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Bochner-Chen Ideal Submanifolds

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Abstract In this paper, we investigate ideal invariant submanifolds, ideal anti-invariant submanifolds and ideal CR-submanifolds of Bochner Kaehler manifolds. Moreover, some characterizations related with the holomorphic sectional curvature and the anti-holomorphic sectional curvature are obtained for Bochner Kaehler manifolds.

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1. Introduction

In 1949, S. Bochner [1] introduced a new tensor as an analogue of the Weyl conformal curvature tensor in a complex local coordinate system in an m complex dimensional Kaehlerian manifold $\widetilde{M}$ with Riemannian metric $\widetilde{g}$ as follows:

$$\begin{aligned} B(X,Y)Z &= \widetilde{R}(X,Y)Z - \frac{1}{2(m+2)}\{\widetilde{g}(Y,Z)\widetilde{Q}(X) - \widetilde{g}(\widetilde{Q}X,Z)Y + \widetilde{g}(\widetilde{Q}Y,Z)X \\ &\quad -\widetilde{g}(X,Z)\widetilde{Q}Y + \widetilde{g}(JY,Z)\widetilde{Q}JX - \widetilde{g}(\widetilde{Q}JX,Z)JY + \widetilde{g}(\widetilde{Q}JY,Z)JX \\ &\quad -\widetilde{g}(JX,Z)\widetilde{Q}JY - 2\widetilde{g}(JX,\widetilde{Q}Y)JZ - 2\widetilde{g}(JX,Y)\widetilde{Q}JZ\} \\ &\quad +\frac{\widetilde{\tau}}{4(m+1)(m+2)}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y + \widetilde{g}(JY,Z)JX \\ &\quad -\widetilde{g}(JX,Z)JY - 2\widetilde{g}(JX,Y)JZ\}, \end{aligned} \tag{1.1}$$

where J is the almost complex structure, $\widetilde{R}$ is the Riemannian curvature tensor, $\widetilde{\tau}$ is the scalar curvature, $\widetilde{Q}$ denotes the Ricci operator defined by

$$\widetilde{g}(\widetilde{Q}X,Y) = \widetilde{\mathrm{Ric}}(X,Y), \tag{1.2}$$

for any $X,Y,Z,W \in \Gamma(T\widetilde{M})$. Later, many mathematicians have obtained some necessary and sufficient conditions for exploring the geometric meaning of vanishing Bochner curvature tensor in some different spaces. For example, in [2], S. Tachibana studied and

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obtained an interesting expression for the Bochner curvature tensor in Kaehler manifold. In [3], M. Sitaramayya and in [4], H. Mori obtained a generalized Bochner curvature tensor as a component in its curvature tensor on Kaehlerian vector spaces. In [5], F. Tricerri and L. Vanhecke generalized this notion, that is, they defined the generalized Bochner curvature tensor as a component of the element of spaces of arbitrary generalized curvature tensors on Hermitian vector space. In [6], L. Vanhecke proved that the Ricci operator is complex linear for an almost Hermitian manifold with vanishing Bochner curvature tensor if and only if the manifold is a para-Kahlerian manifold. In [7], L. Vanhecke and K. Yano showed that the Ricci operator is complex linear if and only if the Riemannian curvature tensor satisfies the following relation:

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z,W) &= \widetilde{g}(\widetilde{R}(JX,JY)Z,W)+\widetilde{g}(\widetilde{R}(JX,Y)JZ,W) \\ &\quad +\widetilde{g}(\widetilde{R}(JX,Y)Z,JW) \end{aligned} \tag{1.3}$$

for all tangent vectors X,Y,Z,W of the manifold. In [8], D. E. Blair stated that every totally geodesic anti-invariant submanifold of a Kaehler manifold of complex dimension > 3 with vanishing Bochner curvature tensor is conformally flat. In [9], H. M. Abood studied the geometric meaning of vanishing the generalized Bochner curvature tensor in nearly Kaehler manifold. In [10], Y. Euh, J.H. Park, K. Sekigawa investigated the local structures of nearly Kaehler manifolds with vanishing Bochner curvature tensor. Bochner Kaehler manifolds were also discussed in [11–21] etc.

Furthermore, B.-Y. Chen and F. Dillen [22] established new simple geometric characterizations of Bochner-Kaehler and Einstein-Kaehler spaces of complex space forms. They stated that the manifold $\widetilde{M}$ is an m-complex dimensional Bochner Kaehler manifold if and only if the following statements satisfy for every orthonormal basis $\{X,Y\}$ of any totally real plane section:

(1) the totally real bisectional curvature $\widetilde{H}(X,Y)$ depends on the totally real plane section $\Pi=\text{Span}\{X,Y\}$ and not on the choice of orthonormal basis X,Y;

(2) $\widetilde{H}(X)+\widetilde{H}(Y)$ depends only on the totally real plain section $\Pi=\text{Span}\{X,Y\}$ and not on the choice of orthonormal basis X,Y;

(3) the sectional curvatures satisfy $\widetilde{K}(X,Y)=\widetilde{K}(X,JY)$;

(4) $\widetilde{H}(X)+\widetilde{H}(Y)=8\widetilde{K}(X,Y)$;

(5) $\widetilde{H}(X)+\widetilde{H}(Y)=4\widetilde{H}(X,Y)$;

(6) $\widetilde{R}(X,JY,JY,Y)=\widetilde{R}(X,JX,JX,Y)$.

