



Strong Convergence of Approximating Fixed Point Sequences for Relatively Nonlinear Mappings

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Abstract : Some recent iteration algorithms to prove strong convergence of approximating fixed point sequences for relatively nonlinear mappings in Banach spaces by using the hybrid methods in mathematical programming are introduced. Also, we establish strong convergence of modified Ishikawa type iteration algorithm for both uniformly equicontinuous and total relatively asymptotically nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces.

Keywords : Strong convergence, modified Ishikawa's iteration, uniformly equicontinuous mappings, total (relatively) asymptotically nonexpansive mappings.

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1 Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \rightarrow C$ be a mapping. Then T is said to be a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called *uniformly k -Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* if $\lim_{n \rightarrow \infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [12] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X , then every asymptotically nonexpansive mapping $T : C \rightarrow C$ has a fixed point.

Recently, Alber et al. [2] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [9]. They say that a mapping $T : C \rightarrow C$ is said to be *total asymptotically nonexpansive* [2] if there exists nonnegative real sequences $\{\mu_n\}$ and $\{\eta_n\}$, $n \geq 1$ with $\mu_n, \eta_n \rightarrow 0$ as $n \rightarrow \infty$ and strictly increasing continuous function

$\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \tau(\|x - y\|) + \eta_n, \quad (1.1)$$

for all $x, y \in C$ and $n \geq 1$.

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T ; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point* of T [28] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. We say that a sequence $\{x_n\}$ in C is said to be an *approximating fixed point* sequence for T if $\|x_n - Tx_n\| \rightarrow 0$.

Let X be a smooth Banach space and let X^* be the dual of X . The function $\phi : X \times X \rightarrow \mathbb{R}^+ := [0, \infty)$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in X$, where J is the normalized duality mapping from X to X^* . We say that a mapping $T : C \rightarrow C$ is *relatively asymptotically nonexpansive* [18] if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$ and, for each $n \geq 1$ there exists a constant $k_n > 0$ such that $\phi(p, T^n x) \leq k_n^2 \phi(p, x)$ for $x \in C$ and $p \in F(T)$, where $\lim_{n \rightarrow \infty} k_n = 1$. In particular, T is called *relatively nonexpansive* [22] if $k_n = 1$ for all n ; see also [3,4,5]. Further, we say that $T : C \rightarrow C$ is a mappings of *relatively asymptotically nonexpansive* in the intermediate sense if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$ and $\lim_{n \rightarrow \infty} c_n = 0$, where $c_n = \sup_{x \in C} c_n(x)$ and

$$c_n(x) = \sup_{p \in F(T)} [\phi(p, T^n x) - \phi(p, x)] \vee 0. \quad (1.2)$$

In this case, (1.2) reduces to

$$\phi(p, T^n x) \leq \phi(p, x) + c_n, \quad (1.3)$$

for all $x \in C$, $p \in F(T)$, and $n \geq 1$. Finally, motivated by the class of total asymptotically nonexpansive mappings due to Alber et al. [2], we also say that T is *total relatively asymptotically nonexpansive* if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$, and there exists nonnegative real sequences $\{\mu_n\}$ and $\{\eta_n\}$, $n \geq 1$ with $\mu_n, \eta_n \rightarrow 0$ as $n \rightarrow \infty$ and strictly increasing continuous function $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \mu_n \tau(\phi(p, x)) + \eta_n, \quad (1.4)$$

for all $x \in C$, $p \in F(T)$, and $n \geq 1$. Note that if $\tau(t) = t$, then (1.4) reduces to

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + \eta_n,$$

for all $x \in C$, $p \in F(T)$, and $n \geq 1$. In addition, if $\eta_n = 0$, $k_n = \sqrt{1 + \mu_n}$ for all $n \geq 1$, then the class of total relatively asymptotically nonexpansive mappings coincides with the class of relatively asymptotically nonexpansive mappings. If $\mu_n = 0$ and $\eta_n = 0$ for all $n \geq 1$, then (1.4) reduces to the class of relatively

nonexpansive mappings, introduced by Matsushita and Takahashi [22]. Also, if we take $\mu_n = 0$ and $\eta_n = c_n$ as above, then (1.4) reduces to (1.3), the class of mappings of relatively asymptotically nonexpansive in the intermediate sense.

