Thai Journal of Mathematics
Volume 18 Number 4 (2020)
Pages 1787–1799

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Coincidence Best Proximity Points for Generalized $\mathcal{MT}$-Proximal Cyclic Contractive Mappings in S-Metric Space

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Abstract In this paper, we use the concept of $\mathcal{MT}$-function to establish the coincidence best proximity point results for a certain class of proximal cyclic contractive mappings in S-metric spaces. Our results extend and improve some known results in the literature. We give an example to analyze and support our main results.

MSC: 47H10; 54H25

Keywords: cyclic mapping; best proximity point; $\mathcal{MT}$-function ($\mathcal{R}$-function); S-metric space

Submission date: 15.07.2019 / Acceptance date: 23.06.2020

1. Introduction

The best approximation results provide an approximate solution to the fixed point equation $Tx = x$, when the nonself-mapping T has no fixed point. In particular, a celebrated best approximation theorem, due to Fan [1], asserts the fact that if K is a nonempty compact convex subset of a Hausdroff locally convex topological vector space X and $T : K \to X$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, Tx) = \inf\{d(y, Tx) : y \in K\}$, where d is a metric on X.

The evolution of best proximity point theory has been extended as a generalization of the concept of the best approximation. The best approximation theorem guarantees the existence of an approximate solution, the best proximity point theorem is considered for solving the problem to find an approximate solution which is optimal.

Let A and B be two nonempty subsets of a metric space (X, d). An element $x \in A$ is said to be a fixed point of a given map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \varnothing$ is a necessary(but not sufficient) condition for the existence of a fixed point of T. If $T(A) \cap A = \varnothing$, then $d(x, Tx) > 0$ for all $x \in A$ that is, the set of fixed points of T is empty. In a such situation, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point analysis has been developed in this direction.

Published by The Mathematical Association of Thailand.

An element $a \in A$ is called a *best proximity point* of T if

$$d(a, Ta) = \mathrm{d}(A, B),$$

where

$$\mathrm{d}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Because of the fact that $d(x, Tx) \to \mathrm{d}(A, B)$ for all $x \in A$, the global minimum of the mapping $x \to d(x, Tx)$ is attained at a best proximity point. Clearly, if the underlying mapping is self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The goal of the best proximity point theorem is to provide sufficient conditions to ascertain the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$. For more details on this approach, we refer the reader to ([2–7]) and references therein.

In the case of cyclic contractive mapping $T : A \cup B \to A \cup B$, a point $x \in A \cup B$ is called the best proximity point if $d(x, Tx) = \mathrm{d}(A, B)$. Notice that a best proximity point x is a fixed point of T whenever $A \cap B \neq \varnothing$. Thus it generalizes the notion of fixed point in case when $A \cap B = \varnothing$. Further [4, 8–14] examine several variants of contractions for the existence of a best proximity point.

On the other hand, Sedghi, Shobe and Aliouche have defined the concept of an S-metric space, Sedghi et al. [15]. This notion is a generalization of a G-metric space [16] and a D^*-metric space [17] , respectively.

In 2003, the concepts of cyclic mapping and best proximity point were innovated by Kirk, Srinavasan and Veeramani, Kirk et al. [18]. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called a cyclic mapping if $T(A) \subset B$ and $T(B) \subset A$. In 2006, Eldered and Veeramani [19] demonstrated some existence results about best proximity points of cyclic contraction mappings.

A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-function (or $\mathcal{R}$-functon). If $\limsup_{s \to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$.

In this paper, we establish some new existence and convergence theorems of iterates of best proximity points for $\mathcal{MT}$-cyclic contractions in S-metric space.

2. Preliminaries

First we recall some necessary definitions and results in this direction.

The notion of S-metric spaces is defined as follows.

Definition 2.1. [15] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$.

(S1) $S(x, y, z) \geq 0$;
(S2) $S(x, y, z) = 0$ if and only if $x = y = z$;
(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the pair (X, S) is called an S-metric space.

Remark 2.2. Note that every S-metric on X induces a metric d_S on X defined by

$$d_S(x, y) = S(x, x, y) + S(y, y, x)$$

for all $x, y \in X$.

The following is an intuitive geometric example for S-metric spaces.

Example 2.3. [15] Let $X = \mathbb{R}$ and d be an ordinary metric on X. Put

$$S(x, y, z) = d(x, y) + d(x, z)$$

for all $x, y, z \in \mathbb{R}$. Then S is an S-metric on X.