The main purpose of the present paper is to continue this work.

2. Riemannian Invariants and Submanifolds

In this section, we recall a number of Riemannian invariants which are the intrinsic characteristics of a Riemannian manifold and affect the behavior of the Riemannian manifold.

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold equipped with a Riemannian metric $\widetilde{g}$ and inner product of the metric $\widetilde{g}$ is denoted by $\langle,\rangle$. Let $\{e_1,\ldots,e_m\}$ be an orthonormal basis for $T_p\widetilde{M}$. The *sectional curvature*, denoted $\widetilde{K}_{ij}$, of the plane section spanned by e_i and e_j at $p\in M$ is given by

$$\widetilde{K}_{ij}\equiv\widetilde{K}_M(e_i,e_j)\equiv\widetilde{R}(e_i,e_j,e_j,e_i)\equiv\widetilde{R}(e_j,e_i,e_i,e_j), \tag{2.1}$$

where $\widetilde{R}$ is the Riemannian curvature tensor.

The *Ricci tensor* $\widetilde{\mathrm{Ric}}$ is defined by

$$\widetilde{\mathrm{Ric}}\,(X,Y)=\sum_{j=1}^{m}\widetilde{R}\,(e_j,X,Y,e_j) \tag{2.2}$$

for any $X,Y\in T_p\widetilde{M}$.

For a fixed $i\in\{1,\ldots,m\}$, we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}\,(e_i,e_i) &= \sum_{j=1}^{m}\widetilde{R}\,(e_j,e_i,e_i,e_j)=\sum_{j=1}^{m}\widetilde{R}\,(e_i,e_j,e_j,e_i)\\ &= \sum_{j\neq i}^{m}\widetilde{R}\,(e_i,e_j,e_j,e_i)=\sum_{j\neq i}^{m}\widetilde{K}_{ij}.\end{aligned}$$

Let u be a unit vector in $T_p\widetilde{M}$. We choose an orthonormal basis $\{e_1,...,e_m\}$ of $T_p\widetilde{M}$ such that $e_1=u$. The Ricci curvature $\widetilde{\mathrm{Ric}}\,(u)$ of u is defined by

$$\widetilde{\mathrm{Ric}}(u)=\widetilde{K}_{12}+\widetilde{K}_{13}+\cdots+\widetilde{K}_{1m}=\sum_{j=2}^{m}\widetilde{K}_{1j}. \tag{2.3}$$

The *scalar curvature* $\widetilde{\tau}$ at p is defined by

$$\widetilde{\tau}\,(p)=\sum_{i<j}\widetilde{K}_{ij}. \tag{2.4}$$

The *Chen invariant* which is certainly an intrinsic character of a (sub)manifold [23] is given by

$$\delta_{\widetilde{M}}(p)=\widetilde{\tau}(p)-(\inf\widetilde{K})(p), \tag{2.5}$$

where

$$(\inf\widetilde{K})(p)=\inf\{\widetilde{K}(\Pi)\,|\,\Pi \text{ is a plane section}\subset \mathrm{T_p\widetilde{M}}\}.$$

We note that new optimal inequalities involving the Chen invariant have been recently proved in [24–38] etc.

Let L^k be a k-plane section of $T_p\widetilde{M}$ and u be a unit vector in L^k. We choose an orthonormal basis $\{e_1,...,e_k\}$ of L^k such that $e_1=u$. The Ricci curvature $\widetilde{\mathrm{Ric}}_{L^k}$ of L^k at u is defined by

$$\widetilde{\mathrm{Ric}}_{L^k}(u)=\widetilde{K}_{12}+\widetilde{K}_{13}+\cdots+\widetilde{K}_{1k}. \tag{2.6}$$

Here, $\widetilde{\mathrm{Ric}}_{L^k}(u)$ is called a *k-Ricci curvature* [39]. Thus for each fixed e_i, $i\in\{1,...,k\}$ we get

$$\widetilde{\mathrm{Ric}}_{L^k}(e_i)=\sum_{j\neq i}^{k}\widetilde{K}_{ij}. \tag{2.7}$$

The scalar curvature $\widetilde{\tau}\left(L^k\right)$ of the k-plane section L^k is given by

$$\widetilde{\tau}\left(L^k\right)=\sum_{1\leq i<j\leq k}\widetilde{K}_{ij}. \tag{2.8}$$

We note that

$$\widetilde{\tau}\left(L^k\right)=\frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}^{k}\widetilde{K}_{ij}=\frac{1}{2}\sum_{i=1}^{n}\widetilde{\mathrm{Ric}}_{L^k}(e_i). \tag{2.9}$$

Let M be an n-dimensional submanifold in a manifold $\tilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. The Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y=\nabla_X Y+\sigma\left(X,Y\right)$$

and

$$\tilde{\nabla}_X N=-A_N X+\nabla^{\perp}_X N$$

for all $X,Y\in TM$ and $N\in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, the induced Riemannian and the induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and σ is the second fundamental form related to the shape operator A by

$$\widetilde{g}\left(\sigma\left(X,Y\right),N\right)=\widetilde{g}\left(A_N X,Y\right).$$

The equation of Gauss is given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W)-\widetilde{g}(\sigma(X,W),\sigma(Y,Z))\\ &\quad+\widetilde{g}(\sigma(X,Z),\sigma(Y,W))\end{aligned} \tag{2.10}$$

for all $X,Y,Z,W\in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M, respectively.

