



An Asymptotic Behavior and a Posteriori Error Estimates for the Generalized Overlapping Domain Decomposition Method for Diffusion Equation with Robin Boundary Condition

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Abstract In this paper, a posteriori error estimates for the generalized overlapping domain decomposition method with respect to the mixed boundary condition for diffusion equation is proved using the finite difference time scheme combined with finite element spatial approximation. In addition, a result of asymptotic behavior in Sobolev norm is given via Benssoussan's algorithm.

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1. INTRODUCTION

Schwarz method in general case can be used to solve the different some boundary value problems in domains resolving from two or more overlapping subdomains (see [1–5]). It was discovered by Hermann Amandus Schwartz in 1890.

The solution to the qualitative problem can be approximated by an infinite sequence of functions that result from solving a series of evolutionary boundary value problems in each subdomain. An extensive analysis of Schwartz's alternative method for nonlinear boundary value problems has been studied extensively over the past three decades (see [6–9]). In addition, for stationary, the a priori estimate case is given in several works, see for instance [2] which we give the weak formulation of the classical Schwarz method. In [9], we studied the geometry convergence are given. Also, in [7], Convergence of circular geometric has been done. These results can be found in recent books on field analysis

methods [10, 11]. Recently, in [12, 13] Schwarz method for highly heterogeneous media has been improved and studied.

Very few works have been studied on standardized standard error analysis of overlapping mismatched network methods of fixed problems in many works for example in [11–14]. The main purpose of this paper, we'll proceed as follows [12]. Precisely, We improve an approach that combines the result of the geometric convergence produced by [4, 8, 15] and the lemma which consists of estimating the standard error between continuous and discrete Schwarz iterations. Then the optimal order of their convergence was proven using Galerkin's standard method and error estimation on the unified rule of linear elliptic equations [2].

Very recent, in [16], the authors presented the criterion-maximum error analysis for a class of nonlinear elliptic problems in the context of overlapping mismatched networks and studied optimal error estimation on a uniform base between discrete Schwarz sequences and the exact solution of partial differential equations, and in [17]. The authors extracted post-error estimates for a generalized nested domain analysis method with Dirichlet boundary conditions on interfaces for Laplace boundary value problems, and showed that continuous-state error estimation is based on differences in traces of subband solutions on interfaces using the finite element method.

In this work, we are going to prove a posteriori error estimates for the following parabolic equation: find $u \in L^2(0, T; H_0^1(\Omega)) \cap C^2(0, T, H^{-1}(\Omega))$ solution of

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta u + a_0 u = f, \quad \text{in } \Sigma, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u(., 0) = u_0, \quad \text{in } \Omega, \end{array} \right. \quad (1.1)$$

where Σ is a set in $R^2 \times R$ defined as $\Sigma = \Omega \times [0, T]$ with $T < +\infty$, where Ω is a smooth bounded domain of R^2 with boundary Γ .

The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative verifies

$$a_0 \leq \beta, \quad \beta > 0. \quad (1.2)$$

f is a regular function satisfies

$$f \in L^2(0, T, L^2(\Omega)) \cap C^1(0, T, H^{-1}(\Omega)).$$

The symbol $(., .)_\Omega$ stands for the inner product in $L^2(\Omega)$.

The outline of the paper is as follows: In Section 2, we introduce some necessary notations, then we prove a weak formulation of the presented problem. In Section 3, a posteriori error estimate is proposed for the convergence of the discretized solution using theta time scheme combined with Galerkin method on subdomains.

2. THE CONTINUOUS PROBLEM

The problem (1.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2(0, T, H_0^1(\Omega))$ solution of

$$\begin{cases} \left(\frac{\partial u}{\partial t}, v \right) + a(u, v) = (f, v) + (\varphi, v)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u^i(x, 0) = u_0^i \text{ in } \Omega, \end{cases}$$

where $a(\cdot, \cdot)$ is the bilinear form defined as:

$$u, v \in H_0^1(\Omega) : a(u, v) = (\nabla u, \nabla v) - (a_0 u, v)$$

and

$$a_0 \in L^2(0, T, L^\infty(\Omega)) \cap C^0(0, T, H^{-1}(\Omega))$$

is sufficiently smooth functions and satisfy the following condition: $a_0(t, x) \geq \beta > 0$, β is a constant.

Let $(\cdot, \cdot)_\Omega$ be the scalar product in $L^2(\Omega)$ and $(\cdot, \cdot)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined as:

$$\Gamma_0 = \{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi > 0, x + \xi \notin \bar{\Omega}\}.$$

In [9], we have treated the overlapping domain decomposition method combined with a finite element approximation for elliptic equation related for Laplace operator Δ , where a Sobolev norm analysis of an overlapping Schwarz method on nonmatching grids has been used, where we proved that the discretization on every subdomain converges in Sobolev norm. Furthermore, a result of asymptotic behavior in uniform norm has been given.

In this paper, we extend the last work for parabolic equation with mixed boundary conditions where we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary conditions on the boundaries for the discrete solutions on subdomains using theta time scheme combined with a finite element spatial approximation, similar to that in [16], which investigated full elliptic operator with Dirichlet boundary condition.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, definitions and fundamental published propositions in the proposed problem then we give the variational formulation of our model. In section 3 and 4, a posteriori error estimate for both continuous and discrete cases are proposed for the convergence of the discrete solution using theta time scheme combined with a finite element method on subdomains.