Example 2.4. [15] Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put

$$S(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in \mathbb{R}$. It is easy to check that S is an S-metric on X.

Lemma 2.5. [15] *Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Remark 2.6. Let (X, S) be an S-metric space. From Definition 2.1 and Lemma 2.5 we have,

$$S(x, x, z) \leq S(x, x, y) + 2S(y, y, z)$$

for all $x, y, z \in X$.

Definition 2.7. [15] Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ is said to *converge to* $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.

(ii) A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

(iii) The S-metric space (X, S) is said to be *complete* if every Cauchy sequence is a convergent sequence

Now we recall the notion of $\mathcal{MT}$- functions introduced in as follows.

Definition 2.8. [20] A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-function (or $\mathcal{R}$-function). If

$$\limsup_{s \to t^+} \varphi(s) < 1 \quad \text{for all } t \in [0, \infty).$$

It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

Example 2.9. [21] $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) = \begin{cases} \frac{\sin(t)}{t} & , t \in (0, \pi/2] \\ 0 & , \text{otherwise.} \end{cases}$$

Since $\lim_{s \to t^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

Very recently, Du [21] first proved some characterizations of $\mathcal{MT}$-functions.

Theorem 2.10. [21] *Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.*

(i) *For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.*

(ii) *φ is a function of contractive factor, for any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.*

In [22], the authors present some definitions about type of proximal contractions.

Definition 2.11. [22] A mapping $T : A \to B$ is called *proximal contraction of the first kind* if there exists $k \in [0, 1)$ such that

$$\left.\begin{array}{rcl} d(u, Tx) & = & d(A, B) \\ d(v, Ty) & = & d(A, B) \end{array}\right\} \implies d(u, v) \leq kd(x, y) \tag{2.1}$$

for all $u, v, x, y \in A$. It is easy to see that a self-mapping is a contraction of the first kind is precisely a contraction. However a non self proximal contraction is not necessarily a contraction.

Definition 2.12. [22] A mapping $T : A \to B$ is called *proximal contraction of the second kind* if there exists $k \in [0, 1)$ such that

$$\left.\begin{array}{rcl} d(u, Tx) & = & d(A, B) \\ d(v, Ty) & = & d(A, B) \end{array}\right\} \implies d(Tu, Tv) \leq kd(Tx, Ty) \tag{2.2}$$

for all $u, v, x, y \in A$.

Definition 2.13. [22] Let (X, d) be a complete metric space. A mapping $g : X \to X$ is called an *isometry* if $d(gx, gy) = d(x, y)$ for all $x, y \in X$.

Definition 2.14. [22] Consider the non-self-mappings $S : A \to B$ and $T : B \to A$, the pair (S, T) is said to form a *proximal cyclic contraction* if there exists a non-negative number $\alpha < 1$ such that

$$\left.\begin{array}{rcl} d(u, Sx) & = & d(A, B) \\ d(v, Ty) & = & d(A, B) \end{array}\right\} \implies d(u, v) \leq \alpha d(x, y) + (1 - \alpha)d(A, B) \tag{2.3}$$

for all $u, v, x, y \in A$.

Definition 2.15. [20] Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(MT1) $T(A) \subset B$ and $T(B) \subset A$.

(MT2) there exists an $\mathbb{R}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big)d(x, y) + \Big(1 - \varphi\big(d(x, y)\big)\Big)\, \mathrm{d}(A, B)$$

for any $x \in A$ and $y \in B$,

then T is called an $\mathcal{MT}$-cyclic contraction with respect to φ on $A \cup B$.

Remark 2.16. It is obvious that (MT2) implies that T satisfies $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$.

Recall that every S-metric on X induces a metric d_S on X defined by

$$d_S(x, y) = S(x, x, y) + S(y, y, x)\ ; \quad \forall x, y \in X$$

Let (X, S) be an S-metric space. Suppose that A and B are nonempty subsets of an S-metric space (X, S). We define the following sets:

$$A_0 = \{x \in A : d_S(x, y) = d_S(A, B) \text{ for some } y \in B\} \quad \text{and}$$

$$B_0 = \{y \in B : d_S(x, y) = d_S(A, B) \text{ for some } x \in A\}$$

where $d_S(A, B) = \inf\{d_S(x, y) : x \in A, y \in B\}$.