For any orthonormal basis $\{e_1,...,e_n\}$ of the tangent space T_pM, the mean curvature vector $\hbar(p)$ is given by

$$\hbar(p)=\frac{1}{n}\sum_{i=1}^{n}\sigma\left(e_i,e_i\right). \tag{2.11}$$

The submanifold M is called *totally geodesic* in $\tilde{M}$ if $\sigma=0$ and it is called *minimal* if $\hbar=0$. If $\sigma\left(X,Y\right)=g\left(X,Y\right)\hbar$ for all $X,Y\in TM$, then the submanifold M is called *totally umbilical* [40].

3. Bochner Kaehler Manifolds

Let $\widetilde{M}$ be an almost Hermitian manifold with complex structure J and Riemannian metric $\widetilde{g}$. If the almost complex structure J satisfies

$$(\widetilde{\nabla}_X J)Y+(\widetilde{\nabla}_Y J)X=0 \tag{3.1}$$

for any vector fields X and Y on $T\widetilde{M}$, then the manifold is called a *nearly Kaehlerian manifold*, if

$$\widetilde{\nabla}_X J=0 \tag{3.2}$$

for all X vectors on $T\widetilde{M}$, then the manifold is called a *Kaehlerian manifold* [41].

Let $\widetilde{M}$ be an m complex dimensional Kaehlerian manifold. Then the Bochner curvature tensor, defined in [2], is given by

$$\begin{aligned}\widetilde{B}(X,Y,Z,W) \;=\;& \widetilde{R}(X,Y,Z,W) - \langle X,W\rangle L(Y,Z) + \langle Y,W\rangle L(X,Z)\\ &-\langle Y,Z\rangle L(X,W) + \langle X,Z\rangle L(Y,W) - \langle JX,W\rangle M(Y,Z)\\ &+\langle JY,W\rangle M(X,Z) - \langle JY,Z\rangle M(X,W) + \langle JX,Z\rangle M(Y,W)\\ &+2\langle JZ,W\rangle M(X,Y) + 2\langle JX,Y\rangle M(Z,W), \end{aligned} \tag{3.3}$$

where

$$L(Y,Z) = \frac{1}{2(m+2)}\widetilde{\mathrm{Ric}}(Y,Z) - \frac{\widetilde{\tau}}{4(m+1)(m+2)}\langle Y,Z\rangle, \tag{3.4}$$

$$L(Y,Z) = L(Z,Y), \quad L(JY,JZ) = L(Y,Z), \quad L(JY,Z) = -L(Y,JZ), \tag{3.5}$$

$$M(Y,Z) = -L(Y,JZ) = -L(JZ,Y) \tag{3.6}$$

for any $X,Y,Z,W \in T\widetilde{M}$. The manifold $\widetilde{M}$ is called a *Bochner Kaehler manifold* if the tensor B vanishes identically.

Let $\Pi = \mathrm{Span}\{X,Y\}$, $X,Y \in T_p\widetilde{M}$, be a 2-dimensional plane section on $T_p\widetilde{M}$. The plane Π is called *anti-holomorphic* if

$$\widetilde{g}(X,Y) = \widetilde{g}(JX,Y) = 0, \tag{3.7}$$

in other words, $J\Pi \subset \Pi^{\perp}$, where $\Pi^{\perp}$ is complementary space of Π in $T\widetilde{M}$. Furthermore, the plane Π is called *holomorphic* if $J\Pi \subset \Pi$. In this case, the holomorphic sectional curvature is given by

$$\widetilde{H}(\Pi) = \widetilde{H}(X) = \widetilde{R}(X,JX,JX,X), \tag{3.8}$$

where $\Pi = \mathrm{Span}\{X,JX\}$, X is a unit vector on Π.

If Π and Π' are two holomorphic plane sections in $T_p\widetilde{M}$, then the holomorphic bisectional curvature $\widetilde{H}(\Pi,\Pi')$ is defined by

$$\widetilde{H}(\Pi,\Pi') = \widetilde{H}(X,Y) = \widetilde{R}(X,JX,JY,Y), \tag{3.9}$$

where X is a unit vector on Π and Y is a unit vector on Π' [42].

On the other hand, if $\Pi \perp \Pi'$, then $\widetilde{H}(X,Y)$ is called totally real bisectional curvature and

$$\widetilde{H}(X,Y) = \widetilde{K}(X,Y) + \widetilde{K}(X,JY) \tag{3.10}$$

for a totally real plane spanned by any vector pair $\{X,Y\}$ [22].

We now recall the following important facts.