The purpose of this paper is firstly to introduce some recent results and open questions relating to strong convergence for modified Mann (or Ishikawa) iteration processes, and secondly to carry the previous results over the wider class of total relatively asymptotically nonexpansive mappings. In section 2, we introduce three famous iteration processes introduced by Halpern [13], Mann [20], and Ishikawa [14], respectively. In section 3, we give some properties of generalized projection relating to the above function $\phi : X \times X \rightarrow \mathbb{R}$, and furthermore, in section 4, we give some recent developments and open questions for strong convergence of approximating fixed point sequences in Hilbert spaces or general Banach spaces. Finally, in section 5, we prove strong convergence of modified Ishikawa type iteration for both uniformly equicontinuous and total relatively asymptotically nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces. Some applications are also added.

2 Three famous iteration algorithms

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing. However, the sequence $\{T^n x\}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping T . The first one is introduced by Halpern [13] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \geq 0, \quad (2.1)$$

where $\{t_n\}$ is a sequence in the interval $[0, 1]$.

The second iteration process is now known as Mann's iteration process [20] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \quad (2.2)$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [14] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 0, \quad (2.3)$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. By taking $\beta_n = 1$ for all $n \geq 0$ in (2.3), Ishikawa's iteration

process reduces to the Mann's iteration process (2.2). It is known in [8] that the process (2.2) may fail to converge while the process (2.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (2.1) has been proved to be strongly convergent in both Hilbert spaces [13, 19, 33] and uniformly smooth Banach spaces [26, 29, 35], while Mann's iteration (2.2) has only weak convergence even in a Hilbert space [11].

3 Some properties of generalized projections

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X . Denote by $\langle \cdot, \cdot \rangle$ the duality product. When $\{x_n\}$ is a sequence in X , we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \rightarrow x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω -limit set of $\{x_n\}$ by $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$. The normalized duality mapping J from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$.

A Banach space X is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$, equivalently, the function $x \mapsto \|x\|^2$ is strictly convex; see Proposition 2.13 of [4]. It is also said to be *uniformly convex* if $\|x_n - y_n\| \rightarrow 0$ for any two sequences $\{x_n\}, \{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \rightarrow 1$.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X . Then the Banach space X is said to be *smooth* provided

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (3.1)$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit in (3.1) is attained uniformly for $x, y \in U$. It is also known that if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X . Some properties of the duality mapping have been given in [10, 27, 31]. A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [10, 31] for more details.

Let X be a smooth Banach space. Recall that the function $\phi : X \times X \rightarrow \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in X$. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad (3.2)$$

for all $x, y \in X$. Further, we have that for any $x, y, z \in X$,

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, J(z) - J(y) \rangle. \quad (3.3)$$

In particular, it is easy to see that if X is strictly convex, for $x, y \in X$, $\phi(y, x) = 0$ if and only if $y = x$ (see, for example, Remark 2.1 of [22]).

Let X be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of X . Then, for any $x \in X$, there exists a unique element $\tilde{x} \in C$ such that

$$\phi(\tilde{x}, x) = \inf_{z \in C} \phi(z, x).$$

Then a mapping $\prod_C : X \rightarrow C$ defined by $\prod_C x = \tilde{x}$ is called the *generalized projection* (see [1, 3, 15]). In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [1, 3, 15]).

Proposition 3.1. ([1, 3, 15]) *Let K be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

- (a) *If X is smooth, then, $\tilde{x} = \prod_K x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$ for $y \in K$.*
- (b) *If X is reflexive, strictly convex and smooth, then $\phi(y, \prod_K x) + \phi(\prod_K x, x) \leq \phi(y, x)$ for all $y \in K$.*

The following subsequent two lemmas are recently given in [16].

Lemma 3.2. ([16]) *Let C be a nonempty closed convex subset of a smooth Banach space X , $x, y, z \in X$ and $\lambda \in [0, 1]$. Given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \phi(v, z) \leq \lambda\phi(v, x) + (1 - \lambda)(\phi(v, y) + a)\}$$

is closed and convex.

Lemma 3.3. ([16]) *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let K be a nonempty closed convex subset of X . Let $x_0 \in X$ and $q := \prod_K x_0$, where \prod_K denotes the generalized projection from X onto K . If $\{x_n\}$ is a sequence in X such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\phi(x_n, x_0) \leq \phi(q, x_0)$$

for all n . Then $x_n \rightarrow q (= \prod_K x_0)$.

Recently, Kamimura and Takahashi [15] proved the following result, which plays a crucial role in our discussion.

