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Limit Distributions for Random Sums of Reciprocals of Independent Random Variables

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Abstract: Let $X_1, X_2, ...$ be a sequence of independent not necessary identically distributed continuous random variables. Conditions are found for the distribution functions of random sums of reciprocals of these random variables converge to a Cauchy distribution function.

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1 Introduction

The problem of limit distribution for convergence of the reciprocals of random variables are investigated by many authors (see for examples, Shapiro([8],[9],[10]), Termwutipong[11], Neammanee([3],[4],[5]) and Neammanee and Suntadkarn [6]. The problem can be stated as follows.

Let $X_1, X_2, ...$ be a sequence of independent continuous random variables. If r is a positive real number, find suitable conditions which guarantees that there exist sequences of real constants $(A_n(r))$ and $(B_n(r))$ such that the distribution functions of the sums

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|X_k|^r} - A_n(r)$$

converge to a limit.

In 1975-1977, Shapiro considered this problem in the case of $X_1, X_2, ...$ are identically distributed random variables. The limit distribution function is the normal distribution function for $0 < r \leq \frac{1}{2}$ and is a stable distribution function with characteristic exponent $\frac{1}{r}$ for $r > \frac{1}{2}$.

In 1986, Termwuttipong[11] considered the problem when $0 < r \leq \frac{1}{2}$ and shown that the limit distribution function is normal.

In 1988, Shapiro[10] considered the problem when the random variables are not necessary identically distributed and the parameter r is greater than $\frac{1}{2}$. In this case, the limit distribution function is also a stable distribution function with characteristic exponent $\frac{1}{r}$.

In this paper, we consider the random sums of the reciprocals of the random variables in the general form

$$\frac{1}{Z_n}\sum_{k=1}^{Z_n}\frac{1}{g(X_k)} - A_{Z_n}$$

where (Z_n) is a sequence of positive integers random variables with independent with (X_n) and g is a continuous function from an interval subset A of $\mathbb{R}$ into $\mathbb{R}$ which satisfied the following conditions:
$(g-1)$ there exists an a in A such that $g(a) = 0$,
$(g-2)$ g is strictly monotone on $A \cap (-\infty, a]$ and $A \cap [a, \infty)$,
$(g-3)$ g' exists and continuous on $(a - \delta^*, a + \delta^*)$ for some $\delta^* > 0$ and $g'(a)$ is positive.

The following is main theorem.

Main theorem *Assume that*

(i) $(f_k(x))$ *is equicontinuous at* a *and*

(ii) *the Cesaro limit* L *of the sequence* $(f_k(a))$ *exists and is positive.*

(iii) ${Z_n}^p \to \infty$.

Then there exists a sequence of real constants (A_n) *such that the distribution functions of the sums*

$$\frac{1}{Z_n}\sum_{k=1}^{Z_n}\frac{1}{g(X_k)} - A_{Z_n}$$

converge weakly to the Cauchy distribution function F *which defined by*

$$F(x) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan\frac{x}{\pi[g^{-1}(0)]'L}\Big).$$

2 Auxiliary Results

To prove the main theorem we often deal with the q-quantiles of the random variable Z_n which defined by $l_n(q) = \max\{k \in \mathbb{N} \mid P(Z_n < k) \le q\}$ for $q \in (0, 1)$. Clearly, l_n is well defined, nondecreasing in q and

$$P(Z_n < l_n(q)) \le q < P(Z_n < l_n(q)).$$

Next, we will give lemma and theorems which be basic in the proof of are main theorem.