Theorem 3.1. [22] *Let $\widetilde{M}$ be a complex dimensional Kaehlerian manifold of complex dimension $m > 1$. Then $\widetilde{M}$ is Bochner-Kaehler manifold if and only if every totally real bisectional curvature $\widetilde{H}(X,Y)$ depends only on the totally real plane section spanned by X,Y but it does not depend on the choice of orthonormal basis X, Y.*

Theorem 3.2. [12] *Let $\widetilde{M}$ be a Kaehler manifold of complex dimension $m > 1$. Then $\widetilde{M}$ is a Bochner-Kaehler manifold if and only if there is a Hermitian quadratic form Q on $\widetilde{M}$ such that the holomorphic sectional curvature $\widetilde{H}(X)$ of the holomorphic plane spanned by X and JX satisfies $\widetilde{H}(X) = Q(X,X)$ for any unit tangent vector X. Furthermore, if such Q exists, then*

$$Q = \frac{4}{(m+2)}\widetilde{Ric} - \frac{2\tau}{(m+1)(m+2)}\widetilde{g}. \tag{3.11}$$

Let $\widetilde{M}$ be an m complex dimensional Bochner Kaehler manifold and $\{e_1,\ldots,e_n,Je_1,\ldots,Je_n\}$ be an orthonormal basis of $T_p\widetilde{M}$ at $p \in M$. Then the scalar curvature $\widetilde{\tau}(p)$ is given by

$$\widetilde{\tau}(p) = \sum_{i=1}^{n} \widetilde{H}(e_i) + 2\sum_{i<j} \widetilde{H}(e_i,e_j). \tag{3.12}$$

If L is a complex subspace of real dimension $r \geq 2$ in $T_p\widetilde{M}$, the r-scalar curvature $\widetilde{\tau}(L)$ of L is defined as

$$\widetilde{\tau}(L) = \sum_{i=1}^{r} \widetilde{H}(e_i) + 2\sum_{i<j}^{r} \widetilde{H}(e_i,e_j). \tag{3.13}$$

From (3.3), (3.5), (3.6), (3.9) and (3.10), we get the following lemma:

Lemma 3.3. *Let $\widetilde{M}$ be an m complex dimensional Bochner Kaehler manifold. Then*

$$\widetilde{H}(X) = 2\langle X,X\rangle L(X,X), \tag{3.14}$$

$$\begin{aligned}\widetilde{H}(X,Y) &= 2\left(\langle Y,Y\rangle M(X,JX) - \langle X,X\rangle M(JY,Y)\right)\\ &= 2\left(\langle Y,Y\rangle L(X,X) + \langle X,X\rangle L(Y,Y)\right),\end{aligned} \tag{3.15}$$

for a totally real plane spanned by any vector pair $\{X,Y\}$.

Taking into consider (3.4) equation and Lemma 3.3, we get the following proposition:

Proposition 3.4. *Let $\widetilde{M}$ be an m complex dimensional Bochner Kaehler manifold. Then we have*

$$\widetilde{H}(X) = \frac{1}{(m+2)}\widetilde{Ric}(X) - \frac{\widetilde{\tau}(p)}{2(m+1)(m+2)}, \tag{3.16}$$

$$\widetilde{H}(X,Y) = \frac{1}{(m+2)}[\widetilde{Ric}(X) + \widetilde{Ric}(Y)] - \frac{\widetilde{\tau}(p)}{2(m+1)(m+2)} \tag{3.17}$$

for a totally real plane spanned by any vector pair $\{X,Y\}$.

4. Submanifolds of a Bochner Kaehler Manifold

Let M be an n-dimensional submanifold of an almost Hermitian manifold $(\widetilde{M}, J, \widetilde{g})$. For any $X \in T_pM$, we decompose JX into tangential and normal parts given by

$$JX = PX + FX, \qquad PX \in T_pM,\ FX \in T_p^{\perp}M; \tag{4.1}$$

thus PX is the tangential part of JX while FX is the normal part of JX. The squared norm of P at $p \in M$ is defined to be

$$\|P\|^2 = \sum_{i,j=1}^{n} \langle Pe_i, e_j\rangle^2, \tag{4.2}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

In an almost Hermitian manifold, its almost complex structure J transforms a vector into a vector perpendicular to it. According to the behavior of the tangent bundle of a submanifold under the action of the almost complex structure J of the ambient manifold, there are two well-known classes of submanifolds, namely, invariant submanifolds and anti-invariant submanifolds.

Let M be a submanifold of an almost Hermitian manifold $(\widetilde{M}, J, \widetilde{g})$. The tangent space of the submanifold remains invariant under the action of the almost complex structure J where as in the second case it is mapped into the normal space. Thus, M is invariant if $F = 0$, and it is anti-invariant if $P = 0$ [41].

In 1978, A. Bejancu [43] generalized the concept of invariant and anti-invariant submanifolds in to a CR-submanifold as follows.

A submanifold M of an almost Hermitian manifold is called a *CR-submanifold* if the tangent bundle TM of M can be decomposed as the direct sum of a holomorphic (invariant) distribution and a totally real (anti-invariant) distribution, that is,

$$TM = D \oplus D^{\perp},$$

where $J(D) = D$ and $J(D^{\perp}) \subset TM^{\perp}$. In fact, we have $D = \ker(F)$ and $D^{\perp} = \ker(P)$. Invariant and anti-invariant submanifolds are CR-submanifolds with $D = \{0\}$ and $D^{\perp} = \{0\}$, respectively.

Let M be an n-dimensional CR-submanifold of an m complex dimensional Bochner Kaehler manifold and $\{e_1, \ldots, e_s, e_{s+1}, \ldots, e_n\}$ be an orthonormal basis of T_pM such that the invariant distribution of T_pM is spanned by the vectors $e_1, \ldots, e_s$ and the anti-invariant distribution of T_pM is spanned by the vectors $e_{s+1}, \ldots, e_n$. Let us define

$$\hbar_D(p) = \frac{1}{s}\sum_{i=1}^{s} \sigma(e_i, e_i) \quad \text{and} \quad \hbar_{D^{\perp}}(p) = \frac{1}{n-s}\sum_{j=s+1}^{n} \sigma(e_j, e_j). \tag{4.3}$$

The submanifold M is called *D-minimal* if $\hbar_D = 0$, called *$D^{\perp}$-minimal* if $\hbar_{D^{\perp}} = 0$ for all point of M [16].