Proposition 3.4. ([15]) *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X . If $\phi(x_n, z_n) \rightarrow 0$ and either $\{x_n\}$ or $\{z_n\}$ is bounded, then $x_n - z_n \rightarrow 0$.*

Also, it is not hard to prove the converse of Proposition 3.4 in smooth Banach spaces; see Proposition 2.7 of [16] for details.

Proposition 3.5. ([16]) *Let X be a smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences in X . If $x_n - z_n \rightarrow 0$ and either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_n) \rightarrow 0$.*

Now combing Proposition 3.4 and 3.5 yields the following equivalent form in uniformly convex and smooth Banach spaces.

Proposition 3.6. ([16]) *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X . If either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_n) \rightarrow 0$ if and only if $x_n - z_n \rightarrow 0$.*

Similarly, we can prove the following variation of Proposition 3.6 in a more general Banach space; see Proposition 2.10 in [16] for details.

Proposition 3.7. ([16]) *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property. If $\{x_n\}$ is a sequence in X and $x (\neq 0) \in X$, then $\phi(x_n, x) \rightarrow 0$ if and only if $x_n \rightarrow x$.*

Finally, concerning the set of fixed points of *continuous* mappings which are total relatively asymptotically nonexpansive, we can prove the following result.

Proposition 3.8. *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, let C be a nonempty closed convex subset of X , and let $T : C \rightarrow C$ be a continuous total relatively asymptotically nonexpansive mapping. Then $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence of $F(T)$ such that $p_n \rightarrow p \in C$. Using (1.4), for this $p \in C$, we have that

$$\phi(p_n, T^n p) \leq \phi(p_n, p) + \mu_n \tau(\phi(p_n, p)) + \eta_n$$

for each $n \geq 1$. As taking the limsup on both sides as $n \rightarrow \infty$, since the right hand side tends to 0, we get $\phi(p_n, T^n p) \rightarrow 0$. This combined with (3.3), since

$$\phi(p, T^n p) = \phi(p, p_n) + \phi(p_n, T^n p) + 2\langle p - p_n, J(p) - J(p_n) \rangle \rightarrow 0$$

as $n \rightarrow \infty$, yields $\phi(p, T^n p) \rightarrow 0$. Immediately, Proposition 3.7 gives $T^n p \rightarrow p$ and so $p \in F(T)$ by continuity of T .

Next, we show that $F(T)$ is convex. For $p, q \in F(T)$ and $\lambda \in (0, 1)$, put $r = \lambda p + (1 - \lambda)q$. It suffices to show that $r \in F(T)$. Indeed, as in [22], we have that for $n \geq 1$,

$$\begin{aligned} \phi(r, T^n r) &= \|r\|^2 - 2\langle \lambda p + (1 - \lambda)q, JT^n r \rangle + \|T^n r\|^2 \\ &= \|r\|^2 - 2\lambda\langle p, JT^n r \rangle - 2(1 - \lambda)\langle q, JT^n r \rangle + \|T^n r\|^2 \\ &= \|r\|^2 + \lambda\phi(p, T^n r) + (1 - \lambda)\phi(q, T^n r) - \lambda\|p\|^2 - (1 - \lambda)\|q\|^2 \\ &= \|r\|^2 + \lambda[\phi(p, r) + \mu_n \tau(\phi(p, r)) + \eta_n] + (1 - \lambda)[\phi(q, r) + \mu_n \tau(\phi(q, r)) + \eta_n] \\ &\quad - \lambda\|p\|^2 - (1 - \lambda)\|q\|^2 \\ &\leq \|r\|^2 + \lambda\phi(p, r) + (1 - \lambda)\phi(q, r) - \lambda\|p\|^2 - (1 - \lambda)\|q\|^2 \\ &\quad + \mu_n[\lambda\tau(\phi(p, r)) + (1 - \lambda)\tau(\phi(q, r))] + \eta_n. \end{aligned}$$

Since $\mu_n, \eta_n \rightarrow 0$ as $n \rightarrow \infty$, together with

$$\begin{aligned} & \|r\|^2 + \lambda\phi(p, r) + (1 - \lambda)\phi(q, r) - \lambda\|p\|^2 - (1 - \lambda)\|q\|^2 \\ = & \|r\|^2 - 2\langle \lambda p + (1 - \lambda)q, Jr \rangle + \|r\|^2 \\ = & \|r\|^2 - 2\langle r, Jr \rangle + \|r\|^2 = 0, \end{aligned}$$

the right hand side of the above inequality converges to 0. By Proposition 3.7 again, we have $T^n r \rightarrow r$ and hence $r \in F(T)$ by the continuity of T . \square