Lemma 2.1. *If $Z_n \overset{p}{\to} \infty$ then $l_n(q) \to \infty$ for every $q \in (0,1)$.*

Proof. Fixes $q \in (0,1)$ and let M be a given positive integer. Since $\lim_{n\to\infty} P(Z_n \geq m) = 1$, there exists n_0 such that $n > n_0$ implies $P(Z_n \geq m) > 1 - q$. Then for $n > n_0$ $P(Z_n < m) < q$ This shows that for $n > n_0, l_n(q) \geq M$ □

Theorem 2.2. *Let (X_{nk}) , $k = 1, 2, \ldots, n = 1, 2, \ldots$ be a double sequence of random variables such that*

(2.2-1) $Z_n, X_{n1}, X_{n2}, \ldots$ are independent,

(2.2-2) there exist sequence (A_n) and (B_n) such that for every $q \in (0,1)$, the distribution functions of the sums of

$$\frac{1}{B_{l_n(q)}}\Big(X_{n1} + X_{n2} + \cdots + X_{nl_n(q)}\Big) - A_{l_n(q)}$$

weakly converge to a distribution function F. Then the distribution functions of random sums of

$$\frac{1}{B_{Z_n}}\Big(X_{n1} + X_{n2} + \cdots + X_{nZ_n}\Big) - A_{Z_n}$$

weakly converge to F.

Proof. For each n, let $F_n^{(j)}$ and $\overline{F_n}$ be the distribution functions of $\frac{1}{B_j}\Big(X_{n1} + X_{n2} + \cdots + X_{nj}\Big) - A_j$ and $\frac{1}{B_{Z_n}}\Big(X_{n1} + X_{n2} + \cdots + X_{nZ_n}\Big) - A_{Z_n}$ respectively. For each n, let Im $Z_n = \{k_{nj}\}$, numbered so that $k_{nj} < k_{n(j+1)}$. For each n and $j \in \mathbb{N}$, let $q_{nj} = \sum_{k=1}^{k_{nj}} P(Z_n = k)$ and $q_{n0} = 0$. Then for each $q \in [q_{n(j-1)}, q_{nj})$, we have $l_n(q) = k_{nj}$. Hence for a continuity point x of $\overline{F_n}$, we have

$$\begin{aligned}
\overline{F_n}(x) &= P\Big(\frac{1}{B_{Z_n}}\Big(X_{n1} + X_{n2} + \cdots + X_{nZ_n}\Big) - A_{Z_n} \leq x\Big) \\
&= \sum_{k_{nj} \in \mathrm{Im} Z_n} P\Big(\frac{1}{B_{k_{nj}}}\Big(X_{n1} + X_{n2} + \cdots + X_{nk_{nj}}\Big) - A_{k_{nj}} \leq x \wedge Z_n = k_{nj}\Big) \\
&= \sum_{k_{nj} \in \mathrm{Im} Z_n} P(Z_n = k_{nj}) F_n^{(k_{nj})}(x) \\
&= \sum_{k_{nj} \in \mathrm{Im} Z_n} (q_{nj} - q_{n(j-1)}) F_n^{(k_{nj})}(x) \\
&= \sum_{k_{nj} \in \mathrm{Im} Z_n} \int_{q \in [q_{n(j-1)}, q_{nj})} F_n^{(l_n(q))}(x)\, dq \\
&= \int_0^1 F_n^{(l_n(q))}(x)\, dq.
\end{aligned}$$

By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \overline{F_n}(x) = \int_0^1 \lim_{n\to\infty} F_n^{(l_n(q))}(x)\ dq = \int_0^1 F(x)\ dq = F(x). \qquad \square$$

In the proofs of main theorem we also need the classical theorems on convergence of a sequence of distribution functions of sums of independent random variables which stated as follows:

Theorem 2.3.(Gnedenko and Kolmogorov, p.116) *In order that for some suitably chosen constants* (A_n) *the distributions of the sums* $X_{n1}+X_{n2}+\cdots+X_{nk_n}-A_n$ *of independent infinitesimal random variables* (X_{nk}) *converge to a limit if and only if there exist non-decreasing functions* $M(x)$ *and* $N(x)$, *defined on the intervals* $(-\infty,0)$ *and* $(0,\infty)$ *respectively with* $M(-\infty)=0$ *and* $N(\infty)=0$ *and a constant* $\sigma \geq 0$ *such that*

(2.3-1) $\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x)$ *for a continuity point* x *of* M,

(2.3-2) $\lim_{n\to\infty} \sum_{k=1}^{k_n} [F_{nk}(x)-1] = N(x)$ *for a continuity point* x *of* N,

(2.3-3)
$$\lim_{\varepsilon\to\infty} \liminf_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_{|x|<\varepsilon} x^2\ dF_{nk}(x) - \left(\int_{|x|<\varepsilon} x\ dF_{nk}(x) \right)^2 \right\}$$
$$= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_{|x|<\varepsilon} x^2\ dF_{nk}(x) - \left(\int_{|x|<\varepsilon} x\ dF_{nk}(x) \right)^2 \right\}$$
$$= \sigma^2$$

where F_{nk} *denotes the distribution function of* X_{nk}.