3. THE DISCRETE PARABOLIC EQUATION

3.1. THE SPACE DISCRETIZATION

Let Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, \dots, m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$

where M_j is a vertex of the considered triangulation. We introduce the following discrete spaces V_h of finite element

$$V^h = \left\{ \begin{array}{l} v \in (L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\bar{\Omega}))) \\ \text{such that } v_h|_K = P_1, k \in \tau_h, \\ v_h(\cdot, 0) = v_{h0} \text{ (initial data) in } \Omega, \frac{\partial v_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, v_h = 0 \text{ in } \Gamma \setminus \Gamma_0, \end{array} \right\} \tag{3.1}$$

where P_1 Lagrangian polynomial of degree less than or equal to 1.

We consider r_h be the usual interpolation operator defined by $r_h v = \sum_{i=1}^{m(h)} v(M_i) \varphi_i(x)$.

The discrete maximum principle assumption (dmp) [15]: We assume the matrices whose coefficients $a(\varphi_i, \varphi_j)$ are M -matrix. For convenience in all the sequels, C will be a generic constant independent on h .

It can be approximated the problem (1.1) by a weakly coupled system of the following parabolic equation $v \in H^1(\Omega)$

$$\left(\frac{\partial u}{\partial t}, v \right)_{\Omega} + a(u, v) = (f, v)_{\Omega} + (\varphi, v)_{\Gamma_0}. \tag{3.2}$$

We discretize in space, i.e., we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset (L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\bar{\Omega})))$, we get the following semi-discrete system of parabolic equation

$$\left(\frac{\partial u_h}{\partial t}, v_h \right)_{\Omega} + a(u_h, v_h) = (f, v_h - u_h)_{\Omega} + (\varphi, v_h)_{\Gamma_0}. \tag{3.3}$$

3.2. THE TIME DISCRETIZATION

Now we apply the θ -scheme in the semi-discrete approximation (3.3). Thus we have, for any $\theta \in [0, 1]$ and $k = 1, \dots, p$

$$\begin{aligned} & (u_h^k - u_h^{k-1}, v_h)_{\Omega} + (\Delta t) a(u_h^{\theta, k}, v_h) \\ & = (\Delta t) \left[(f^{i, \theta, k}, v_h - u_h^{i, \theta, k})_{\Omega} + (\varphi^{i, \theta, k}, v_h - u_h^{i, \theta, k})_{\Gamma_0} \right], \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} u_h^{\theta, k} &= \theta u_h^k + (1 - \theta) u_h^{k-1} \\ f^{\theta, k} &= \theta f^k + (1 - \theta) f^{k-1} \end{aligned} \tag{3.5}$$

and

$$\varphi^{\theta, k} = \theta \varphi^k + (1 - \theta) \varphi^{k-1}. \tag{3.6}$$

By multiplying and dividing by θ and by adding $\left(\frac{u_h^{k-1}}{\theta\Delta t}, v_h - u_h^{\theta,k}\right)$ to both parties of the inequalities (3.4), we get

$$\left(\frac{u_h^{\theta,k}}{\theta\Delta t}, v_h - u_h^{\theta,k}\right)_\Omega + a(u_h^{\theta,k}, v_h) = \left(f^{\theta,k} + \frac{u_h^{\theta,k-1}}{\theta\Delta t}, v_h\right)_\Omega + (\varphi^{\theta,k}, v_h)_{\Gamma_0}, \quad v_h \in V_h. \tag{3.7}$$

Then, the problem (3.7) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$b(u_h^{\theta,k}, v_h) = (f^{i,\theta,k} + \mu u_h^{k-1}, v_h)_\Omega + (\varphi^{\theta,k}, v_h)_{\Gamma_0}, \quad v_h, u_h^{\theta,k} \in V_h, \tag{3.8}$$

where

$$\begin{cases} b^i(u_h^{i,\theta,k}, v_h) = \mu(u_h^{i,\theta,k}, v_h)_\Omega + a(u_h^{i,\theta,k}, v_h), \quad v_h \in V_h^i, \\ \mu = \frac{1}{\theta\Delta t} = \frac{p}{\theta T}. \end{cases} \tag{3.9}$$

3.3. THE SPACE CONTINUOUS FOR THE GENERALIZED SCHWARZ METHOD

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \Omega_{12}$, $\partial\Omega_s \cap \Omega_t = \Gamma_s$, $s \neq t$ and $s, t = 1, 2$. We need the spaces

$$V_s = H^1(\Omega) \cap H^1(\Omega_s) = \{v \in H^1(\Omega_i) : v_{\partial\Omega_i \cap \partial\Omega} = 0\}$$

and

$$W_s = H_0^{U^*(F)bd}(\Gamma_s) = \{v_{\Gamma_s}, v \in V_s \text{ and } v = 0 \text{ on } \partial\Omega_s \setminus \Gamma_s\},$$

which is a subspace of $H^{U^*(F)bd}(\Gamma_s) = \{\psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s} \text{ for } \varphi \in V_s, s = 1, 2\}$, with its norm $\|\varphi\|_{W_s} = \inf_{v \in V_s, v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}$.

We define the continuous counterparts of the continuous Schwarz sequences defined in (3.9), respectively by $u_1^{k,m+1} \in H_0^1(\Omega)$, $m = 0, 1, 2, \dots, i = 1, \dots, M$ solution of

$$\begin{cases} c(u_1^{\theta,k,m+1}, v) = \left(F^\theta(u_1^{\theta,k-1,m+1}), v\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0}, \\ u_1^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\ \frac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1} + \alpha_1 u_1^{\theta,k,m+1} = \frac{\partial u_2^{\theta,k,m}}{\partial\eta_1} + \alpha_1 u_1^{\theta,k,m} \text{ on } \Gamma_1 \end{cases} \tag{3.10}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s = 1, 2$.