Theorem 4.1. *Let (M, g) be an n-dimensional invariant submanifold of an m complex dimensional Bochner Kaehler manifold $(\widetilde{M}, \widetilde{g})$. For all mutually orthogonal unit vectors $X, Y \in TM$, we have*

$$\begin{aligned} K(X,Y) &= \frac{3}{(m+2)}\widetilde{Ric}(JX, Y) - \frac{3\widetilde{\tau}(p)}{2(m+1)(m+2)} - \|\sigma(X,Y)\|^2 \\ &\quad + g(\sigma(X,X), \sigma(Y,Y)) + \frac{1}{2}\left(\widetilde{H}(X) + \widetilde{H}(Y)\right). \end{aligned} \tag{4.4}$$

where σ is the second fundamental form is given in the Gauss and Weingarten formulas.

Proof. Let $\{e_1,\dots,e_m,Je_1,\dots,Je_m\}$ be an orthonormal basis of $T_p\widetilde{M}$. By doing straightforward computation in (3.3), we get

$$\widetilde{K}(e_i,e_j)=L(e_i,e_i)+L(e_j,e_j)+6\langle Je_i,e_j\rangle L(Je_i,e_j). \tag{4.5}$$

If we put (3.14) in (4.5), we have

$$\widetilde{K}(e_i,e_j)=\frac{1}{2}\left(\widetilde{H}(e_i)+\widetilde{H}(e_j)\right)+6\langle Je_i,e_j\rangle L(Je_i,e_j), \tag{4.6}$$

for $i\neq j\in\{1,\dots,m\}$. Using (2.10), (3.4) and (4.6) equalities, we get

$$\begin{aligned} K(e_i,e_j) &= \frac{3}{(m+2)}\widetilde{Ric}(Je_i,e_j)-\frac{3\widetilde{\tau}(p)}{2(m+1)(m+2)}+\widetilde{g}(\sigma(e_i,e_i),\sigma(e_j,e_j)) \\ &\quad -\|\sigma(e_i,e_j)\|^2. \end{aligned} \tag{4.7}$$

Finally, putting $e_i=X$ and $e_j=Y$, we have (4.4). ■

We recall now the following theorem of B.-Y. Chen in [44] for future uses.

Theorem 4.2. *Let M be an n-dimensional $(n\geq 3)$ submanifold in a real space form $R(c)$ of constant sectional curvature c. Then, for each point $p\in M$ we have*

$$\delta(2)\leq\frac{n^2(n-2)}{2(n-1)}\|\hbar(p)\|^2+\frac{1}{2}(n+1)(n-2)c. \tag{4.8}$$

The equality in (4.8) holds at $p\in M$ if and only if there exist an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\dots,e_m\}$ of $T_p^{\perp}M$ such that **(a)** $\Pi=\mathrm{Span}\{e_1,e_2\}$ *and* **(b)** *the forms of shape operators A_{e_r}, $r=n+1,\dots,m$, become*

$$A_{e_{n+1}}=\begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & \mu I_{n-2}\end{pmatrix},\quad \mu=a+b, \tag{4.9}$$

$$A_{e_r}=\begin{pmatrix} c_r & d_r & 0\\ d_r & -c_r & 0\\ 0 & 0 & 0_{n-2}\end{pmatrix},\qquad r\in\{n+2,\dots,m\}. \tag{4.10}$$

Next, we give a generalization of Theorem 4.2 in terms of the Chen's invariant in submanifolds of any Riemannian manifold. We note that the following theorem is a special case of Theorem 3.1 in [45] of B.-Y. Chen:

Theorem 4.3. *Let M be an n-dimensional $(n\geq 3)$ submanifold in an m-dimensional Riemannian manifold $\widetilde{M}$. Then, for each point $p\in M$ and each plane section $\Pi\subset T_pM$, we have*

$$\delta(2)\leq\frac{n^2(n-2)}{2(n-1)}\|\hbar(p)\|^2+\widetilde{\tau}(T_pM)-\widetilde{K}(\Pi_2). \tag{4.11}$$

The equality in (4.11) holds at $p\in M$ if and only if there exist an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\dots,e_m\}$ of $T_p^{\perp}M$ such that the forms of shape operators A_{e_r}, $r=n+1,\dots,m$, become as (4.9) and (4.10).