Definition 2.17. [13] Let (X, d) be a metric space and $A, B \subseteq X$, let $g : A \to A$ and $f : A \to B$ be mappings then a point $x \in A$ is a *best proximity coincidence point* of the pair (g, f) if $d(gx, fx) = d(A, B)$.

3. Main Result

Definition 3.1. Let A and B be two non-empty subsets of S-metric space (X, S). Let $f : A \to B$ is called *generalized S-$\mathcal{MT}$-proximal cyclic contraction of the first kind with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x, u, a, b, y, v \in A$

$$\left.\begin{array}{rcl} d_S(u, fx) &=& d_S(A, B) \\ d_S(b, fa) &=& d_S(A, B) \\ d_S(v, fy) &=& d_S(A, B) \end{array}\right\} \implies S(u, b, v) \leq \varphi(S(x, a, y)) \frac{S(x, a, y)S(y, u, b)}{S(x, a, y) + 2S(x, a, b)} \tag{3.1}$$

where $S(x, a, y) + 2S(x, a, b) \neq 0$.

Definition 3.2. Let A and B be two non-empty subsets of S-metric space (X, S). Let $f : A \to B$ is called *generalized S-$\mathcal{MT}$-proximal cyclic contraction of the second kind with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x, u, a, b, y, v \in A$,

$$\left.\begin{array}{rcl} d_S(u, fx) &=& d_S(A, B) \\ d_S(b, fa) &=& d_S(A, B) \\ d_S(v, fy) &=& d_S(A, B) \end{array}\right\} \implies \tag{3.2}$$

$$S(fu, fb, fv) \leq \varphi(S(fx, fa, fy)) \frac{S(fx, fa, fy)S(fy, fu, fb)}{S(fx, fa, fy) + 2S(fx, fa, fb)}$$

where $S(fx, fa, fy) + 2S(fx, fa, fb) \neq 0$.

Definition 3.3. Let A and B be two non-empty subsets of S-metric space (X, S). Suppose that $f : A \to B$ and $\hat{f} : B \to A$ are mappings. The pair $(f, \hat{f})$ is called *S-$\mathcal{MT}$-proximal cyclic contraction with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x, u \in A$ and $y, v \in B$

$$\left.\begin{array}{rcl} d_S(u, fx) &=& d_S(A, B) \\ d_S(v, \hat{f}y) &=& d_S(A, B) \end{array}\right\} \implies \tag{3.3}$$

$$d_S(u, v) \leq \varphi\big(d_S(x, y)\big) d_S(x, y) + \Big(1 - \varphi\big(d_S(x, y)\big)\Big) d_S(A, B).$$

Definition 3.4. Let (X, S) be a complete S-metric space. A mapping $g : X \to X$ is called an *isometry* if $S(gx, gy, gz) = S(x, y, z)$ for all $x, y, z \in X$.

Definition 3.5. Let $f : A \to B$ and $g : A \to A$ be an isometry. The mapping f is said to *preserve the isometric distance with respect to g* if $S(fgx, fgy, fgz) = S(fx, fy, fz)$ for all $x, y, z \in A$.

Theorem 3.6. *Let A, B be two nonempty subsets of a S-metric space (X, S) such that $A_0, B_0 \neq \varnothing$. Let $f : A \to B, \hat{f} : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(i) *f and $\hat{f}$ are generalized S-$\mathcal{MT}$-proximal cyclic contractions of the first kind;*

(ii) *g is an isometry;*
(iii) *the pair* $(f, \hat{f})$ *is generalized* S-$\mathcal{MT}$*-proximal cyclic contraction;*
(iv) $f(A_0) \subseteq B_0$ *and* $\hat{f}(B_0) \subseteq A_0$*;*
(v) $A_0 \subseteq g(A_0)$ *and* $B_0 \subseteq g(B_0)$*.*

Then there exists a point $x \in A$ *and there exists a point* $y \in B$ *such that*

$$d_S(gx, fx) = d_S(gy, \hat{f}y) = d_S(x, y) = d_S(A, B).$$

Moreover, for any best proximity point $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by*

$$d_S(gx_{n+1}, fx_n) = d_S(A, B)$$

converges to the element x*.*
Similarly, for any best proximity point $y_0 \in B_0$*, the sequence* $\{y_n\}$ *defined by*

$$d_S(gy_{n+1}, \hat{f}y_n) = d_S(A, B)$$

converges to the element y*.*

Proof. Let $x_0 \in A_0$, since $f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $d_S(gx_1, fx_0) = d_S(A, B)$. Also, since $fx_1 \in B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $d_S(gx_2, fx_1) = d_S(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d_S(gx_{n+1}, fx_n) \; = \; d_S(A, B) \; , \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