In the next section, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplace operator with the new boundary conditions

of generalized Schwarz alternating method, we get

$$\begin{aligned} \left(-\Delta u_1^{k,m+1}, v_1\right)_{\Omega_1} &= \left(\nabla u_1^{\theta,k,m+1}, \nabla(v_1)\right)_{\Omega_1} - \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\partial \Omega_1 - \Gamma_1} \\ &\quad + \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla(v_1)\right)_{\Omega_1} - \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1} \end{aligned}$$

thus we can deduce

$$\begin{aligned} \left(-\Delta u_1^{\theta,k,m+1}, v_1\right)_{\Omega_1} &= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} - \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\partial \Omega_1 - \Gamma_1} \\ &\quad + \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1}, v_1\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} \\ &\quad - \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_2} + \alpha_1 u_2^{\theta,k,m} - \alpha_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\alpha_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla v_1\right)_{\Omega_1} + \left(\alpha_1 u_1^{\theta,k,m+1}, v_1\right)_{\Gamma_1} \\ &\quad - \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, v_1\right)_{\Gamma_1}, \end{aligned}$$

thus the problem (3.10) equivalent to; find $u_1^{\theta,k,m+1} \in V_1$ such that

$$\begin{aligned} c(u_1^{\theta,k,m+1}, v_1) + \left(\alpha_1 u_1^{\theta,k,m}, v_1\right)_{\Gamma_1} &= \left(F^\theta(u_1^{\theta,k-1,m+1}), v_1\right)_{\Omega_1} + (\varphi, v)_{\Gamma_0} \\ &\quad + \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, v_1\right)_{\Gamma_1}, \forall v_1 \in V_1 \end{aligned} \tag{3.11}$$

and we have $u_2^{\theta,k,m+1} \in V_2$

$$\begin{aligned} c(u_2^{\theta,k,m+1}, v_2) + \left(\alpha_2 u_2^{\theta,k,m+1}, v_2\right)_{\Gamma_2} &= \left(F(u_2^{\theta,k-1,m+1}), v_2\right)_{\Omega_2} + (\varphi, v)_{\Gamma_0} \\ &\quad + \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_2} + \alpha_2 u_1^{\theta,k,m}, v_2\right)_{\Gamma_2}. \end{aligned} \tag{3.12}$$

4. A POSTERIORI ERROR ESTIMATE IN CONTINUOUS CASE

We define these auxiliary problems by of (3.10) with another problem in a nonoverlapping way over Ω . These auxiliary problems are needed for analysis and not for the computation section.

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \emptyset \quad \Omega = \Omega_2 \cup \Omega_4, \text{ and } \Omega_2 \cap \Omega_4 = \emptyset.$$

Let $(u_1^{k,m}, u_2^{k,m})$ be the solution of problem (3.10), we define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution of the following nonoverlapping problems

$$\left\{ \begin{array}{l} \frac{u_1^{k,m+1} - u_1^{k-1,m+1}}{\Delta t} - \Delta u_1^{\theta,k,m+1} + a_0^k u_1^{\theta,k,m+1} = F^\theta \left(u_1^{\theta,k-1,m+1} \right) \text{ in } \Omega_1, \\ u_1^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_1 \cap \partial\Omega, \quad k = 1, \dots, n, \\ \frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1} + \alpha u_1^{i,\theta,k,m} = \frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha u_2^{\theta,k,m}, \text{ on } \Gamma_1 \end{array} \right. \tag{4.1}$$

and

$$\left\{ \begin{array}{l} \frac{u_3^{k,m+1} - u_3^{k-1,m+1}}{\Delta t} - \Delta u_3^{\theta,k,m+1} + a_0^k u_3^{\theta,k,m+1} = F^\theta \left(u_3^{\theta,k-1,m+1} \right) \text{ in } \Omega_3, \\ u_3^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_3 \cap \partial\Omega, \\ \frac{\partial u_3^{\theta,k,m+1}}{\partial \eta_3} + \alpha u_3^{\theta,k,m} \text{ on } \Gamma_2 = \frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_3} + \alpha u_1^{\theta,k,m}, \text{ on } \Gamma_1. \end{array} \right. \tag{4.2}$$

It can be taken $\epsilon_1^{\theta,k,m} = u_2^{\theta,k,m+1} - u_3^{\theta,k,m+1}$ on Γ_1 , the difference between the overlapping and the nonoverlapping solutions $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ of the problem (3.10) and (resp., (4.1) and (4.2)) in Ω_3 . Because both overlapping and the nonoverlapping problems converge see [16] that is, $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ tend to $u_3^{\theta,k}$ (resp. $u_3^{\theta,k}$), then $\epsilon_1^{\theta,k,m}$ should tend to naught when m tends to infinity in V_2 .