Using Theorem 4.1 and Theorem 4.3, we have the following theorem:

Theorem 4.4. *Let (M, g) be an invariant submanifold of a Bochner Kaehler manifold $(\widetilde{M}, \widetilde{g})$ and $\Pi = Span\{e_1, e_2\}$ is a 2-dimensional plane section T_pM. Then we have*

$$\begin{aligned}\tau(p) &\leq \frac{3}{(m+2)}\widetilde{Ric}(JX,Y) - \frac{3\widetilde{\tau}(p)}{2(m+1)(m+2)} + \frac{n^2(n-2)}{2(n-1)}\|\hbar(p)\|^2 \\ &+ \widetilde{\tau}_{T_pM}(p) - \widetilde{K}(\Pi) + 4\|\hbar|_{\Pi}(p)\|^2 + \frac{1}{2}\left(\widetilde{H}(e_1) + \widetilde{H}(e_2)\right), \qquad (4.12)\end{aligned}$$

where

$$\|\hbar|_{\Pi}(p)\|^2 = \frac{\widetilde{g}(\sigma(e_1,e_1),\sigma(e_2,e_2))}{4}. \qquad (4.13)$$

The equality in (4.12) holds at $p \in M$ if and only if the shape operators take the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \mu I_{n-2} \end{pmatrix}, \quad \mu = a + b, \qquad (4.14)$$

$$A_{e_r} = \begin{pmatrix} c_r & 0 & 0 \\ 0 & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \quad r \in \{n+2, \ldots, m\}. \qquad (4.15)$$

Now we state the following definition:

Definition 4.5. Let (M, g) be an invariant submanifold of a Bochner Kaehler manifold $(\widetilde{M}, \widetilde{g})$. We call the manifold M as a *Bochner-Chen ideal invariant submanifold* if the shape operators take the form as (4.14) and (4.15).

Let (M, g) be a Bochner-Chen ideal invariant submanifold of a Bochner Kaehler manifold $(\widetilde{M}, \widetilde{g})$ and $\Pi = \mathrm{Span}\{e_1, e_2\}$. Then, from (4.14) and (4.15), we get

$$K(e_1, e_2) = \widetilde{K}(e_1, e_2) + ab - \sum_{r=n+2}^{m} (c_r)^2, \qquad (4.16)$$

$$K(e_1, e_j) = \widetilde{K}(e_1, e_j) + a\mu, \qquad (4.17)$$

$$K(e_2, e_j) = \widetilde{K}(e_2, e_j) + b\mu, \qquad (4.18)$$

$$K(e_i, e_j) = \widetilde{K}(e_i, e_j) + \mu^2, \qquad (4.19)$$

$$\mathrm{Ric}(e_1) = \widetilde{\mathrm{Ric}}_{T_pM}(e_1) + ab - \sum_{r=n+2}^{m} (c_r)^2 + (n-2)a\mu^2, \qquad (4.20)$$

$$\mathrm{Ric}(e_2) = \widetilde{\mathrm{Ric}}_{T_pM}(e_2) + ab - \sum_{r=n+2}^{m} (c_r)^2 + (n-2)b\mu^2, \qquad (4.21)$$

$$\mathrm{Ric}(e_i) = \widetilde{\mathrm{Ric}}_{T_pM}(e_i) + (n-2)\mu^2, \qquad (4.22)$$

where $i, j > 2$ and $\widetilde{\mathrm{Ric}}_{T_pM}$ is n-plane Ricci curvature given in (2.7).

Taking trace in (4.5), we have the following lemma:

Lemma 4.6. *Let (M,g) be an n-dimensional submanifold of a Bochner Kaehler manifold $(\widetilde{M},\widetilde{g})$ and $\{e_1,\ldots,e_n\}$ be an orthonormal basis of T_pM. Then we have*

$$\widetilde{Ric}_{T_pM}(e_j) = trace(L|_M) + (n-2)L(e_j,e_j) + 6\sum_{i=1}^{n}\langle Je_i,e_j\rangle L(e_i,e_j). \tag{4.23}$$

Taking into consideration (4.20), (4.21) equalities and Lemma 4.6, we get the following theorem:

Theorem 4.7. *Let (M,g) be an n-dimensional Bochner-Chen ideal invariant submanifold of Bochner Kaehler manifold $(\widetilde{M},\widetilde{g})$. If the submanifold (M,g) is Einstein, then there exist a plane section spanned by unit vectors X,Y such that*

$$L(X,X) - L(Y,Y) = b^2 - a^2, \tag{4.24}$$

where a,b given in (4.14).

Proof. Let $\Pi = \text{Span}\{e_1,e_2\}$ and the submanifold (M,g) be Einstein. Then

$$\widetilde{\text{Ric}}_{T_pM}(e_1) - \widetilde{\text{Ric}}_{T_pM}(e_2) = (n-2)(b^2-a^2). \tag{4.25}$$

Furthermore, from Lemma 4.6, we have

$$\widetilde{\text{Ric}}_{T_pM}(e_1) - \widetilde{\text{Ric}}_{T_pM}(e_2) = (n-2)\left(L(e_1,e_1) - L(e_2,e_2)\right). \tag{4.26}$$

Puting $e_1 = X$ and $e_2 = Y$, we obtain (4.24). ∎

Lemma 4.8. *Let (M,g) be an n-dimensional anti-invariant submanifold of a Bochner Kaehler manifold $(\widetilde{M},\widetilde{g})$. Then we have*

$$\widetilde{H}(X,Y) = 2\widetilde{K}(X,Y) \tag{4.27}$$

where X,Y are vector fields on M such that Span$\{X,Y\}$ is an anti-holomorphic plane section.

