega_2,c_1^*}^{T-1}\mathcal{B} & \mathcal{M}_{\omega_2,c_1^*}^{T-2}\mathcal{B} & \cdots & \mathcal{B}_{\omega_2,c_1^*} & \mathcal{M}_{\omega_2,c_2^*}^{T-1}\mathcal{B} & \mathcal{M}_{\omega_2,c_2^*}^{T-2}\mathcal{B} & \cdots & \mathcal{B}_{\omega_2,c_2^*} & \cdots & \mathcal{B}_{\omega_2,c_p^*} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathcal{M}_{\omega_n,c_1^*}^{T-1}\mathcal{B} & \mathcal{M}_{\omega_n,c_1^*}^{T-2}\mathcal{B} & \cdots & \mathcal{B}_{\omega_n,c_1^*} & \mathcal{M}_{\omega_n,c_2^*}^{T-1}\mathcal{B} & \mathcal{M}_{\omega_n,c_2^*}^{T-2}\mathcal{B} & \cdots & \mathcal{B}_{\omega_n,c_2^*} & \cdots & \mathcal{B}_{\omega_n,c_p^*} \end{bmatrix},$$

$$\mathbf{W} = \begin{bmatrix} u_0^{c_1^*} & \cdots & u_T^{c_1^*} & u_0^{c_2^*} & \cdots & u_T^{c_2^*} & \cdots & u_0^{c_p^*} & \cdots & u_T^{c_p^*} \end{bmatrix}', \quad \text{and}$$

$$\mathbf{B} = \begin{cases} \left(z_T - \mathcal{M}_1^T z_0\right)\Big|_\omega & \text{if periodic boundary conditions are stated,} \\ \left(z_T - \mathcal{M}_2^T z_0\right)\Big|_\omega & \text{if reflective boundary conditions are stated,} \\ \left(z_T - \mathcal{M}_3^T z_0 + \left(\sum_{i=1}^T \mathcal{M}_3^{T-i}\right) K\right)\Big|_\omega & \text{if the boundary cells are fixed at } s. \end{cases}$$

Thus, we have the result as the following theorems.

Theorem 3.14. *Let $\mathbf{A}$ and $\mathbf{B}$ be the $n \times (pT)$ and $N_{\mathcal{L}} \times 1$ matrices, respectively, defined by equation (3.26) and let $\tilde{\mathbf{A}} = [\mathbf{A}|\mathbf{B}]$ be the augmented matrix of the system (3.26). Then cellular automaton $\mathcal{A}_2$ is regionally controllable if and only if $\mathrm{rank}\mathbf{A} = \mathrm{rank}\tilde{\mathbf{A}}$.*

Theorem 3.15. *Let $\mathbf{A}$ be the matrix defined by equation (3.26). Then the cellular automaton $\mathcal{A}_2$ is regionally controllable if $\mathrm{rank}\mathbf{A} = n$.*

Theorem 3.16. *Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices defined by equation (3.26). Then the cellular automaton $\mathcal{A}_2$ is regionally controllable with the control vector*

$$\mathbf{U} = \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{B} + \left(I - \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{A}\right)\mathbf{Y}, \tag{3.27}$$

for any $pT \times 1$ matrix $\mathbf{Y}$ *if and only if* $\text{rank}\mathbf{A} = \text{rank}\tilde{\mathbf{A}}$ *and* $\text{rank}\mathbf{A} < pT$.

Theorem 3.17. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be the matrices defined by equation* (3.26). *Then the cellular automaton* $\mathcal{A}_2$ *is regionally controllable with the control vector* $\mathbf{U} = \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{B}$, *which is the least-norm vector, if* $\text{rank}\mathbf{A} = \text{rank}\tilde{\mathbf{A}}$ *and* $\text{rank}\mathbf{A} < pT$.

Theorem 3.18. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be the and matrices defined by equation* (3.26). *Then the cellular automaton* $\mathcal{A}_2$ *is regionally controllable with a unique control vector* $\mathbf{U} = \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{B}$ *if and only if* $\text{rank}\mathbf{A} = \text{rank}\tilde{\mathbf{A}}$ *and* $\text{rank}\mathbf{A} < pT$.

Simulation Examples for Two-Dimensional Additive Real-Valued CA

Consider the additive cellular automaton $\mathcal{A}_2$ with square lattice $\mathcal{L} = \{(i,j)|i = 1,...,10, j = 1,...,10\}$. Each cell's state can take a value in $\mathbb{R}$. The local transition function f with Von Neumann neighbourhood of radius $r = 1$ and periodic boundary conditions is defined by:

$$s_{t+1}(i,j) = f(s_t(\mathcal{N}(i,j))) = \sum_{c^* \in \mathcal{N}(i,j)} s_t(c^*).$$

Let the control be active in cells $(3,2)$ and $(8,7)$ and let $\mathcal{M} = \mathcal{B}$ and $T = 5$. Suppose that s_0 is the initial configuration, which the initial states are randomly generated between -5 to 5, and s_d is zero configuration, restrict to ω.