By taking

$$\begin{aligned} \Lambda_3^{k,m} &= \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} + \alpha u_2^{\theta,k,m}, \quad \Lambda_1^{k,m} = \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha u_1^{\theta,k,m}, \\ \Lambda_3^{k,m} &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha u_3^{\theta,k,m} + \frac{\partial \epsilon_1^{\theta,k,m}}{\partial \eta_1} + \alpha \epsilon_1^{\theta,k,m}, \\ \Lambda_1^{k,m} &= \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha u_1^{\theta,k,m}. \end{aligned} \tag{4.3}$$

Using Green formula, (4.1) and (4.2) can be reformulated to the following system of elliptic variational equations

$$c(u_1^{\theta,k,m+1}, v_1) + \left(\alpha_1 u_1^{\theta,k,m}, v_1 \right)_{\Gamma_1} = \left(F^\theta(u_1^{\theta,k-1,m+1}), v_1 \right)_{\Omega_1} + (\varphi, v)_{\Gamma_0} + \left(\Lambda_3^{k,m}, v_1 \right)_{\Gamma_1}, \forall v_1 \in V_1 \quad (4.4)$$

and

$$c(u_3^{\theta,k,m+1}, v_3) + \left(\alpha_3 u_3^{\theta,k,m+1}, v_3 \right)_{\Gamma_1} = \left(F^\theta(u_3^{\theta,k-1,m+1}), v_3 \right)_{\Omega_3} + (\varphi, v)_{\Gamma_0} + \left(\Lambda_1^{k,m}, v_3 - u_3^{\theta,k,m+1} \right)_{\Gamma_1}, \forall v_3 \in V_3. \quad (4.5)$$

On the other hand by taking

$$\theta_1^{k,m} = \frac{\partial \epsilon_1^{\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{\theta,k,m}, \quad (4.6)$$

we get

$$\begin{aligned} \Lambda_3^{\theta,k,m} &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \frac{\partial(u_2^{\theta,k,m} - u_3^{\theta,k,m})}{\partial \eta_1} + \alpha_1(u_2^{\theta,k,m} - u_3^{\theta,k,m}) \\ &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \frac{\partial \epsilon_1^{k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{k,m} \\ &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m}. \end{aligned} \quad (4.7)$$

Using (4.6) we have

$$\begin{aligned} \Lambda_3^{k,m+1} &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\ &= -\frac{\partial u_3^{\theta,k,m}}{\partial \eta_3} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\ &= \alpha_3 u_3^{\theta,k,m} - \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} - \alpha_3 u_1^{\theta,k,m} + \alpha_1 u_3^{\theta,k,m} + \theta_1^{k,m+1} \\ &= (\alpha_1 + \alpha_3) u_3^{\theta,k,m} - \Lambda_1^{k,m} + \theta_1^{k,m+1} \end{aligned} \quad (4.8)$$

and the last equation in (4.8), we have

$$\begin{aligned} \Lambda_1^{k,m+1} &= -\frac{\partial u_1^{\theta,k,m}}{\partial \eta_1} + \alpha_3 u_1^{\theta,k,m} \\ &= \alpha_1 u_1^{\theta,k,m} - \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} - \alpha_1 u_2^{\theta,k,m} + \alpha_3 u_1^{\theta,k,m} + \alpha_3 u_1^{\theta,k,m} \\ &= (\alpha_1 + \alpha_3) u_1^{\theta,k,m} - \Lambda_3^{k,m} + \theta_3^{k,m+1}. \end{aligned} \quad (4.9)$$

Lemma 4.1. Let $u_s^k = u_{\Omega_s}^k$, $e_s^{\theta,k,m+1} = u_s^{\theta,k,m+1} - u_s^k$ and $\eta_s^{k,m+1} = \Lambda_s^{k,m+1} - \Lambda_s^k$. Then for $s, t = 1, 3, s \neq t$, we have

$$\begin{aligned} & c_s(e_s^{i\theta,k,m+1}, v_s - e_s^{\theta,k,m+1}) + (\alpha_s e_s^{\theta,k,m+1}, v_s - e_s^{k,m+1})_{\Gamma_s} \\ &= \left(\eta_t^{k,m}, v_s - e_s^{i,k,m+1} \right)_{\Gamma_s}, \forall v_s \in V_s \end{aligned} \tag{4.10}$$

and

$$\left(\eta_s^{k,m+1}, \psi^i \right)_{\Gamma_s} = ((\alpha_s + \alpha_t) e_s^{k,m+1}, v_s)_{\Gamma_s} - \left(\eta_t^{k,m}, \psi \right)_{\Gamma_s} + \left(\theta_t^{k,m+1}, \psi \right)_{\Gamma_s}, \forall \psi \in V_s. \tag{4.11}$$

Proof. The proof is very similar to that in [9]. ■

Lemma 4.2. By letting C be a generic constant which has different values at different places, we get for $s, t = 1, 3, s \neq t$

$$\left(\eta_s^{k,m-1} - \alpha_s e_s^{k,m}, w \right)_{\Gamma_1} \leq C \|e_s^{k,m}\|_{1,\Omega_s} \|w\|_{W_1} \tag{4.12}$$

and

$$\left(\alpha_s w_s + \theta_1^{k,m+1}, e_s^{k,m+1} \right)_{\Gamma_1} \leq C \|e_s^{k,m+1}\|_{1,\Omega_s} \|w\|_{W_1}. \tag{4.13}$$

where C is a constant independent of h and k .

Proof. The proof is very similar to that in [9]. ■

Proposition 4.3. [9] For the sequences $(u_1^{\theta,k,m+1}, u_3^{\theta,k,m+1})_{m \in U^*(F)_{2115}}$ solutions of (4.1) and (4.2) we have the following a posteriori error estimation

$$\left\| u_1^{\theta,k,m+1} - u_1^k \right\|_{1,\Omega_1} + \left\| u_3^{\theta,k,m+1} - u_3^k \right\|_{3,\Omega_3} \leq C \left\| u_1^{\theta,k,m+1} - u_3^{k,m} \right\|_{W_1}.$$

where C is a constant independent of h and k .

