Thai Journal of Mathematics
Volume 18 Number 4 (2020)
Pages 1743–1751

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Convergence Theorem of Inertial P-iteration Method for a Family of Nonexpansive Mappings with Applications

Attapol Kaewkhao[1,*], Limpapat Bussaban[2] and Suthep Suantai[3]

[1] *Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : akaewkhao@yahoo.com

[2] *Graduate Ph.D. Degree Program in Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : lim.bussaban@gmail.com

[3] *Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : suthep.s@cmu.ac.th

Abstract In this paper, an algorithm called inertial P-iteration method is proposed for finding a common fixed point of a family of nonexpansive mappings and proved under some suitable conditions that the sequence generated by the proposed method weakly converges to a common fixed point of a family of nonexpansive mappings. In application, our method can be applied for finding zero point of sum of two monotone operators and minizer of sum of two convex functions.

MSC: 47H09; 47H10; 65F10
Keywords: P-iteration; nonexpansive mapping

Submission date: 10.05.2019 / Acceptance date: 17.12.2020

1. Introduction

Let H be a Hilbert space together with norm $\|\cdot\|$ and X be a nonempty subset of H. A mapping $T : X \to X$ is said to have a fixed point if there exists a point $\bar{x} \in X$ such that $\bar{x} = T\bar{x}$. The fixed point problem is mathematical problem to find a fixed point of specific mapping, e.g. contraction, L-Lipschitz, nonexpansive mapping. The classical theorem in fixed point theory is Banach contraction theorem which guarantee existence and uniqueness fixed point of a contraction mapping in complete metric space. In 1965, the existence of a fixed point of nonexpansive mapping on Hilbert space was proved by Browder theorem. In this work, we focus on approximation problem of a fixed point of nonexpansive mapping on Hilbert space. In order to find a fixed point $\bar{x}$, many

*Corresponding author.

Published by The Mathematical Association of Thailand.

researchers proposed various methods for finding the approximate solution. One of most popular iterative methods, called *Picard iteration method*, was defined by:

$$x_{n+1} = Tx_n, \tag{1.1}$$

where initial point x_1 is chosen randomly. In addition, other iterative methods for improving picard iteration method have been studied extensively such as follows.

Mann iteration method [1] is defined by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \geq 1, \tag{1.2}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$ is a sequence in $[0,1]$. In case of $\alpha_n = 1$ for all $n \geq 1$, this iteration method reduces to the Picard iteration method.

Ishikawa iteration method [2] is defined by:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, n \geq 1, \end{cases} \tag{1.3}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. This iteration method reduces to the Mann iteration method when $\beta_n = 0$ for all $n \geq 1$.

S-iteration method [3] is defined by:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, n \geq 1, \end{cases} \tag{1.4}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. In 2017, Agarwal, O'Regan and Sahu proved that this iteration method is independent of Mann and Ishikawa iteration method and converges faster than both of them.

SP-iteration method [4] is defined by:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n, n \geq 1, \end{cases} \tag{1.5}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. In 2011, Phuengrattana and Suantai proved convergence theorem of the SP-iteration of continuous function on arbitary interval and compared the convergence speed which obtained the results that SP-iteration method is faster than Mann, Ishikawa, Noor iteration methods. In 2015, P-iteration method was introduced and studied by Sainuan.

P-iteration method [5] is defined by:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)Tz_n + \alpha_n Ty_n, n \geq 1, \end{cases} \tag{1.6}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. Sainuan proved the convergence theorem of P-iteration method and compared the rate of convergence between P-iteration ans S-iteration. However, the methods mentioned above have a badly convergence rate. Thus, to speed up, the technique for improving speed and giving a better convergence behavior was introduced firstly by [6] by adding an *inertial step*. Motivated by those works mentioned above, the idea of P-iteration method will be combined with inertial step. Thus, a novel algorithm called *inertial P-iteration method* is constructed and defined by:

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}) \\ z_n = (1-\gamma_n)u_n + \gamma_n T u_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)T z_n + \alpha_n T y_n, n \geq 1, \end{cases} \tag{1.7}$$

where initial point x_0, x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\theta_n\}$ are sequences in $[0,1]$. This paper is organized as follows: The basic concept and lemmas will be ginven in Section 2. The convergence theorem of our method will be proved in Section 3. Finally, in Section 4, we apply the proposed method for zero point of monotone operators problem and minimization problem.