4 Recent Developments and Open Questions

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \rightarrow C$ be a mapping with $F(T) \neq \emptyset$. Recalling that a sequence $\{x_n\}$ in C is said to be an *approximating fixed point* sequence for T if $\|x_n - Tx_n\| \rightarrow 0$, there are several ways to construct an approximating fixed point sequences for a nonexpansive mapping T . We now introduce two cases mentioned in Xu [36]. Firstly we can use Banach's contraction principle to obtain a sequence $\{x_n\}$ in C such that

$$x_n = t_n x_0 + (1 - t_n)Tx_n, \quad n \geq 1$$

where the initial guess x_0 is taken arbitrarily in C and $\{t_n\}$ is a sequence in the interval $(0, 1)$ such that $t_n \rightarrow 0$ as $n \rightarrow \infty$, which is called as a Halpern's iteration process (2.1). Due to the assumption that $F(T) \neq \emptyset$, this sequence $\{x_n\}$ is bounded (indeed $\|x_n - p\| \leq \|x_0 - p\|$ for all $p \in F(T)$). Hence

$$\|x_n - Tx_n\| = t_n \|x_0 - Tx_n\| \rightarrow 0$$

and $\{x_n\}$ is an approximating fixed point sequence for T .

Secondly, we recall a sequence $\{x_n\}$ in C generated by Mann's iteration process (2.2) in a recursive way. This sequence $\{x_n\}$ is bounded since, for any $p \in F(T)$, we have

$$\|x_{n+1} - p\| \leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|Tx_n - p\| \leq \|x_n - p\|.$$

That is, $\{\|x_n - p\|\}$ is a nonincreasing sequence. Moreover, since

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - Tx_{n+1}\| \\ &= \|\alpha_n(x_n - Tx_n) + (Tx_n - Tx_{n+1})\| \\ &\leq \alpha_n \|x_n - Tx_n\| + \|x_n - x_{n+1}\| = \|x_n - Tx_n\|, \end{aligned}$$

the sequence $\{\|x_n - Tx_n\|\}$ is also nonincreasing and hence $\lim_{n \rightarrow \infty} \|x_n - Tx_n\|$ exists.

However, it is not known whether this sequence $\{x_n\}$ is always an approximating fixed point sequence for T . Only partial answers have been obtained. Indeed,

if the space X is uniformly convex and if the control sequence $\{\alpha_n\}$ satisfies the condition

$$\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty,$$

then Reich [25] showed that the sequence $\{x_n\}$ generated by Mann's iteration process (2.2) is an approximating fixed point sequence for T . For the sake of completeness, we include a brief proof to this fact. Let δ_X be the modulus of convexity of X . Pick a $p \in F(T)$. Assuming $\|x_n - p\| > 0$ and noticing $\|Tx_n - p\| \leq \|x_n - p\|$, we deduce that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \left[1 - 2\alpha_n(1 - \alpha_n)\delta_X \left(\frac{\|x_n - Tx_n\|}{\|x_n - p\|} \right) \right].$$

Hence

$$\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n)\|x_n - p\|\delta_X \left(\frac{\|x_n - Tx_n\|}{\|x_n - p\|} \right) \leq \|x_0 - p\| < \infty. \quad (4.1)$$

Put $\|x_n - p\| \rightarrow r$. If $r = 0$, we are done. So assume $r > 0$. If $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, we obtain from (3.1) that $\delta_X(\|x_n - Tx_n\|/r) \rightarrow 0$. This implies that $\|x_n - Tx_n\| \rightarrow 0$ and $\{x_n\}$ is an approximating sequence for T .

Recently, numerous attempts to modify the Mann iteration method (2.2) or the Ishikawa iteration method (2.3) so that strong convergence is guaranteed have recently been made.

Firstly, motivated by Solodov and Svaiter [30], Nakajo and Takahashi [24] proposed the following modification of Mann's iteration process (2.2) for a single nonexpansive mapping T with $F(T) \neq \emptyset$ and also proved the existence of an approximating fixed point sequence for T and strong convergence of such a sequence as follows.