Proposition 4.4. For the sequences $(u_2^{\theta,k,m+1}, u_4^{\theta,k,m+1})_{m \in U^*(F)_{2115}}$. We get the the similar following a posteriori error estimation

$$\left\| u_2^{\theta,k,m+1} - u_2^{i,\theta,k} \right\|_{2,\Omega_2} + \left\| u_4^{\theta,k,m+1} - u_4^{\theta,k} \right\|_{4,\Omega_4} \leq C \left\| u_2^{\theta,k,m+1} - u_4^{\theta,k,m+1} \right\|_{W_2}. \tag{4.14}$$

where C is a constant independent of h and k .

Proof. The proof is very similar to proof of Proposition 4.3 which proved in our published paper on [9]. ■

Theorem 4.5. [9] Let $u_s^{\theta,k} = u_{\Omega_s}^{\theta,k}$, $s = 1, 2$. For the sequences $(u_1^{\theta,k,m+1}, u_2^{\theta,k,m+1})_{m \in U^*(F)_{2115}}$ solutions of problems (3.11) and (3.12), one have the following result

$$\begin{aligned} & \left\| u_1^{\theta,k,m+1} - u_1^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_2^{\theta,k,m} - u_2^{\theta,k} \right\|_{2,\Omega_2} \\ & \leq C \left(\left\| u_1^{\theta,k,m+1} - u_2^{\theta,k,m} \right\|_{W_1} + \left\| u_1^{\theta,k,m} - u_{i_1}^{\theta,k,m+1} \right\|_{W_2} + \left\| e_1^{k,m} \right\|_{W_1} + \left\| e_2^{k,m+1} \right\|_{W_2} \right), \end{aligned}$$

where C is a constant independent of h and k .

5. A POSTERIORI ERROR ESTIMATE: DISCRETE CASE

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_s $s = \{1, \dots, m(h)\}$ defined by $\varphi_l(M_j) = \delta_{lj}$, where M_j is a vertex of the considered triangulation.

In the first step, we approach the space H_0^1 by a suitable discretization space of finite dimensional $V^h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we search a sequence of elements $u_h^{\theta,n} \in V^h$ which approaches $u_h(t_n, \cdot)$, $t_n = n\Delta t$, $k = 1, \dots, n$, with initial data $u_h^0 = u_{0h}$.

Let $u_h^{\theta,k,m+1} \in V^h$ be the solution of the discrete problem associated with (3.10), $u_{s,h}^{\theta,k,m+1} = u_{h,\Omega_s}^{\theta,k,m+1}$.

We construct the sequences $(u_{s,h}^{\theta,k,m+1})_{m \in U^*(F)_{2115}}, u_{s,h}^{\theta,k,m+1} \in V_s^h, (s = 1, 2)$ solutions of discrete problems associated with (4.4).

We define the discrete space K_h is a suitable set given by

$$K_h = \left\{ \begin{array}{l} u_h \in (L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\Omega))), \\ u_h = 0 \text{ in } \Gamma, \quad \frac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \quad u_h = 0 \text{ in } \Gamma \setminus \Gamma_0, \end{array} \right.$$

where r_h is the usual interpolation operator defined by $r_h v = \sum_{i=1}^{m(h)} v(M_j) \varphi_i(x)$.

In similar manner to that of the previous section, we introduce two auxiliary problems, we define for (Ω_1, Ω_3) the following full-discrete problems: find $u_{1,h}^{\theta,k,m+1} \in K_h$ solution of

$$\left\{ \begin{array}{l} c(u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h}) + (\alpha_{1,h} u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h})_{\Gamma_1}, \\ = (F^\theta(u_{1,h}^{\theta,k-1,m+1}), \tilde{v}_{1,h})_{\Omega_1} + (\varphi, v)_{\Gamma_0}, \\ u_{1,h}^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_1 \cap \partial\Omega, \quad \tilde{v}_{1,h} \in K_h \\ \frac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{\theta,k,m+1} = \frac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0, \end{array} \right. \tag{5.1}$$

by taking the trial function $\tilde{v}_{1,h} = v_{1,h} - u_{1,h}^{i,\theta,k,m+1}$ in (5.1), we get

$$\left\{ \begin{array}{l} c(u_{1,h}^{\theta,k,m+1}, v_{1,h}) + (\alpha_{1,h} u_{1,h}^{\theta,k,m+1}, v_{1,h})_{\Gamma_1} = (F(u_{1,h}^{\theta,k-1,m+1}), v_{1,h})_{\Omega_1} + (\varphi, v_{1,h})_{\Gamma_0}, \\ u_{1,h}^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_1 \cap \partial\Omega, \quad v_{1,h} \in K_h, \\ \frac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{\theta,k,m+1} = \frac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0. \end{array} \right. \tag{5.2}$$

Similarly, we get

$$\left\{ \begin{array}{l} c(u_{3,h}^{\theta,k,m+1}, v_{1,h}) + (\alpha_{3,h} u_{3,h}^{\theta,k,m+1}, v_{1,h})_{\Gamma_1} = (F^\theta(u_{3,h}^{\theta,k-1,m+1}), v_{1,h})_{\Omega_3} + (\varphi, v_{1,h})_{\Gamma_0}, \\ u_{3,h}^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_3 \cap \partial\Omega, \\ \frac{\partial u_{3,h}^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_{3,h}^{\theta,k,m+1} = \frac{\partial u_{1,h}^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_{1,h}^{\theta,k,m}, \text{ on } \Gamma_1 - \Gamma_0. \end{array} \right.$$

$$(5.3)$$

For (Ω_2, Ω_4) , are similar in (5.2) and (5.3).