This shows that

$$\begin{aligned} d_S(u, fx) &= d_S(A, B) \\ d_S(b, fa) &= d_S(A, B) \\ d_S(v, fy) &= d_S(A, B) \end{aligned} \tag{3.5}$$

where $u = gx_{n+1} = b, x = x_n = a, v = gx_n$ and $y = x_{n-1}$. From (3.1), we have

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) &\leq \varphi\big(S(x_n, x_n, x_{n-1})\big) \frac{S(x_n, x_n, x_{n-1}) S(x_{n-1}, gx_{n+1}, gx_{n+1})}{S(x_n, x_n, x_{n-1}) + 2S(x_n, x_n, gx_{n+1})} \\ &\leq \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big) \frac{S(x_n, x_n, x_{n-1}) S(x_{n-1}, gx_{n+1}, gx_{n+1})}{S(x_{n-1}, x_{n-1}, gx_{n+1})} \\ &\leq \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big) S(x_n, x_n, x_{n-1}). \end{aligned} \tag{3.6}$$

From (3.6) we have

$$S(x_n, x_n, x_{n+1}) \leq \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big) S(x_{n-1}, x_{n-1}, x_n). \tag{3.7}$$

Since φ is an $\mathcal{MT}$-function, then from Theorem 2.10,

$$0 \leq \sup_{n \in \mathbb{N}} \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big) < 1. \tag{3.8}$$

Let $\lambda := \sup_{n \in \mathbb{N}} \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big)$ then

$$0 \leq \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big) \leq \lambda < 1 \tag{3.9}$$

for all $n \in \mathbb{N}$. From (3.7) we have

$$S(x_n, x_n, x_{n+1}) \leq \lambda S(x_{n-1}, x_{n-1}, x_n) \leq S(x_{n-1}, x_{n-1}, x_n). \tag{3.10}$$

So the sequence $\{S(x_n, x_n, x_{n+1})\}$ is non-increasing sequence in $[0, \infty)$ and thus $\lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} S(x_n, x_n, x_{n+1})$ exists.

From the first inequality of (3.10), we obtain

$$S(x_n, x_n, x_{n+1}) \leq \lambda^n S(x_0, x_0, x_1)\ ; \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Due to $\lambda \in [0, 1)$, $\lim\limits_{n\to\infty} \lambda^n = 0$. By taking limit in (3.11) as $n \to \infty$, we deduce

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3.12}$$

Suppose $n, m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_{m-1}, x_{m-1}, x_m) + S(x_n, x_n, x_{m-1}) \\ &\leq 2S(x_{m-1}, x_{m-1}, x_m) + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_n, x_n, x_{m-2}) \\ &\leq 2S(x_{m-1}, x_{m-1}, x_m) + 2S(x_{m-2}, x_{m-2}, x_{m-1}) \cdots + S(x_n, x_n, x_{n+1}). \end{aligned}$$

Now, for $m = n + r; r \geq 1$ and (3.11), we obtain

$$S(x_n, x_n, x_{n+r}) \leq 2\lambda^{n+r-1} S(x_0, x_0, x_1) + 2\lambda^{n+r-2} S(x_0, x_0, x_1) + \cdots + \lambda^n S(x_0, x_0, x_1).$$

By taking limit as $n \to \infty$, we deduce

$$\lim_{n\to\infty} S(x_n, x_n, x_m) = 0. \tag{3.13}$$

That is, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (A, S) is a complete S-metric space, so there exists $x \in A$ such that $x_n \to x$ as $n \to \infty$. Similarly, since $\hat{f}(B_0) \subseteq A_0$ and $B_0 \subseteq g(B_0)$, there exists a sequence $\{y_n\}$ such that it converges to some element $y \in B$. Since the pair $(f, \hat{f})$ is S-$\mathcal{MT}$-proximal cyclic contraction and g is isometry, we have for $x_{n+1} \in A, y_{n+1} \in B$,

$$d_S(gx_{n+1}, fx_n) = d_S(A, B) \quad \text{and} \quad d_S(gy_{n+1}, \hat{f}y_n) = d_S(A, B).$$