Theorem 4.9. *Let (M,g) be an n-dimensional anti-invariant submanifold of a Bochner Kaehler manifold $(\widetilde{M},\widetilde{g})$. Then, for each point $p\in M$ and each plane section $\Pi =$ Span$\{e_1,e_2\}$, we have*

$$\tau(p) - K(\Pi) \leq \frac{n^2(n-2)}{2(n-1)}\|\hbar(p)\|^2 + \widetilde{\tau}(T_pM) - \frac{1}{2}\widetilde{H}(e_1,e_2). \tag{4.28}$$

The equality in (4.28) holds at $p\in M$ if and only if there exist an orthonormal basis $\{e_1,\ldots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_m\}$ of $T_p^{\perp}M$ such that the forms of shape operators A_{e_r}, $r=n+1,\ldots,m$, become

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)\,I_{n-2}\end{pmatrix}, \tag{4.29}$$

$$A_{e_r} = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2}\end{pmatrix}, \qquad r\in\{n+2,\ldots,m\}. \tag{4.30}$$

Definition 4.10. Let (M, g) be an anti invariant submanifold of a Bochner Kaehler manifold $(\widetilde{M}, \widetilde{g})$. We call the submanifold (M, g) as a *Bochner-Chen ideal anti invariant submanifold* if the shape operators take the form as (4.29) and (4.30).

Let M be an n-dimensional CR-submanifold of an m-dimensional Bochner Kaehler manifold and $\{e_1, \dots, e_s, e_{s+1}, \dots, e_n\}$ be an orthonormal basis of T_pM such that the invariant distribution of T_pM is spanned by the vectors $e_1, \dots, e_s$ and the anti-invariant distribution of T_pM is spanned by the vectors $e_{s+1}, \dots, e_n$. Let $P_i : TM \to D_i$, $i \in \{1, 2\}$ be be orthogonal projections. For any mutually orthogonal unit vector fields $X, Y \in TM$, it can be written that

$$X = P_1X + P_2X, \quad \text{and} \quad Y = P_1Y + P_2Y. \tag{4.31}$$

Then we have

$$\begin{aligned}
\widetilde{R}(X, Y, Y, X) &= \widetilde{R}(P_1X + P_2X, P_1Y + P_2Y, P_1Y + P_2Y, P_1X + P_2X) \\
&= \widetilde{R}(P_1X, P_1Y, P_1Y, P_1X) + \widetilde{R}(P_1X, P_2Y, P_1Y, P_1X) \\
&\quad + \widetilde{R}(P_1X, P_1Y, P_2Y, P_1X) + \widetilde{R}(P_1X, P_2Y, P_1Y, P_2X) \\
&\quad + \widetilde{R}(P_1X, P_1Y, P_1Y, P_2X) + \widetilde{R}(P_1X, P_2Y, P_2Y, P_1X) \\
&\quad + \widetilde{R}(P_1X, P_1Y, P_2Y, P_2X) + \widetilde{R}(P_1X, P_2Y, P_2Y, P_2X) \\
&\quad + \widetilde{R}(P_2X, P_1Y, P_1Y, P_1X) + \widetilde{R}(P_2X, P_1Y, P_1Y, P_2X) \\
&\quad + \widetilde{R}(P_2X, P_1Y, P_2Y, P_1X) + \widetilde{R}(P_2X, P_1Y, P_2Y, P_2X) \\
&\quad + \widetilde{R}(P_2X, P_2Y, P_1Y, P_1X) + \widetilde{R}(P_2X, P_2Y, P_1Y, P_2X) \\
&\quad + \widetilde{R}(P_2X, P_2Y, P_2Y, P_1X) + \widetilde{R}(P_2X, P_2Y, P_2Y, P_2X).
\end{aligned}$$

From (3.3), we obtain the followings:

$$\begin{aligned}
\widetilde{R}(P_1X, P_1Y, P_1Y, P_1X) &= \langle P_1X, P_1X\rangle L(P_1Y, P_1Y) + \langle P_1Y, P_1Y\rangle L(P_1X, P_1X) \\
&\quad + 6\langle JP_1X, P_1Y\rangle L(P_1X, P_1Y),
\end{aligned}$$

$$\widetilde{R}(P_1X, P_1Y, P_2Y, P_1X) = \langle P_1X, P_1X\rangle L(P_1Y, P_2Y) + 3\langle JP_1X, P_1Y\rangle L(JP_1X, P_2Y),$$

$$\widetilde{R}(P_1X, P_2Y, P_1Y, P_2X) = \langle JP_1X, P_1Y\rangle L(JP_2X, P_2Y),$$

$$\widetilde{R}(P_1X, P_1Y, P_1Y, P_2X) = \langle P_1Y, P_1Y\rangle L(P_1X, P_2X) + 3\langle JP_1X, P_1Y\rangle L(JP_2X, P_1Y),$$

$$\widetilde{R}(P_1X, P_2Y, P_2Y, P_1X) = \langle P_1X, P_1X\rangle L(P_2Y, P_2Y) + \langle P_2Y, P_2Y\rangle L(P_1X, P_1X),$$

$$\widetilde{R}(P_1X, P_1Y, P_2Y, P_2X) = 2\langle JP_1X, P_1Y\rangle L(JP_2X, P_2Y),$$

$$\widetilde{R}(P_1X, P_2Y, P_2Y, P_2X) = \langle P_2Y, P_2Y\rangle L(P_1X, P_2X),$$

$$\widetilde{R}(P_2X, P_1Y, P_1Y, P_2X) = \langle P_2X, P_2X\rangle L(P_1Y, P_1Y) + \langle P_1Y, P_1Y\rangle L(P_2X, P_2X),$$

$$\widetilde{R}(P_2X, P_1Y, P_2Y, P_2X) = \langle P_2X, P_2X\rangle L(P_1Y, P_2Y),$$

$$\widetilde{R}(P_2X, P_2Y, P_2Y, P_2X) = \langle P_2X, P_2X\rangle L(P_2Y, P_2Y) + \langle P_2Y, P_2Y\rangle L(P_2X, P_2X).$$