2. Preliminaries

Let X be a nonempty subset of a Hilbert space H. A mapping $T : X \to X$ is said to be *L-Lipschtiz operator* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\,\|x - y\|$$

for any $x, y \in X$. An L-Lipschtiz operator is called *nonexpansive operator* if $L = 1$. A point x is said to be *fixed point* of T if $x = Tx$. The set of all fixed point of T is denoted by $F(T)$. Let $\{T_n\}$ be a family of nonexpansive mappings. A point x is called a *common fixed point* of T_n if $x \in \bigcap_{n=1}^{\infty} F(T_n)$.

Definition 2.1. [7, 8] Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive operators such that $\emptyset \neq F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Then, $\{T_n\}$ is said to satisfy *NST-condition(I) with* $\mathcal{T}$ if for each bounded sequence $\{x_n\}$,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \text{ implies } \lim_{n\to\infty} \|x_n - T x_n\| = 0 \text{ for all } T \in \mathcal{T}.$$

If $\mathcal{T}$ is singleton, i.e. $\mathcal{T} = \{T\}$, then $\{T_n\}$ is said to satisfy NST-condition(I) with T.

Lemma 2.2. [9] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 (Opial lemma). *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H such that there exists a nonempty subset Ω of H satisfying the following conditions:*

- *for all $y \in \Omega$, $\lim_{n\to\infty} \|x_n - y\|$ exists,*
- *Any weak-cluster point of $\{x_n\}$ belongs to Ω.*

Then, there exists $\bar{x} \in \Omega$ such that $x_n \rightharpoonup \bar{x}$.

3. Main Results

Let H be a Hilbert space together with norm $\|\cdot\|$ and inner product $\langle\cdot|\cdot\rangle$ and $\{T_n\}$ be a family of nonexpansive mappings on H. In this section, we generalize the inertail P-iteration method (1.7) as follows.

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}) \\ z_n = (1-\gamma_n)u_n + \gamma_n T_n u_n, \\ y_n = (1-\beta_n)z_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)T_n z_n + \alpha_n T_n y_n, n \geq 1, \end{cases} \tag{3.1}$$

where initial point x_0, x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\theta_n\}$ are sequences in $[0,1]$.

Theorem 3.1. *Let a family of nonexpansive mappings $\{T_n\}$ on a Hilbert space H and a nonexpansive mapping T on H be such that $\{T_n\}$ satisfies NST-condition(I) with T. Suppose that $\emptyset \neq F(T) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{x_n\}$ be a sequence in H generated by (3.1) such that*

(a). *x_0, x_1 are choosen randomly,*
(b). *$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$,*
(c). *$0 < p < \beta_n < q < 1$,*

Then, $\{x_n\}$ converges weakly to a point in $F(T)$.

Proof. Let $x^* \in F(T)$ and let $\{x_n\}$ be a sequence in H generated by (3.1). Then,

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|$$

and

$$\begin{aligned} \|z_n - x^*\| &\leq (1-\gamma_n)\|u_n - x^*\| + \gamma_n \|T_n u_n - x^*\| \leq \|u_n - x^*\| \\ \|y_n - x^*\| &\leq (1-\beta_n)\|z_n - x^*\| + \beta_n \|T_n z_n - x^*\| \leq \|z_n - x^*\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|T_n z_n - x^*\| + \alpha_n \|T_n y_n - x^*\| \\ &\leq (1-\alpha_n)\|u_n - x^*\| + \alpha_n \|u_n - x^*\| \\ &\leq \|u_n - x^*\| \\ &\leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$, by using Lemma 2.2, we obtain that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Thus, $\{x_n\}$ is bounded which implies that $\{u_n\}$ is also bounded. From (3.1), we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|x_n - x^* + \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - x^* | x_n - x_{n-1}\rangle \\ &\leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\|. \end{aligned}$$