Theorem NT. ([24]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T)$ is nonempty. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (4.2)$$

where P_K denotes the metric projection from H onto a closed convex subset K of H . If the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ generated by (4.2) is an approximating fixed point sequence for T and strongly convergent to $P_{F(T)}x_0$.

As a special case, taking $\alpha_n = 0$ for all n in Theorem NT, the above iteration scheme (4.2) reduces to the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ C_n = \{z \in C : \|Tx_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \quad (4.3)$$

Recently, Kim and Xu [17] generalized Nakajo and Takahashi's iteration process (4.2) to the following iteration process for an asymptotically nonexpansive mapping T in a Hilbert space, under the hypothesis of boundedness of C .

Theorem KX. ([17]) *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \leq a$ for some $0 < a < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (4.4)$$

where

$$\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\text{diam } C)^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.5)$$

Then $\{x_n\}$ is an approximating fixed point sequence for T and strongly convergent to $P_{F(T)} x_0$.

Very recently, Martinez-Yanez and Xu [21] generalized Nakajo and Takahashi's iteration process (4.2) to the following modification of Ishikawa's iteration process (2.3) for a nonexpansive mapping $T : C \rightarrow C$ with $F(T) \neq \emptyset$ in a Hilbert space H :

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \\ z_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 \\ \quad + (1 - \alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v \rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (4.6)$$

and proved that the sequence $\{x_n\}$ generated by (4.6) converges strongly to $P_{F(T)} x_0$ provided the sequence $\{\alpha_n\}$ is bounded above from one and $\lim_{n \rightarrow \infty} \beta_n = 1$.

Kamimura and Takahashi [15] considered the problem of finding an element u of a Banach space X satisfying $0 \in Au$, where $A \subset X \times X^*$ is a maximal monotone

operator and X^* is the dual space of X . They studied the following algorithm:

$$\begin{cases} x_0 \in X \text{ chosen arbitrarily,} \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), v_n \in Ay_n, \\ H_n = \{z \in X : \langle y_n - z, v \rangle \geq 0\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases} \quad (4.7)$$

where J is the duality mapping on X , $\{r_n\}$ is a sequence of positive real numbers and \prod_K denotes the generalized projection from X onto a closed convex subset K of X ; see the section 2 for more details. They proved that if $A^{-1}0 \neq \emptyset$ and $\liminf_{n \rightarrow \infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (4.7) converges strongly to an element of $A^{-1}0$. This generalizes the result due to Solodov and Svaiter [30] in a Hilbert space.

Question 1. *Can we carry Theorem NT in Hilbert spaces over more general Banach spaces?*

The crucial key to solve this question is to show the convexity of C_n in (4.2) in general, which is not easy to prove it in Banach spaces. Professor H. K. Xu raised the following question to me:

Question 2. *Let C be a nonempty closed convex subset of a normed linear space X . For any choice of $a, b \in C$,*

$$C_{a,b} = \{z \in C : \|a - z\| \leq \|b - z\|\}$$

is a convex subset of C if and only if X is a Hilbert space.

Note that if X is a Hilbert space, then

$$z \in C_{a,b} \Leftrightarrow \langle b - a, z \rangle \leq \frac{1}{2} (\|b\|^2 - \|a\|^2).$$

So, $C_{a,b}$ is convex in C . However, the proof of the converse still remains open. Owing to these troubles, we need another hypotheses for mappings T . In view of this point, for relatively nonexpansive mappings, Matsushita and Takahashi [22] recently extended Nakajo and Takahashi's iteration process (4.2) to the following modification of Mann's iteration process in general Banach spaces.

Theorem MT. ([22]) *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X , let $T : C \rightarrow C$ be a relatively nonexpansive mapping with $F(T) \neq \emptyset$, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases} \quad (4.8)$$

where J is the normalized duality mapping. Then $\{x_n\}$ generated by (4.8) is an approximating fixed point sequence for T and strongly convergent to $\prod_{F(T)} x_0$, where \prod_K denotes the generalized projection from X onto a closed convex subset K of X .

As a special case, taking $\alpha_n = 0$ for all n in (4.8), the iteration scheme reduces to the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ H_n = \{z \in C : \phi(z, Tx_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases} \quad (4.9)$$

which generalizes the iteration scheme (4.3) in Hilbert spaces. Also, they established that even though the condition of uniformly smooth of X is only weakened by the smooth condition of X , the sequence $\{x_n\}$ generated by (4.9) still converges strongly to $\prod_{F(T)} x_0$.