Then

$$\begin{aligned} d_S(x_{n+1}, y_{n+1}) &= d_S(gx_{n+1}, gy_{n+1}) \\ &\leq \varphi\big(d_S(x_n, y_n)\big) d_S(x_n, y_n) + \Big(1 - \varphi\big(d_S(x_n, y_n)\big)\Big) d_S(A, B). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$d_S(x, y) \leq \varphi\big(d_S(x, y)\big) d_S(x, y) + \Big(1 - \varphi\big(d_S(x, y)\big)\Big) d_S(A, B)$$

$$\Big(1 - \varphi\big(d_S(x, y)\big)\Big) d_S(x, y) \leq \Big(1 - \varphi\big(d_S(x, y)\big)\Big) d_S(A, B),$$

yields

$$d_S(x, y) = d_S(A, B). \tag{3.14}$$

Thus, $x \in A_0$ and $y \in B_0$. Since $f(A_0) \subseteq B_0$ and $\hat{f}(B_0) \subseteq A_0$, there exist $u \in A$ and $v \in B$ such that

$$d_S(u, fx) = d_S(A, B) \quad \text{and} \quad d_S(v, \hat{f}y) = d_S(A, B). \tag{3.15}$$

Since f is S-$\mathcal{MT}$-proximal cyclic contraction of the first kind, we get from $d_S(u, fx) = d_S(A, B)$ and $d_S(gx_{n+1}, fx_n) = d_S(A, B)$ as

$$S(u, u, gx_{n+1}) \leq \varphi(S(x, x, x_n)) \frac{S(x, x, x_n) S(x_n, u, u)}{S(x, x, x_n) + 2S(x, x, u)} \leq S(x, x, x_n).$$

Taking limit as $n \to \infty$, we have $S(u, u, gx) = 0$ and so $u = gx$. Therefore

$$d_S(gx, fx) = d_S(A, B). \tag{3.16}$$

Similarly, we have $v = gy$ and so

$$d_S(gy, \hat{f}y) = d_S(A, B). \tag{3.17}$$

Thus, from (3.14),(3.16) and (3.17), we get

$$d_S(x, y) = d_S(gx, fx) = d_S(gy, \hat{f}y) = d_S(A, B).$$

This completes the proof. ∎

Example 3.7. Consider the space $X = \mathbb{R}^2$ with S-metric given in Example then $S(x, y, z) = \frac{1}{2}\max\{|x-y|, |x-z|, |y-z|\}$ for all $x, y, z \in X$. where $|a-b| = |a_1 - b_1| + |a_2 - b_2|$ for $(a_1, a_2), (b_1, b_2) \in X$. Then $S(x, x, y) = \frac{1}{2}\big|x - y\big|$ and $d_S(x, y) = \big|x - y\big|$. Let

$$A = \left\{(-1, x) : x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\} \quad \text{and} \quad B = \left\{(1, y) : y \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\}.$$

Define the mappings $f : A \to B$ and $\hat{f} : B \to A$ as follows:

$$f\big((-1, x)\big) = \begin{cases} \left(1, \frac{x}{4}\right); & \text{if } x \in \left[-\frac{1}{2}, 0\right] \\ (1, 0); & \text{otherwise} \end{cases}, \quad \hat{f}\big((1, y)\big) = \begin{cases} \left(-1, \frac{y}{4}\right); & \text{if } y \in \left[-\frac{1}{2}, 0\right] \\ (-1, 0); & \text{otherwise} \end{cases}$$

and define $g : A \cup B \to A \cup B$ by $g\big((x, y)\big) = (x, -y)$ for all $x, y \in A \cup B$. Then $d_S(A, B) = 2, A_0 = A, B_0 = B$,

$$f(A_0) = \left\{(1, x) : x \in \left[-\tfrac{1}{8}, 0\right]\right\} \subseteq \left\{(1, x) : x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\} = B_0,$$
$$\hat{f}(B_0) = \left\{(-1, y) : y \in \left[-\tfrac{1}{8}, 0\right]\right\} \subseteq \left\{(-1, y) : y \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\} = A_0,$$

and g is an isometry. Define $\varphi(t) = \frac{4}{5}; \forall t \in [0, \infty)$, we will show that f and $\hat{f}$ are generalized S-$\mathcal{MT}$-proximal cyclic contractions.