Theorem 5.1. [12] *The solution of the system of parabolic equations (5.2) and (5.3) is the maximum element the set of discrete subsolutions.*

We can obtain the discrete counterparts of Propositions 4.3 and 4.4 by doing almost the same analysis as in section above (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

$$\left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{3,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k} \right\|_{1,\Omega_3} \leq C \left\| u_{1,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k,m} \right\|_{W_1} \quad (5.4)$$

and

$$\left\| u_{2,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} + \left\| u_{4,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k} \right\|_{1,\Omega_4} \leq C \left\| u_{2,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k,m} \right\|_{W_2}. \quad (5.5)$$

Similar to that in the proof of Theorem 4.5 we get the following discrete estimates

$$\begin{aligned} & \left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{\theta,k,m} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} \\ & \leq C \left(\left\| u_{1,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k,m} \right\|_{W_1} + \left\| u_{2,h}^{\theta,k,m} - u_{1,h}^{\theta,k,m} \right\|_{W_2} + \left\| e_{1,h}^{k+1,m} \right\|_{W_1} + \left\| e_{2,h}^{k+1,m} \right\|_{W_2} \right). \end{aligned}$$

Next we will obtain an error estimate between the approximated solution $u_{s,h}^{\theta,k,m+1}$ and the semi discrete solution in time $u^{i,\theta,k}$. We introduce some necessary notations. We denote by

$$\varepsilon_h = \{E \in T : T \in \tau_h \text{ and } E \notin \partial\Omega\}$$

and for every $T \in \tau_h$ and $E \in \varepsilon_h$, we define as

$$\omega_T = \{T' \in \tau_h : T' \cap T \neq \emptyset\},$$

and

$$\omega_E = \{T' \in \tau_h : T' \cap E \neq \emptyset\}.$$

The right hand side f is not necessarily continuous function across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E .

We have the following theorem which gives an a posteriori error estimate for the discrete GODDM.

6. AN ASYMPTOTIC BEHAVIOR FOR THE PROBLEM

Theorem 6.1. *Let $u_s^{\theta,k} = u^{\theta,k} |_{\Omega_s}$ where u is the solution of problem (1.1), the sequences $\left(u_{1,h}^{\theta,k,m+1}, u_{2,h}^{\theta,k,m} \right)_{m \in U^*(F)_{2115}}$ are solutions of the discrete problems (4.4) and (4.5). Then there exists a constant C independent of h such that*

$$\left\| u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{\theta,k,m} - u_{2,h}^{\theta,k} \right\|_{1,\Omega_2} \leq C \left\{ \sum_{i=1}^2 \sum_{T \in \tau_h} (\eta_i^T) + \eta_{\Gamma_s} \right\},$$

where

$$\eta_{\Gamma_s} = \left\| u_{h,s}^{\theta,k,*} - u_{h,t}^{i,\theta,k,*-1} \right\|_{W_{h,s}} + \left\| e_{i,h}^{\theta,k,*} \right\|_{W_{h,s}}$$

and

$$\eta_s^T = h_T \left\| \begin{array}{c} F(u_{h,s}^{\theta,k-1,*}) + u_{h,s}^{\theta,k-1} + \\ \Delta u_{h,s}^{\theta,k,*} - (1 + \lambda a_{h0}^k) u_{h,s}^{\theta,k} \end{array} \right\|_{0,T} + \sum_{E \in \epsilon_h} h_E^{U^*(F)bd} \left\| \left[\frac{\partial u_{h,s}^{\theta,k,*}}{\partial \eta_E} \right] \right\|_{0,E},$$

where C is a constant independent of h and k and the symbol $*$ is corresponds to $m + 1$ when $s = 1$ and to m when $s = 2$.

Proof. The proof is based on the technique of the residual a posteriori estimation see [16] and Theorem 5.1. We give the main steps by the triangle inequality we have

$$\sum_{s=1}^2 \left\| u_s^{\theta,k} - u_{h,s}^{\theta,k,*} \right\|_{1,\Omega_s} \leq \sum_{s=1}^2 \left\| u_s^{\theta,k} - u_{h,s}^{\theta,k} \right\|_{1,\Omega_s} + \sum_{s=1}^2 \left\| u_{h,s}^{\theta,k} - u_{s,h}^* \right\|_{1,\Omega_s}. \tag{6.1}$$

The second term on the right hand side of (6.1) is bounded by

$$\sum_{s=1}^2 \sum_{i=1}^2 \left\| u_{h,s}^{\theta,k} - u_{s,h}^* \right\|_{1,\Omega_s} \leq C \sum_{s=1}^2 \eta_{\Gamma_s}.$$