Recently, Kim and Takahashi [18] generalized Matsushita and Takahashi's iteration process (4.8) to the following iteration process for a uniformly k -Lipschitzian mapping T which is relatively asymptotically nonexpansive.

Theorem KT. ([18]) *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \rightarrow C$ be a uniformly k -Lipschitzian mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\beta_n \rightarrow 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n), \\ z_n = \beta_n x_n + (1 - \beta_n)T^n x_n, \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n) + \eta_n\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases} \quad (4.10)$$

where J is the normalized duality mapping and

$$\eta_n = (1 - \alpha_n)(k_n^2 - 1) \cdot \sup\{\phi(p, z_n) : p \in F(T)\}.$$

Then $\{x_n\}$ generated by (4.10) is an approximating fixed point sequence for T and strongly convergent to $\prod_{F(T)} x_0$, where $\prod_{F(T)}$ is the generalized projection from X onto $F(T)$.

Let C be a closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then, after noticing that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$, we see that $\|T^n x - T^n y\| \leq k_n \|x -$

$y\|$ is equivalent to $\phi(T^n x, T^n y) \leq k_n^2 \phi(x, y)$. It is therefore easy to show that every asymptotically nonexpansive mapping is both uniformly k -Lipschitzian and relatively asymptotically nonexpansive. In fact, it suffices to show that $\hat{F}(T) \subset F(T)$. The inclusion follows easily from the well-known demiclosedness at zero of $I - T$ (c.f., [34]), where I denotes the identity operator.

Can we remove the hypothesis of *boundedness* of C in Theorem KX in Hilbert spaces? The question still remains open. However, if $F(T)$ is a nonempty bounded subset of C , we now give a partial answer with the following η_n instead of θ_n in (4.5), that is, a Hilbert space's version in a case when $\beta_n = 1$ for all n in Theorem KT.

Corollary KT. ([18]) *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Assume that $F(T)$ is a nonempty bounded subset of C . Assume also that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \eta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (4.11)$$

where

$$\eta_n = (1 - \alpha_n)(k_n^2 - 1) \cdot \sup\{\|x_n - p\|^2 : p \in F(T)\},$$

then $\{x_n\}$ in C generated by (4.11) is an approximating fixed point sequence for T and strongly convergent to $P_{F(T)} x_0$.

The following question is naturally invoked.

Question 3. *Can we extend the modified Ishikawa type iteration (see Theorem KT) due to Kim and Takahashi over a wider class of mappings?*

5 An Answer for Question 3

Recall that $T : C \rightarrow C$ is said to be *uniformly equicontinuous* provided for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in C, \|x - y\| < \delta \Rightarrow \|T^n x - T^n y\| \leq \epsilon$$

for all $n \geq 1$. In particular, it is uniformly continuous, and furthermore

$$\|x_n - y_n\| \rightarrow 0 \Rightarrow \|T^n x_n - T^n y_n\| \rightarrow 0. \quad (5.1)$$

In general, every uniformly continuous mapping does not satisfy the property (5.1). For example, consider $T : [0, 1] \rightarrow [0, 1]$ defined by

$$Tx = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

It is easy to check that $T^n x \rightarrow 1$ for $0 < x < 1/2$, and so if we take $x_n = \frac{1}{n+2}$ and $y_n = 0$ for all n , then $|T^n x_n - T^n 0| \rightarrow 1$, which shows that T does not satisfy (5.1).

Note that every uniformly k -Lipschitzian mapping is uniformly equicontinuous. Obviously, all asymptotically nonexpansive mappings are uniformly k -Lipschitzian (hence uniformly equicontinuous).

In this section, we give an answer for Question 3, that is, strong convergence of the following modified Ishikawa's iterative algorithm of (2.3), motivated by the idea due to [21, 22], for both uniformly equicontinuous and total relatively asymptotically nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces.

Theorem 5.1. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \rightarrow C$ be a uniformly continuous and total relatively asymptotically nonexpansive mapping satisfying (5.1). Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\beta_n \rightarrow 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n), \\ z_n = \beta_n x_n + (1 - \beta_n)T^n x_n, \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)[\phi(v, z_n) + \mu_n \tau_n + \eta_n]\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

where $\tau_n = \sup\{\tau(\phi(p, z_n)) : p \in F(T)\}$ with μ_n, η_n , and τ in (1.4), and J is the normalized duality mapping. Then $\{x_n\}$ converges in norm to $\prod_{F(T)} x_0$, where $\prod_{F(T)}$ is the generalized projection from X onto $F(T)$.