Then,

$$\begin{aligned}\|z_n - x^*\|^2 &= \|(1-\gamma_n)(u_n - x^*) + \gamma_n(T_n u_n - x^*)\|^2 \\ &= (1-\gamma_n)\|u_n - x^*\|^2 + \gamma_n\|T_n u_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|u_n - T_n u_n\|^2 \\ &\le \|u_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|u_n - T_n u_n\|^2,\end{aligned}$$

and

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(1-\beta_n)(z_n - x^*) + \beta_n(T_n z_n - x^*)\|^2 \\ &= (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|T_n z_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - T_n z_n\|^2 \\ &\le \|z_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - T_n z_n\|^2 \\ &\le \|z_n - x^*\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1-\alpha_n)\|T_n z_n - x^*\|^2 + \alpha_n\|T_n y_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|T_n z_n - T_n y_n\|^2 \\ &\le (1-\alpha_n)\|T_n z_n - x^*\|^2 + \alpha_n\|T_n y_n - x^*\|^2 \\ &\le \|z_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x^*\|\|x_n - x_{n-1}\| \\ &\quad - \beta_n(1-\beta_n)\|u_n - T_n u_n\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, it follows that $\|u_n - T_n u_n\| \to 0$. Since $\{u_n\}$ is bounded and $\{T_n\}$ satisfies NST-conditon(I) with T, we get $\|u_n - Tu_n\| \to 0$. From

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - u_n\| + \|u_n - Tu_n\| + \|Tu_n - Tx_n\| \\ &\le 2\|x_n - u_n\| + \|u_n - Tu_n\| \\ &\le 2\theta_n\|x_n - x_{n-1}\| + \|u_n - Tu_n\|,\end{aligned}$$

we obtain $\|x_n - Tx_n\| \to 0$. Let w be a weak cluster point of $\{x_n\}$. Then $w \in F(T)$ by demicloseness of $I - T$ at 0. Hence, by using Opial lemma, we conclude that there exists $\bar{x} \in F(T)$ such that $x_n \rightharpoonup \bar{x}$. ■

4. Applications

In this section, we apply our result to find a zero point of sum of two monotone operators and minimizer of sum of two convex functions.

4.1. Zero Point of Sum of Two Monotone Operators

Let H be a Hilbert space. A mapping $A : H \to 2^H$ is called *monotone operator* if

$$\langle x - y | u - v\rangle \ge 0,$$

for any $(x,u),(y,v) \in graA$, where $graA = \{(x,y) \in H \times H : x \in H, y \in Ax\}$ is the graph of A. A monotone operator A is called *maximal monotone operator* if the graph $graA$ is not properly contained in the graph of any other monotone operator.

Let $A : H \to 2^H$ be a maximal monotone operator and $c > 0$. The *resolvent operator* of A is defined by $J_A = (I + A)^{-1}$ where I is an identity operator. The zero of sum of two monotone operators problem is to find a point $x \in H$ such that

$$0 \in Ax + Bx \tag{4.1}$$

where $A : H \to 2^H, B : H \to 2^H$ are two monotone operators. The set of all zero point of $A + B$ is denoted by $zer(A + B) = \{x \in H : 0 \in Ax + Bx\}$.

Let $A : H \to 2^H$ be maximal monotone operator and $B : H \to H$ be an L-Lipschitz operator. By Proposition 26.1(iv)(a) in [10], we see that a point $\bar{x} \in H$ is a solution of (4.1) if and only if $\bar{x} \in F(T)$ where $T = J_{cA}(I - cB)$ and $c \in (0, \frac{2}{L})$.

Proposition 4.1. *Let H be a Hilbert space and let $A : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be an L-Lipschitz operator. Let $\alpha > 0$ and $x, p \in H$. Setting $\tilde{A}_\alpha = \frac{1}{\alpha}(I - J_{\alpha A}(I - \alpha B))$. Then, the following hold:*

(i) $\tilde{A}_\alpha x \in AJ_{\alpha A}(I - \alpha B)x + Bx$.
(ii) $p \in \tilde{A}_\alpha x$ *if and only if* $(x - \alpha p, p - Bx) \in graA$.
(iii) $\left\|\tilde{A}_\alpha x\right\| \leq \|Ax + Bx\|$ *where* $\|Ax + Bx\| := \inf\{\|z\| : z \in Ax + Bx\}$.

Proof.

(*i*) Let $u = \tilde{A}_\alpha x$. Then $x - \alpha u = J_{\alpha A}(I - \alpha B)x$ which implies that $u - Bx \in A(x - \alpha u)$. Thus, $\tilde{A}_\alpha x \in AJ_{\alpha A}(I - \alpha B)x + Bx$.