To bound the first term on the right hand side of (6.1) we use the residual equation and apply the technique of the residual a posteriori error estimation [16], to get for $v_h \in V^h$

$$\left\{ \begin{array}{l} c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) = c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s - v_{h,s}) \\ \leq \sum_{T \subset \Omega_s} \int \left(\begin{array}{c} F^\theta(u_{h,s}^{\theta,k-1}) + u_{h,s}^{\theta,k-1} + \mu \Delta u_{h,s}^{\theta,k} - \\ (1 + \mu a_{h0}^k) u_{h,s}^{\theta,k} \end{array} \right) (v_s - v_{h,s}) ds \\ - \sum_{E \subset \Omega_s} \int \left[\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_E} \right] (v_s - v_{h,s}) ds \\ - \sum_{E \subset \Gamma_s} \int \frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_E} (v_s - v_{h,s}) ds' \\ + \sum_{E \subset \Omega_s} \int \left(F^\theta(u_s^{\theta,k}) - F^\theta(u_{h,s}^{\theta,k}) \right) (v_s - v_{h,s}) d\sigma \\ + \left(\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_s}, v_s - v_{h,s} \right)_{\Gamma_s}, \end{array} \right.$$

where $F^\theta(u_{h,s}^{\theta,k})$ is any approximation of $F^\theta(u_s^{\theta,k})$. Therefore

$$\begin{aligned} & \sum_{s=1}^2 c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) \\ & \leq \sum_{s=1}^2 \sum_{T \subset \Omega_s} \left\| \begin{array}{c} F^\theta(u_{h,s}^{\theta,k}) + u_{h,s}^{\theta,k-1} + \mu \Delta u_{h,s}^{\theta,k} \\ - (1 + \mu a_{h0}^k) u_{h,s}^{\theta,k} \end{array} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T} \\ & + \sum_{s=1}^2 \sum_{E \subset \Omega_s} \left\| \left[\frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_E} \right] \right\|_{0,E} \|v_s - v_{h,s}\|_{0,E} + \sum_{s=1}^2 \sum_{E \subset \Gamma_s} \left\| \frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_E} \right\|_{0,E} \|v_s - v_{h,s}\|_{0,E} \\ & + \sum_{s=1}^2 \sum_{T \subset \Omega_s} c \left\| u_s^{\theta,k} - u_{h,s}^{\theta,k} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T} + \sum_{s=1}^2 \sum_{T \subset \Omega_s} \left\| \frac{\partial u_{h,s}^{\theta,k}}{\partial \eta_s} \right\|_{0,T} \|v_s - v_{h,s}\|_{0,T}, \end{aligned}$$

(6.2)

Using the following fact

$$\left\| u_s^{\theta,k} - u_{h,s}^{\theta,k} \right\|_{1,\Omega_s} \leq \sup_{v_s^i \in K} \frac{c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s + ch_s^T)}{\|v_s + ch_s^T\|_{1,\Omega_i}},$$

we get

$$\sum_{s=1}^2 c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s + ch_s^{i,T}) \leq \sum_{s=1}^2 \left(\sum_{T \subset \Omega_s} \eta_s^{i,T} \right) \sum_{s=1}^2 \|v_s\|_{1,\Omega_s}. \tag{6.3}$$

Finally, by combining 5.5, 6.1 and 6.2 the required result follows. ■

6.1. A FIXED POINT MAPPING ASSOCIATED WITH DISCRETE PROBLEM

We define the following mapping

$$\begin{aligned} T_h: V_h &\longrightarrow H_0^1(\Omega_i) \\ W_i &\longrightarrow TW_i = \xi_h^{k,m+1} = \partial_h(F(w)), \end{aligned} \tag{6.4}$$

where ξ_h^k is the solution of the following problem

$$\left\{ \begin{aligned} &b_i(\xi_h^{k,m+1}, v_i) + \left(\alpha_{i,h} \xi_h^{k,m+1}, v_{i,h} \right)_{\Gamma_i} = (F(w), v_h)_{\Omega_i}, \\ &\xi_h^{k,m+1} = 0, \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ &\frac{\partial \xi_h^{k,m+1}}{\partial \eta} + \alpha_i \xi_h^{k,m+1} = \frac{\partial \xi_h^{k,m}}{\partial \eta} + \alpha_i \xi_h^{k,m}, \quad \text{on } \Gamma_i. \end{aligned} \right. \tag{6.5}$$

6.2. AN ITERATIVE DISCRETE ALGORITHM

Choosing $u_h^0 = u_{h0}$ the solution of the following discrete equation

$$b(u_h^0, v_h) = (g^0, v_h), \quad v_h \in V_h, \tag{6.6}$$

where g^0 is a regular function.

Now we give the following discrete algorithm

$$u_{i,h}^{k,m+1} = T_h u_h^{k-1,m+1}, \quad k = 1, \dots, n, \quad i = 1, \dots, 4,$$

where u_h^k is the solution of the problem (6.5).

Proposition 6.2. *Let ξ_h^k be a solution of the problem (6.5) with the right hand side $F^i(w_i)$ and the boundary condition $\frac{\partial \xi_h^{k,m+1}}{\partial \eta} + \alpha_i \xi_h^{k,m+1}$, $\tilde{\xi}_h^k$ the solution for \tilde{F} and $\frac{\partial \tilde{\xi}_h^{k,m+1}}{\partial \eta} +$*

$\alpha_i \tilde{\xi}_h^{k,m+1}$. The mapping T_h is a contraction in V_h with the rate of contraction $\frac{\lambda}{\beta + \lambda}$. Therefore, T_h admits a unique fixed point which coincides with the solution of the problem (6.5).