Let $(-1, x_1), (-1, x_2), (-1, a_1), (-1, a_2) \in A$ satisfying

$$d_S\big((-1, a_1), f(-1, x_1)\big) = 2 \quad \text{and} \quad d_S\big((-1, a_2), f(-1, x_2)\big) = 2.$$

<u>Case I: $x_1, x_2 \in \left[-\frac{1}{2}, 0\right]$ and $x_1 > x_2$</u> We have,

$$S\big((-1, a_1), (-1, a_1), (-1, a_2)\big) = \tfrac{1}{2}\left|\tfrac{x_1}{4} - \tfrac{x_2}{4}\right| = \tfrac{1}{8}\left|x_1 - x_2\right|,$$
$$k := \varphi\Big(S\big((-1, x_1), (-1, x_1), (-1, x_2)\big)\Big) = \tfrac{4}{5},$$
$$S\big((-1, x_1), (-1, x_1), (-1, x_2)\big) = \tfrac{1}{2}\left|x_1 - x_2\right|,$$
$$S\big((-1, x_2), (-1, a_1), (-1, a_1)\big) = \tfrac{1}{2}\left|\tfrac{x_1}{4} - x_2\right| \ \left(\geq \tfrac{1}{2}(|x_1 - x_2|\right),$$
$$S\big((-1, x_1), (-1, a_2), (-1, a_1)\big) = \tfrac{1}{2}\left|x_1 - \tfrac{x_2}{4}\right| \ \left(\leq \tfrac{1}{2}(|x_1 - x_2|\right).$$

Consider

$$\begin{aligned} &k \cdot \frac{S\big((-1, x_1), (-1, x_1), (-1, x_2)\big) S\big((-1, x_2), (-1, a_1), (-1, a_1)\big)}{S\big((-1, x_1), (-1, x_1), (-1, x_2)\big) + 2S\big((-1, x_1), (-1, a_2), (-1, a_1)\big)} \\ &= k \cdot \frac{\frac{1}{2}\left|x_1 - x_2\right| \cdot \frac{1}{2}\left|\frac{x_1}{4} - x_2\right|}{\frac{1}{2}\left|x_1 - x_2\right| + \left|x_1 - \frac{x_2}{4}\right|} \\ &\geq k \cdot \frac{\frac{1}{4}\left|x_1 - x_2\right|^2}{\frac{3}{2}\left|x_1 - x_2\right|} = \frac{2}{15}\left|x_1 - x_2\right| \\ &\geq \frac{1}{8}\left|x_1 - x_2\right| = S\big((-1, a_1), (-1, a_1), (-1, a_2)\big). \end{aligned}$$

<u>Case II: $x_1, x_2 \notin \left[-\frac{1}{2}, 0\right]$ and $x_1 > x_2$</u> We have,

$$S\big((-1, a_1), (-1, a_1), (-1, a_2)\big) = 0,$$

$$k := \varphi\Big(S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)\Big) = \tfrac{4}{5},$$
$$S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) = \tfrac{1}{2}\left|x_1 - x_2\right|,$$
$$S\big((-1,x_2),(-1,a_1),(-1,a_1)\big) = \tfrac{1}{2}\left|x_2\right|,$$
$$S\big((-1,x_1),(-1,a_2),(-1,a_1)\big) = \tfrac{1}{2}\left|x_1\right|.$$

Consider

$$k \cdot \frac{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)S\big((-1,x_2),(-1,a_1),(-1,a_1)\big)}{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) + 2S\big((-1,x_1),(-1,a_2),(-1,a_1)\big)}$$
$$= k \cdot \frac{\frac{1}{2}\left|x_1 - x_2\right| \cdot \frac{1}{2}\left|x_2\right|}{\frac{1}{2}\left|x_1 - x_2\right| + \left|x_1\right|} \geq 0 = S\big((-1,a_1),(-1,a_1),(-1,a_2)\big)$$

Case III: $x_2 \in \left[-\frac{1}{2}, 0\right]$ and $x_1 \notin \left[-\frac{1}{2}, 0\right]$ We have,

$$S\big((-1,a_1),(-1,a_1),(-1,a_2)\big) = \tfrac{1}{2}\left|\tfrac{x_2}{4}\right| = \tfrac{1}{8}\left|x_2\right|,$$
$$k := \varphi\Big(S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)\Big) = \tfrac{4}{5},$$
$$S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) = \tfrac{1}{2}\left|x_1 - x_2\right|,$$
$$S\big((-1,x_2),(-1,a_1),(-1,a_1)\big) = \tfrac{1}{2}\left|x_2\right| \ \big(\leq \tfrac{1}{2}(\left|x_1 - x_2\right|\big),$$
$$S\big((-1,x_1),(-1,a_2),(-1,a_1)\big) = \tfrac{1}{2}\left|\tfrac{x_2}{4}\right| = \ \tfrac{1}{8}\left|x_2\right|.$$