Let $\omega = \{(3,2),(4,1),(4,2),(4,3),(5,2)\}$, $u = \{u_0, u_1, u_2, u_3, u_4\}$ where $u_0 = [u_0^{(3,2)}, u_0^{(8,7)}]$, $u_1 = [u_1^{(3,2)}, u_1^{(8,7)}]$, $u_2 = [u_2^{(3,2)}, u_2^{(8,7)}]$, $u_3 = [u_3^{(3,2)}, u_3^{(8,7)}]$, $u_4 = [u_4^{(3,2)}, u_4^{(8,7)}]$, and $z_d = Vec(s_d) = [0,0,0,0,0]'$.

In this case, we obtain the system $\mathbf{AW} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} \mathcal{M}^5_{(3,2),(3,2)} & \mathcal{M}^4_{(3,2),(3,2)} & \cdots & \mathcal{M}_{(3,2),(3,2)} & \mathcal{M}^5_{(3,2),(8,7)} & \mathcal{M}^4_{(3,2),(8,7)} & \cdots & \mathcal{M}_{(3,2),(8,7)} \\ \mathcal{M}^5_{(4,1),(3,2)} & \mathcal{M}^4_{(4,1),(3,2)} & \cdots & \mathcal{M}_{(4,1),(3,2)} & \mathcal{M}^5_{(4,1),(8,7)} & \mathcal{M}^4_{(4,1),(8,7)} & \cdots & \mathcal{M}_{(4,1),(8,7)} \\ \mathcal{M}^5_{(4,2),(3,2)} & \mathcal{M}^4_{(4,2),(3,2)} & \cdots & \mathcal{M}_{(4,2),(3,2)} & \mathcal{M}^5_{(4,2),(8,7)} & \mathcal{M}^4_{(4,2),(8,7)} & \cdots & \mathcal{M}_{(4,2),(8,7)} \\ \mathcal{M}^5_{(4,3),(3,2)} & \mathcal{M}^4_{(4,3),(3,2)} & \cdots & \mathcal{M}_{(4,3),(3,2)} & \mathcal{M}^5_{(4,3),(8,7)} & \mathcal{M}^4_{(4,3),(8,7)} & \cdots & \mathcal{M}_{(4,3),(8,7)} \\ \mathcal{M}^5_{(5,2),(3,2)} & \mathcal{M}^4_{(5,2),(3,2)} & \cdots & \mathcal{M}_{(5,2),(3,2)} & \mathcal{M}^5_{(5,2),(8,7)} & \mathcal{M}^4_{(5,2),(8,7)} & \cdots & \mathcal{M}_{(5,2),(8,7)} \end{bmatrix}$$

$$= \begin{bmatrix} 4.38 & 0.81 & -1.38 & -0.75 & 0.50 & 0 & 0 & 0 & 0 & 0 \\ 2.88 & 0.69 & -0.75 & -0.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.44 & -2.00 & -0.75 & 0.50 & 0.50 & 0 & 0 & 0 & 0 & 0 \\ -3.13 & -0.75 & 0.75 & 0.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.19 & -0.63 & 0.38 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W} = \left[u_0^{(3,2)}, u_1^{(3,2)}, u_2^{(3,2)}, u_3^{(3,2)}, u_4^{(3,2)}, u_0^{(8,7)}, u_1^{(8,7)}, u_2^{(8,7)}, u_3^{(8,7)}, u_4^{(8,7)} \right]',$$

$$\mathbf{B} = z_d - \left(\mathcal{M}^5_1 z_0 \right)|_\omega = [-11.45, -23.61, 10.70, 14.41, 32.12]'.$$

Since $\text{rank}\mathbf{A} = \text{rank}\tilde{\mathbf{A}} = 5 < 10 = pT$, then by Theorem 3.16, $\mathcal{A}_2$ is regionally controllable with $\mathbf{U} = \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{B} + \left(I - \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{A} \right)\mathbf{Y}$. In this case, it has infinitely many control vector sequence u, depending on vector $\mathbf{Y}$. Next, we give two examples for

calculation of the control vector sequence u with different $\mathbf{Y}$, which are presented below.