Let us choose $P_1X = \frac{1}{\sqrt{2}}e_\alpha$, $P_1Y = \frac{1}{\sqrt{2}}e_\beta$, $P_2X = \frac{1}{\sqrt{2}}e_\gamma$, $P_2Y = \frac{1}{\sqrt{2}}e_\omega$ for any $\alpha, \beta \in \{1, \ldots, s\}$ and $\gamma, \omega \in \{s+1, \ldots, n\}$. Then we have

$$\begin{aligned} \widetilde{R}(X,Y,Y,X) &= \frac{1}{2}\left[L(e_\alpha,e_\alpha) + L(e_\beta,e_\beta)L(e_\gamma,e_\gamma) + L(e_\omega,e_\omega)\right.\\ &\quad \left.+2L(e_\alpha,e_\gamma) + 2L(e_\beta,e_\omega)\right] + 6\langle Je_\alpha, e_\beta\rangle \left(L(Je_\alpha,e_\beta)\right.\\ &\quad \left.+L(Je_\alpha,e_\omega) + L(Je_\gamma,e_\omega) + L(Je_\gamma,e_\beta)\right). \end{aligned} \tag{4.32}$$

Using the equation of Gauss, (3.4) and (4.32), we state the following lemma:

Lemma 4.11. *Let M be an n-dimensional CR-submanifold of an m complex dimensional Bochner Kaehler manifold. For any mutually orthogonal unit vectors $X, Y \in T_pM$ and $\Pi = Span\{X, Y\}$, we have*

$$\begin{aligned} \widetilde{K}(\Pi) &= \frac{1}{(m+2)}\Big[\widetilde{Ric}(P_1X) + \widetilde{Ric}(P_2X) + \widetilde{Ric}(P_1Y) + \widetilde{Ric}(P_2Y)\\ &\quad 2\widetilde{Ric}(P_1X,P_2X) + 2\widetilde{Ric}(P_1Y,P_2Y)\Big] + \frac{3}{(m+2)}\langle JP_1X, P_1Y\rangle\\ &\quad \Big(\widetilde{Ric}(JP_1X,P_1Y) + \widetilde{Ric}(JP_1X,P_2Y) + \widetilde{Ric}(JP_2X,P_1Y)\\ &\quad +\widetilde{Ric}(JP_2X,P_2Y)\Big) - \frac{2\widetilde{\tau}(p)}{(m+1)(m+2)}. \end{aligned} \tag{4.33}$$

From Lemma 4.11, we get the following theorems:

Theorem 4.12. *Let M be an n-dimensional CR-submanifold of an m complex dimensional Bochner Kaehler manifold.*

i) *If $\Pi = Span\{X, Y\}$ is a plane section in $\Gamma(D)$, then*

$$\widetilde{K}(\Pi) = \frac{1}{(m+2)}\left[\widetilde{Ric}(X) + \widetilde{Ric}(Y) + 3\langle JX, Y\rangle \widetilde{Ric}(X,Y) - \frac{\widetilde{\tau}(p)}{2(m+1)}\right]. \tag{4.34}$$

ii) *If $\Pi = Span\{X, Y\}$ is a plane section in $\Gamma(D^\perp)$, then*

$$\widetilde{K}(\Pi) = \frac{1}{(m+2)}\left[\widetilde{Ric}(X) + \widetilde{Ric}(Y) - \frac{\widetilde{\tau}(p)}{2(m+1)(m+2)}\right]. \tag{4.35}$$

Theorem 4.13. *Let M be an n-dimensional CR-submanifold of an m complex dimensional Bochner Kaehler manifold. Then, for each point $p \in M$ and each plane section $\Pi = \mathrm{Span}\,\{X, Y\}$ of TM, we have*

$$\begin{aligned} \tau(p) - K(\Pi) &\leq \frac{n^2(n-2)}{2(n-1)}\|\hbar(p)\|^2 + \left(1 + \frac{1}{(m+1)(m+2)}\right)\widetilde{\tau}(p)\\ &\quad -\frac{1}{(m+2)}\Big[\widetilde{Ric}(P_1X) + \widetilde{Ric}(P_2X) + \widetilde{Ric}(P_1Y) + \widetilde{Ric}(P_2Y)\\ &\quad + 2\widetilde{Ric}(P_1X,P_2X) + 2\widetilde{Ric}(P_1Y,P_2Y)\Big] - \frac{3}{(m+2)}\langle JP_1X, P_1Y\rangle\\ &\quad \Big(\widetilde{Ric}(JP_1X,P_1Y) + \widetilde{Ric}(JP_1X,P_2Y) + \widetilde{Ric}(JP_2X,P_1Y)\\ &\quad +\widetilde{Ric}(JP_2X,P_2Y)\Big). \end{aligned} \tag{4.36}$$

The equality in (4.36) holds at $p \in M$ if and only if the forms of shape operators A_{e_r}, $r = n+1, \ldots, m$, become as (4.9) and (4.10).

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