Consider

$$k \cdot \frac{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)S\big((-1,x_2),(-1,a_1),(-1,a_1)\big)}{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) + 2S\big((-1,x_1),(-1,a_1),(-1,a_1)\big)}$$
$$= k \cdot \frac{\frac{1}{2}\left|x_1 - x_2\right| \cdot \frac{1}{2}\left|x_2\right|}{\frac{1}{2}\left|x_1 - x_2\right| + \frac{1}{4}\left|x_2\right|}$$
$$\geq k \cdot \frac{\frac{1}{4}\left|x_2\right|}{\frac{3}{2}\left|x_1 - x_2\right|}$$
$$\geq \big(\tfrac{4}{5}\left|x_1 - x_2\right|\big) \frac{\frac{1}{4}\left|x_2\right|}{\frac{3}{2}\left|x_1 - x_2\right|} = \tfrac{2}{15}\left|x_2\right|$$
$$\geq \tfrac{1}{8}\left|x_2\right| = S\big((-1,a_1),(-1,a_1),(-1,a_2)\big).$$

From all the above cases, we conclude that f is a generalized S-$\mathcal{MT}$-proximal cyclic contractions. Similarly, we can show that $\hat{f}$ is a generalized S-$\mathcal{MT}$-proximal cyclic contractions too. Next, we show that the pair $(f, \hat{f})$ is a S-$\mathcal{MT}$-proximal cyclic contraction. Let $(-1,u),(-1,x) \in A$ and $(1,v),(1,y) \in B$ be such that

$$d_S\big((-1,u), f(-1,x)\big) = d_S(A,B) = 2, \ d_S\big((1,v), \hat{f}(1,y)\big) = d_S(A,B) = 2.$$

Then we get

$$u = \begin{cases} \frac{x}{4}; & \text{if } x \in \left[-\frac{1}{2}, 0\right] \\ 0; & \text{otherwise} \end{cases}, \ v = \begin{cases} \frac{y}{4}; & \text{if } y \in \left[-\frac{1}{2}, 0\right] \\ 0; & \text{otherwise} \end{cases}.$$

Case I: $x, y \in \left[-\frac{1}{2}, 0\right]$ and $x > y$ We have $d_S\big((-1,u),(1,v)\big) = \frac{1}{4}|x - y|$

Case II: $x, y \notin \left[-\frac{1}{2}, 0\right]$ and $x > y$ We have, $d_S\big((-1,u),(1,v)\big) = 0$

Case III: $y \in \left[-\frac{1}{2}, 0\right]$ and $x \notin \left[-\frac{1}{2}, 0\right]$ We have, $d_S\big((-1,u),(1,v)\big) = \frac{1}{4}|y|$

From all the above cases, we get

$$\begin{aligned} d_S\big((-1,u),(1,v)\big) &\leq \tfrac{4}{5}\,|x-y| + 2\left(\tfrac{1}{5}\right) \\ &= k d_S\big((-1,x),(1,y)\big) + (1-k)\, d_S\big(A,B\big) \end{aligned}$$

where $k = \varphi\big(d_S((-1,x),(1,y))\big)$. Hence the pair $(f,\hat{f})$ is a S-$\mathcal{MT}$-proximal cyclic contraction. Therefore, all the hypotheses of Theorem 3.6 are satisfied. Thus, $(-1,0) \in A$ and $(1,0) \in B$ are elements such that

$$d_S\big(g(1,0), f(1,0)\big) = d_S\big(g(-1,0), \hat{f}(-1,0)\big) = d_S\big((-1,0),(-1,0)\big) = d_S(A,B).$$

If g is the identity mapping in Theorem 3.6, we obtain the following best proximity point result.