(*ii*) By using definition of $\tilde{A}_\alpha$ and $J_{\alpha A}$, we have

$$\begin{aligned} p = \tilde{A}_\alpha x &\Leftrightarrow x - \alpha p = J_{\alpha A}(I - \alpha B)x \Leftrightarrow (I - \alpha B)x \in (I + \alpha A)(x - \alpha p) \\ &\Leftrightarrow p - Bx \in A(x - \alpha p) \Leftrightarrow (x - \alpha p, p - Bx) \in graA. \end{aligned}$$

(*iii*) Let $w = \tilde{A}_\alpha x$ and $u \in Ax + Bx$. Then, by monotonic of A, we have

$$\langle u - w | w \rangle = \frac{1}{\alpha} \langle (u - Bx) - (w - Bx) | x - (x - \alpha w) \rangle \geq 0.$$

By CauchySchwarz inequality, we obtain that $\|w\| \leq \|u\|$. Thus,

$$\left\|\tilde{A}_\alpha x\right\| = \inf\{\|z\| : z \in \tilde{A}_\alpha x\} \leq \inf\{\|z\| : z \in Ax + Bx\} = \|Ax + Bx\|.$$

∎

Lemma 4.2. *Let H be a Hilbert space. Let $A : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be an L-Lipschitz operator. Let $\alpha, \beta > 0$. Then,*

$$\frac{1}{\beta} \|J_{\alpha A}(I - \alpha B)x - J_{\beta A}(I - \beta B)J_{\alpha A}(I - \alpha B)x\| \leq \frac{1 + \alpha L}{\alpha} \|x - J_{\alpha A}(I - \alpha B)x\|,$$

for every $x \in H$.

Proof. Let $x \in H$. Set $\tilde{A}_\alpha = \frac{1}{\alpha}(I - J_{\alpha A}(I - \alpha B))$. Then, by using Proposition 4.1, we obtain

$$\begin{aligned}
&\frac{1}{\beta}\left\| J_{\alpha A}(I-\alpha B)x - J_{\beta A}(I-\beta B)J_{\alpha A}(I-\alpha B)x \right\| \\
&= \left\| \tilde{A}_\beta J_{\alpha A}(I-\alpha B)x \right\| \\
&\leq \left\| AJ_{\alpha A}(I-\alpha B)x + BJ_{\alpha A}(I-\alpha B)x \right\| \\
&\leq \left\| AJ_{\alpha A}(I-\alpha B)x + Bx \right\| + \left\| Bx - BJ_{\alpha A}(I-\alpha B)x \right\| \\
&\leq \left\| \tilde{A}_\alpha x \right\| + L\left\| x - J_{\alpha A}(I-\alpha B)x \right\| \\
&= \frac{1+\alpha L}{\alpha}\left\| x - J_{\alpha A}(I-\alpha B)x \right\|.
\end{aligned}$$

■

Theorem 4.3. *Let H be a Hilbert space. Let $A : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be an L-Lipschitz operator. Let $c \in (0, \frac{2}{L})$ and $\{c_n\} \subset (a, \frac{2}{L})$ for some $a > 0$. Define $T_n = J_{c_n A}(I - c_n B)$. Then, $\{T_n\}$ satisfies the NST-condition(I) with T_c where $T_c = J_{cA}(I - cB)$.*

Proof. Let $\{x_n\}$ be a bounded sequence in H. Suppose that $\|x_n - T_n x_n\| \to 0$. Since T_n and T_c are nonexpansive for all $n \in \mathbb{N}$, see Theorem 26.14 in [10] for detail, by Lemma 4.2, we obtain that

$$\begin{aligned}
\|x_n - T_c x_n\| &= \|x_n - J_{cA}(I-cB)x_n\| \\
&\leq \|x_n - J_{c_n A}(I-c_n B)x_n\| \\
&\quad + \|J_{c_n A}(I-c_n B)x_n - J_{cA}(I-cB)J_{c_n A}(I-c_n B)x_n\| \\
&\quad + \|J_{cA}(I-cB)J_{c_n A}(I-c_n B)x_n - J_{cA}(I-cB)x_n\| \\
&\leq 2\|x_n - J_{c_n A}(I-c_n B)x_n\| + \frac{c(1+c_n L)}{c_n}\|x_n - J_{c_n A}(I-c_n B)x_n\| \\
&= (2 + \frac{c(1+c_n L)}{c_n})\|x_n - T_n x_n\| \\
&\leq (2 + \frac{3c}{a})\|x_n - T_n x_n\| \to 0
\end{aligned}$$

Thus, $\{T_n\}$ satisfies the NST-condition(I) with T_c. ■

By Theorem 4.3, we can apply our method for finding a solution of (4.1) as follows.