Case(i) Let $\mathbf{Y} = [100, 100, 100, 100, 100, 100, 100, 100, 100, 100]'$. Then

$$\mathbf{U} = \left[u_0^{(3,2)}, u_1^{(3,2)}, u_2^{(3,2)}, u_3^{(3,2)}, u_4^{(3,2)}, u_0^{(8,7)}, u_1^{(8,7)}, u_2^{(8,7)}, u_3^{(8,7)}, u_4^{(8,7)}\right]$$

$$= [-1835.47, 4289.05, -8695.66, 8234.17, -2294.01, 100.00, 100.00, 100.00, 100.00, 100.00].$$

Thus, we obtain $u = \{u_0, u_1, u_2, u_3, u_4\}$, where $u_0 = [-1835.47, 100.00]$, $u_1 = [4289.05, 100.00]$, $u_2 = [-8695.66, 100.00]$, $u_3 = [8234.17, 100.00]$, $u_4 = [-2294.01, 100.00]$ and $\|\mathbf{U}\|_2 = 13057$.

Case(ii) Let $\mathbf{Y} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]'$, then we get

$$\mathbf{U} = \left[u_0^{(3,2)}, u_1^{(3,2)}, u_2^{(3,2)}, u_3^{(3,2)}, u_4^{(3,2)}, u_0^{(8,7)}, u_1^{(8,7)}, u_2^{(8,7)}, u_3^{(8,7)}, u_4^{(8,7)}\right]$$

$$= [164.50, -511.00, 904.30, -1065.80, 256.00, 0, 0, 0, 0, 0].$$

This indicates that $u_0 = [164.50, 0]$, $u_1 = [-511.00, 0]$, $u_2 = [904.30, 0]$, $u_3 = [-1065.80, 0]$, $u_4 = [256.00, 0]$ and $\|\mathbf{U}\|_2 = 13055$. The corresponding space-time dynamics of $\mathcal{A}_2$ using $u = \{u_0, u_1, u_2, u_3, u_4\}$ are shown in Figure 3.
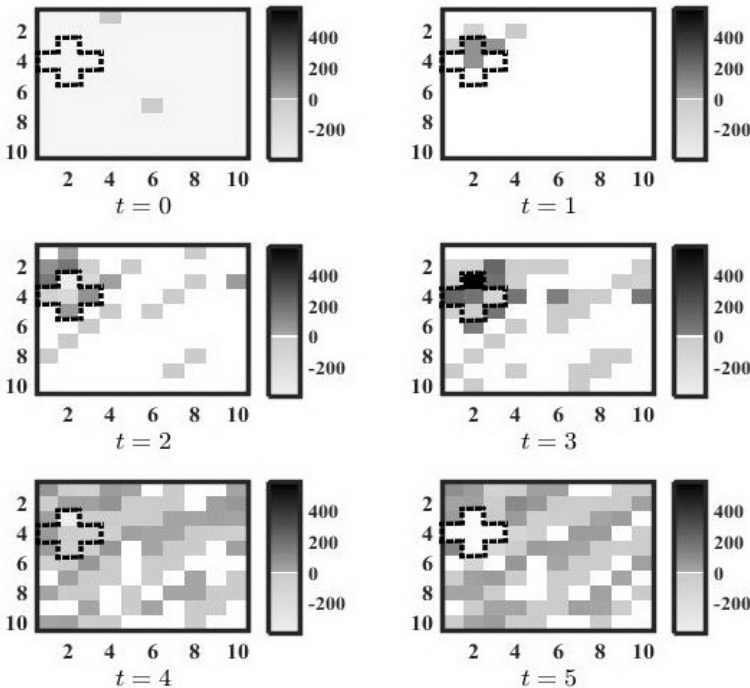

FIGURE 3. A space-time illustration of the dynamics of cellular automaton $\mathcal{A}_2$ for case(ii) with control vector sequence $u = \{u_0, u_1, u_2, u_3, u_4\}$ for $t = 0, 1, 2, 3, 4, 5$ on a lattice of 10×10 cells.

Remark 3.19. The rank conditions of our study can be used to determine the regional controllability of a system that is modeled using one- or two-dimensional additive real-valued cellular automata with periodic, fixed, and reflective boundary conditions.

4. Conclusions

The regional controllability of cellular automata deals with finding the control vector sequence u, which results in a given desired configuration on subregion ω at the final time step T. Our focus in this study is the regional controllability of the one- and two-dimension additive real-valued cellular automata with periodic, fixed, and reflective boundary conditions based on theory of linear equations. We constructed the global transition functions of one- and two-dimensional additive real-valued cellular automata in the matrix form to update the configuration. By applying the global transition formulae, the sufficient conditions for regional controllability of the additive cellular automata were derived and proved. Moreover, we obtained the formulae of the control vectors and control vectors with least-norm vector for which the additive cellular automata are regionally controllable. Finally, we provided some numerical examples to support the theoretical results.

Acknowledgements

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