Proof. We note that

$$\|W\|_{H_0^1(\Omega_i)} = \|W\|_1.$$

Setting

$$\phi = \frac{1}{\beta + \lambda} \|F(w_i) - F(\tilde{w}_i)\|_1.$$

Then, for $\xi_h^{k,m+1} + \phi$ is solution of

$$\begin{cases} b \left(\xi_h^{k,m+1} + \phi, (v_h + \phi) \right) = (F(w) + \alpha\phi, (v_h + \phi)), \\ \xi_h^{k,m+1} = 0, \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ \frac{\partial \xi_h^{k,m+1}}{\partial \eta_i} + \alpha_i \xi_h^{k,m+1} = \frac{\partial \xi_h^{k,m}}{\partial \eta} + \alpha_i \xi_h^{k,m}, \quad \text{on } \Gamma, \end{cases}$$

From assumption (1.2), we have

$$\begin{aligned} F(w) &\leq F(\tilde{w}) + \|F(w) - F(\tilde{w})\|_1 \\ &\leq F(\tilde{w}) + \frac{\alpha}{\beta + \lambda} \|F(w) - F(\tilde{w})\|_1 \\ &\leq F(\tilde{w}) + \alpha\phi. \end{aligned}$$

It is very clear that if $F(w) \geq F(\tilde{w})$ then $\xi_h^{k,m+1} \geq \tilde{\xi}_h^{k,m+1}$. Thus

$$\xi_h^{k,m+1} \leq \tilde{\xi}_h^{k,m+1} + \phi.$$

But the role of w_i and \tilde{w}_i are symmetrical, thus we have the similar prof

$$\tilde{\xi}_h^{k,m+1} \leq \xi_h^{k,m+1} + \phi,$$

yields

$$\begin{aligned} \|T(w) - T(\tilde{w})\|_1 &\leq \frac{1}{\beta + \lambda} \|F(w_i) - F(\tilde{w}_i)\|_1 \\ &= \frac{1}{\beta + \lambda} \|f^i + \lambda w_i - f^i - \lambda \tilde{w}_i\|_1 \\ &\leq \frac{\lambda}{\beta + \lambda} \|w_i - \tilde{w}_i\|_1. \end{aligned}$$

■

Proposition 6.3. *Under the previous hypotheses and notations, we have the following estimate of convergent*

$$\left\| u_h^{n,m+1} - u_{i,h}^{\infty,m+1} \right\|_1 \leq \left(\frac{1}{1 + \beta\theta(\Delta t)} \right)^n \left\| u_h^{\infty,m+1} - u_{h_0} \right\|_1, \quad k = 0, \dots, n, \quad (6.7)$$

where $u^{\infty,m+1}$ is an asymptotic continuous solution. and u_{h_0} solution of (6.5).

Proof. We have

$$u_h^\infty = T_h u_h^{i,\infty},$$

$$\left\| u_h^{1,m+1} - u_h^{\infty,m+1} \right\|_1 = \left\| T_h u_h^{0,m+1} - T_h u_h^{\infty,m+1} \right\|_1 \leq \left(\frac{1}{1 + \beta\theta(\Delta t)} \right) \left\| u_h^{i,0} - u_h^{\infty,m+1} \right\|_1$$

and for $n + 1$, we have

$$\left\| u_h^{n+1,m+1} - u_h^{i,\infty} \right\|_1 = \left\| T_h u_h^{n,m+1} - T_h u_h^{\infty,m+1} \right\|_1 \leq \left(\frac{1}{1 + \beta\theta(\Delta t)} \right) \left\| u_h^{n,m+1} - u_h^{i,\infty} \right\|_1,$$

then

$$\left\| u_h^{n,m+1} - u_{i,h}^\infty \right\|_1 \leq \left(\frac{1}{1 + \beta\theta(\Delta t)} \right)^n \left\| u_h^{\infty,m+1} - u_{h_0} \right\|_1. \quad \blacksquare$$

Now we evaluate the variation in H_0^1 -norm between $u(T, x)$, the discrete solution calculated at the moment $T = n\Delta t$ and u^∞ , the asymptotic continuous solution (1.1).

Theorem 6.4. *Under the previous hypotheses, notations, results, we have*

$$\left\| u_h^{n,m+1} - u^\infty \right\|_1 \leq C \left[\begin{aligned} & \left\| u_{1,h}^{k,m+1} - u_{2,h}^{k,m} \right\|_{W_1} + \left\| u_{2,h}^{k,m} - u_{1,h}^{k,m-1} \right\|_{W_2} + \left\| e_{1,h}^{n+1,m} \right\|_{W_1} \\ & + \left\| e_{2,h}^{n+1,m-1} \right\|_{W_2} + \left(\frac{1}{1 + \beta\theta(\Delta t)} \right)^n \end{aligned} \right] \quad (6.8)$$

and

$$\left\| u_h^{n,m+1} - u^\infty \right\|_1 \leq C \left[h^2 |\log h| + \left(\frac{1}{1 + \beta\theta(\Delta t)} \right)^n \right]. \quad (6.9)$$

Proof. It can be easily proved this theorem by using the results of Theorem 6.1 and Proposition 6.3. ■

7. CONCLUSION

In this paper, A posteriori error estimates for the generalized overlapping domain decomposition method with mixed boundary boundary conditions on the interfaces for parabolic equation with second order boundary value problems are studied using theta time scheme combined with a Galerkin approximation. Furthermore, a result of an asymptotic behavior using H_0^1 -norm is presented using Benssoussan-Lions' Algorithm. In future, . The geometrical convergence of both the continuous and discrete corresponding Schwarz algorithms error estimate for linear elliptic PDEs will be established and the results of some numerical experiments will be presented to support the theory.

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