Corollary 4.4. *Let H be a Hilbert space. Let $A : H \to 2^H$ be maximal monotone operator and $B : H \to H$ be an L-Lipschitz operator. Let $c \in (0, \frac{2}{L})$ and $\{c_n\} \subset (a, \frac{2}{L})$ for some $a > 0$. Define $T_n = J_{c_n A}(I - c_n B)$ and $T = J_{cA}(I - cB)$. Suppose that $zer(A + B) \neq \emptyset$. Let $\{x_n\}$ be a sequence in H generated by (3.1). Then, $\{x_n\}$ converges weakly to a point in $zer(A + B)$.*

4.2. Minimization of Sum of Two Convex Functions

Let $f : H \to \mathbb{R}$ be a smooth convex function with gradient having Lipschtiz constant L and $g : H \to \mathbb{R}$ be a convex smooth (or possible non-smooth) function. The minimization problem of $f+g$ is to find a point $x \in H$ such that

$$f(x) + g(x) = \min_{y\in H} f(y) + g(y). \tag{4.2}$$

This problem can be apply to image processing problems, machine learning problems etc. Note that $argmin(f+g)$ is the set of all solutions of (4.2). As in [11], $\bar{x}$ is a solution of (4.2) if and only if x is a fixed point of forward-backward operator, i.e. $\bar{x} = J_{c\partial g}(I - c\nabla f)(\bar{x})$ where $c > 0$. The following corollary obtained by setting $A = \partial g$ and $B = \nabla f$ as Corollary 4.4.

Corollary 4.5. *Let H be a Hilbert space. Let $g \in \Gamma_0(H)$ and $f : H \to \mathbb{R}$ be convex and differentiable with an L-Lipschitz continuous gradient, let $c \in (0, \frac{2}{L})$ and $\{c_n\} \subset (a, \frac{2}{L})$ for some $a > 0$. Define $T_n = J_{c_n\partial g}(I - c_n\nabla f)$ and $T = J_{c\partial g}(I - c\nabla f)$. Suppose that $argmin(f+g) \neq \emptyset$. Let $\{x_n\}$ be a sequence in H generated by (3.1). Then, $\{x_n\}$ converges weakly to a point in $argmin(f+g)$.*

Acknowledgements

This research was supported by Chiang Mai University. The first author would like to thanks the Thailand Research Fund through the Royal Golden Jubilee (RGJ) PhD Programme (Grant No. PHD/0184/2560).

References

[1] W.R. Mann, Mean value methods in iteration, Proceedings of the American Mathematical Society 4 (1953) 506–510.

[2] S. Ishikawa, Fixed points by a new iteration method, Proceedings of the American Mathematical Society 44 (1974) 147–150.

[3] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed point of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61–79.

[4] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235 (2011) 3006–3014.

[5] P. Sainuan, Rate of Convergence of P-Iteration and S-Iteration for Continuous Functions on Closed Intervals, Thai J. Math. 13 (2) (2015) 449–457.

[6] B. Polyak, Some methods of speeding up the convergence of iteration methods, USSR Computational Mathematics and Mathematical Physics 4 (1964) 1–17.

[7] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence to a common fixed point of families of nonexpansive mappings in banach spaces, J. Nonlinear Convex Anal. 8 (2007) 11–34.

[8] W. Takahashi, Viscosity approximation methods for countable families of nonexpansive mappings in banach spaces, Nonlinear Anal., Theory Methods Appl. 70 (2009) 719–734.

[9] K. Tan, H. Xu, Approximating fixed points of nonexpansive mappings by the ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.

[10] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces. 2nd ed., Springer Publishing Company, Incorporated, 2017.

[11] N. Parikh, S. Boyd, Proximal algorithms, Found. Trends Optim. 1 (2014) 127–239.