Proof. First, observe that H_n is closed and convex by Lemma 3.2, and that W_n is obviously closed and convex for each $n \geq 0$. Next we show that $F(T) \subset H_n$ for all n . Indeed, for $p \in F(T)$, using the convexity of $\|\cdot\|^2$ for the first inequality and (1.4) for the second inequality, we get

$$\begin{aligned} & \phi(p, y_n) = \phi(p, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1 - \alpha_n)JT^n z_n \rangle + \|\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2(1 - \alpha_n) \langle p, JT^n z_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|T^n z_n\|^2 \\ &= \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, T^n z_n) \\ &\leq \alpha_n \phi(p, x_n) + (1 - \alpha_n) [\phi(p, z_n) + \mu_n \tau(\phi(p, z_n)) + \eta_n] \\ &= \alpha_n \phi(p, x_n) + (1 - \alpha_n) [\phi(p, z_n) + \mu_n \tau_n + \eta_n]. \end{aligned}$$

So $p \in H_n$ for all n . Moreover, we show that

$$F(T) \subset H_n \cap W_n \tag{5.2}$$

for all $n \geq 0$. It suffices to show that $F(T) \subset W_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F(T) \subset C = W_0$. Assume that $F(T) \subset W_k$ for some $k \geq 1$. Since x_{k+1} is the generalized projection of x_0 onto $H_k \cap W_k$, by Proposition 3.1 (a) we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1} \rangle \geq 0$$

for all $z \in H_k \cap W_k$. As $F(T) \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of W_{k+1} implies that $F(T) \subset W_{k+1}$. Hence (5.2) holds for all $n \geq 0$. So, $\{x_n\}$ is well defined. Obviously, since $x_n = \prod_{W_n} x_0$ by the definition of W_n and Proposition 3.1 (a), and since $F(T) \subset W_n$, we have $\phi(x_n, x_0) \leq \phi(p, x_0)$ for all $p \in F(T)$. In particular, we obtain, for all $n \geq 0$,

$$\phi(x_n, x_0) \leq \phi(q, x_0), \quad \text{where } q := \prod_{F(T)} x_0. \quad (5.3)$$

Therefore, $\{\phi(x_n, x_0)\}$ is bounded; so is $\{x_n\}$ by (3.2). Consequently, $\{T^n x_n\}$ is bounded. Indeed, using (1.4) and (3.2), we have

$$(\|T^n x_n\| - \|q\|)^2 \leq \phi(q, T^n x_n) \leq \phi(q, x_n) + \mu_n \tau(\phi(q, x_n)) + \eta_n$$

for $n \geq 1$. Since $\{x_n\}$ is bounded and τ is increasing, the right hand side of above inequality should be bounded and so is $\{T^n x_n\}$. So $\{z_n\}$ is bounded. Therefore, $\{\tau_n\}$ is also bounded.

Noticing that $x_n = \prod_{W_n} x_0$ again and the fact that $x_{n+1} \in H_n \cap W_n \subset W_n$, we get

$$\phi(x_n, x_0) = \min_{z \in W_n} \phi(z, x_0) \leq \phi(x_{n+1}, x_0),$$

which shows that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing and so the $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists. Simultaneously, from Proposition 3.1 (b), we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi\left(x_{n+1}, \prod_{W_n} x_0\right) \leq \phi(x_{n+1}, x_0) - \phi\left(\prod_{W_n} x_0, x_0\right) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \rightarrow 0. \end{aligned} \quad (5.4)$$

By Proposition 3.6, we have

$$\|x_{n+1} - x_n\| \rightarrow 0. \quad (5.5)$$

Now since $x_{n+1} \in H_n$, we have

$$\begin{aligned} \phi(x_{n+1}, y_n) &\leq \alpha_n \phi(x_{n+1}, x_n) + (1 - \alpha_n)[\phi(x_{n+1}, z_n) + \mu_n \tau_n + \eta_n] \\ &= \phi(x_{n+1}, x_n) + (1 - \alpha_n)[\phi(x_{n+1}, z_n) - \phi(x_{n+1}, x_n) + \mu_n \tau_n + \eta_n] \\ &= \phi(x_{n+1}, x_n) + (1 - \alpha_n)[2\langle x_{n+1}, Jx_n - Jz_n \rangle + \|z_n\|^2 - \|x_n\|^2 \\ &\quad + \mu_n \tau_n + \eta_n]. \end{aligned} \quad (5.6)$$