Theorem 3.8. *Let A, B be two nonempty subsets of a S-metric space (X,S) such that $A_0, B_0 \neq \varnothing$. Let $f : A \to B$, $\hat{f} : B \to A$ satisfy the following conditions:*

(i) *f and $\hat{f}$ are generalized S-$\mathcal{MT}$-proximal cyclic contractions of the first kind;*
(ii) *the pair $(f,\hat{f})$ is generalized S-$\mathcal{MT}$-proximal cyclic contraction;*
(iii) *$f(A_0) \subseteq B_0$, $\hat{f}(B_0) \subseteq A_0$;*

Then there exists a point $x \in A$ and there exists a point $y \in B$ such that

$$d_S(x,fx) = d_S(y,\hat{f}y) = d_S(x,y) = d_S(A,B).$$

Moreover, for any best proximity point $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d_S(x_{n+1}, fx_n) = d_S(A,B)$$

converges to the element x.
Similarly, for any best proximity point $y_0 \in B_0$, the sequence $\{y_n\}$ defined by

$$d_S(y_{n+1}, \hat{f}y_n) = d_S(A,B)$$

converges to the element y.

The following is the best proximity point theorem for non self-mappings which are generalized S-$\mathcal{MT}$-proximal cyclic contractions of the first kind and second kind:

Theorem 3.9. *Let A, B be two nonempty subsets of a S-metric space (X,S) such that $A_0, B_0 \neq \varnothing$. Let $f : A \to B$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(i) *f is a generalized S-$\mathcal{MT}$-proximal cyclic contraction of the first kind and second kind;*
(ii) *g is an isometry;*
(iii) *$f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.*

Then there exists a point $x \in A$ such that

$$d_S(gx, fx) = d_S(A,B).$$

Moreover, for any best proximity point $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d_S(gx_{n+1}, fx_n) = d_S(A,B)$$

converges to the element x.

Proof. Proceeding as in Theorem 3.6, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d_S(gx_{n+1}, fx_n) \ = \ d_S(A,B)\ , \quad \forall n \in \mathbb{N}\cup\{0\}. \tag{3.18}$$

Since f is an S-$\mathcal{MT}$-proximal cyclic contraction of the first kind, we have

$$S(x_n, x_n, x_{n+1}) \leq \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big)S(x_{n-1}, x_{n-1}, x_n) \tag{3.19}$$

for all $n \geq 1$. Again, similarly, we can show that the sequence $\{x_n\}$ is a Cauchy sequence and so it converges to some $x \in A$.

Since f is S-$\mathcal{MT}$-proximal cyclic contraction of the second kind,

$$\begin{aligned} S(fx_n, fx_n, fx_{n+1}) &\leq \varphi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big)S(fx_{n-1}, fx_{n-1}, fx_n) \\ &\leq S(fx_{n-1}, fx_{n-1}, fx_n). \end{aligned} \tag{3.20}$$

This shows that $\{S(fx_n, fx_n, fx_{n+1})\}$ is a decreasing sequence and bounded below. Hence there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} S(fx_n, fx_n, fx_{n+1}) = r$. Suppose that $r > 0$. From (3.20) observe that

$$\frac{S(fx_n, fx_n, fx_{n+1})}{S(fx_{n-1}, fx_{n-1}, fx_n)} \ \leq \ \varphi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big).$$

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} \varphi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big) \ = \ 1$$

which is a contradiction to φ is an $\mathcal{MT}$-function so $r = 0$, then we get,

$$\lim_{n\to\infty} S(fx_n, fx_n, fx_{n+1}) = 0.$$

similarly, in the proof of Theorem 3.6, we can show that $\{fx_n\}$ is a Cauchy sequence and converges to some element $y \in B$. Therefore, we can conclude that $d_S(gx, y) = \lim\limits_{n\to\infty} d_S(gx_{n+1}, fx_n) = d_S(A,B)$, which implies that $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $S(gx, gx, gz) = 0$. By the fact that g is an isometry, we have $S(x, x, z) = S(gx, gx, gz) = 0$. Hence $x = z$ and so $x \in A_0$. Since $f(A_0) \subseteq B_0$, there exists $gx \in A$ such that

$$d_S(gx, fx) = d_S(A,B).$$ ■

Corollary 3.10. *Let A, B be two nonempty subsets of a S-metric space (X, S) such that $A_0, B_0 \neq \varnothing$. Let $f : A \to B$ satisfy the following conditions:*

(i) *f is a generalized S-$\mathcal{MT}$-proximal cyclic contraction of the first kind and second kind;*
(ii) *$f(A_0) \subseteq B_0$.*

Then there exists a point $x \in A$ such that

$$d_S(x, fx) = d_S(A,B).$$

Moreover, for any best proximity point $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d_S(x_{n+1}, fx_n) = d_S(A,B)$$

converges to the element x.

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