On the other hand, since $\beta_n \rightarrow 0$, and $\{x_n\}, \{T^n x_n\}$ are bounded, we have

$$\|z_n - x_n\| = (1 - \beta_n)\|x_n - T^n x_n\| \rightarrow 0. \quad (5.7)$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have $\|Jx_n - Jz_n\| \rightarrow 0$. Hence, we have

$$\begin{aligned} & |2\langle x_{n+1}, Jx_n - Jz_n \rangle + \|z_n\|^2 - \|x_n\|^2| \\ & \leq 2\|x_{n+1}\| \cdot \|Jx_n - Jz_n\| + (\|z_n\| + \|x_n\|)(\|z_n - x_n\|) \rightarrow 0. \end{aligned} \quad (5.8)$$

Using (5.4), (5.8), and $\mu_n, \tau_n \rightarrow 0$ together with boundedness of $\{\tau_n\}$, we readily see that the right hand of (5.6) converges to 0; hence $\phi(x_{n+1}, y_n) \rightarrow 0$. Using Proposition 3.6 again, we obtain $\|x_{n+1} - y_n\| \rightarrow 0$. This, together with (5.5), yields that $\|x_n - y_n\| \rightarrow 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have $\|Jx_n - Jy_n\| \rightarrow 0$. Combining with $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and

$$\begin{aligned} Jx_n - Jy_n &= Jx_n - JJ^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n) \\ &= Jx_n - (\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n) \\ &= (1 - \alpha_n)(Jx_n - JT^n z_n) \end{aligned}$$

(from the definition of y_n) yields

$$\|Jx_n - JT^n z_n\| = \frac{1}{1 - \alpha_n} \|Jx_n - Jy_n\| \rightarrow 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have $\|x_n - T^n z_n\| \rightarrow 0$. This combined together with (5.1) and (5.7) yields that

$$\|x_n - T^n x_n\| \leq \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \rightarrow 0. \quad (5.9)$$

Then, uniform continuity of T again combined with (5.5) and (5.9) gives

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|TT^n x_n - Tx_n\| \rightarrow 0. \end{aligned} \quad (5.10)$$

By (5.10), $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. This, combined with (5.3) and Lemma 3.3 (with $K = F(T)$), guarantees that $x_n \rightarrow q = \prod_{F(T)} x_0$. \square

As a direct consequence of Theorem 5.1, Theorem KT due to Kim and Takahashi [18] will be modified as follows.

Corollary 5.2. [18] *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \rightarrow C$ be a uniformly equicontinuous mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ and $\{\beta_n\}$ are*

sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\beta_n \rightarrow 1$. Define a sequence $\{x_n\}$ in C by the algorithm:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n), \\ z_n = \beta_n x_n + (1 - \beta_n)T^n x_n, \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)[\phi(v, z_n) + \zeta_n]\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping and

$$\zeta_n = (k_n^2 - 1) \cdot \sup\{\phi(p, z_n) : p \in F(T)\}.$$

Then $\{x_n\}$ converges in norm to $\prod_{F(T)} x_0$, where $\prod_{F(T)}$ is the generalized projection from X onto $F(T)$.

Proof. If T is relatively asymptotically nonexpansive, $k_n = \sqrt{1 + \mu_n}$ is equivalent to $k_n^2 - 1 = \mu_n$. In this case, $\tau(t) = t$ for all $t \geq 0$ and $\eta_n = 0$ in (1.4). Therefore, $\mu_n \tau_n = \zeta_n$. Now the conclusion is immediately obtained by Theorem 5.1 \square

As another application of Theorem 5.1, we have the following.

Corollary 5.3. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \rightarrow C$ be a uniformly equicontinuous mapping which is relatively asymptotically nonexpansive in the intermediate sense. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\beta_n \rightarrow 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n z_n), \\ z_n = \beta_n x_n + (1 - \beta_n)T^n x_n, \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)[\phi(v, z_n) + c_n]\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping. Then $\{x_n\}$ converges in norm to $\prod_{F(T)} x_0$, where $\prod_{F(T)}$ is the generalized projection from X onto $F(T)$.

Proof. Noticing that $\mu_n = 0$ and $\eta_n = c_n$, the conclusion is immediately fulfilled from Theorem 5.1. \square

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