A constant A_n *may be chosen according to the formula*

$$A_n = \sum_{k=1}^{n} \int_{|x|<\tau} x dF_{nk}(x)$$

where $-\tau$ *and* τ *are continuity points of* M *and* N *respectively.*

The logarithms of the characteristic function of the limit distribution is defined by

$$\ln \psi(t) = -\frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big) dM(x) + \int_{0}^{\infty} \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big) dN(x). \tag{1}$$

3 Proof of the Main Theorem

We shall prove main theorem only the case of g are both strictly increasing on $A\cap(-\infty,a]$ and $A\cap[a,\infty)$. In the other cases, we can use the same arguments. For each n and k, we let $X_{nk}=\dfrac{1}{l_n(q)g(X_k)}$ and F_{nk} be the distribution function of X_{nk}. We divide the proof into 5 steps as follows.

Step 1 We will show that

$$F_{nk}(x) = P(X_{nk}\le x) = P\Big(\frac{1}{l_n(q)g(X_k)}\le x\Big) = \begin{cases} F_k(a)-F_k\Big(g^{-1}\Big(\dfrac{1}{l_n(q)x}\Big)\Big) & \text{if } x<0 \text{ and } \dfrac{1}{l_n(q)x}\in \operatorname{Im} g \\ F_k(a) & \text{if } x=0 \\ 1+F_k(a)-F_k\Big(g^{-1}\Big(\dfrac{1}{l_n(q)x}\Big)\Big) & \text{if } x>0 \text{ and } \dfrac{1}{l_n(q)x}\in \operatorname{Im} g. \end{cases} \tag{2}$$

Case $x=0$

$$\begin{aligned} F_{nk}(x) &= P\Big(\frac{1}{l_n(q)g(X_{nk})}\le 0\Big)\\ &= P(g(X_{nk})\le 0)\\ &= P(X_k\le g^{-1}(0))\\ &= P(X_k\le a)\\ &= F_{nk}(a). \end{aligned}$$

Case $x<0$ and $\dfrac{1}{l_n(q)x}\in \operatorname{Im} g$

$$\begin{aligned} F_{nk}(x) &= P\Big(\frac{1}{l_n(q)g(X_{nk})}\le x\Big)\\ &= P\Big(\frac{1}{l_n(q)x}\le g(X_{nk})<0\Big)\\ &= P\Big(g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\le X_k<g^{-1}(0)\Big)\\ &= F_k(a)-F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big). \end{aligned}$$

Case $x > 0$ and $\dfrac{1}{l_n(q)x} \in \operatorname{Im} g$

$$\begin{aligned}F_{nk}(x) &= P\Big(\frac{1}{l_n(q)g(X_{nk})} \le 0\Big) + P\Big(0 < \frac{1}{l_n(q)g(X_{nk})} \le x\Big)\\ &= F_k(a) + P\Big(\frac{1}{l_n(q)x} \le g(X_{nk})\Big)\\ &= F_k(a) + P\Big(X_k \ge g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)\\ &= F_k(a) + 1 - F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big).\end{aligned}$$

Step 2 We show that $\lim\limits_{n\to\infty} \dfrac{F_k(a_n) - F_k(b_n)}{a_n - b_n} = f_k(a)$ uniform on k where $b_n < a_n$, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 1$ and f_k, F_k be the density functions and distribution functions of X_k, respectively.

Let ε be any positive number such that $\varepsilon < \min\{-g(a-\delta^*), g(a+\delta^*)\}$. Since $(f_k(x))$ is equicontinuous at a for and g^{-1} is continuous at 0, we have a positive number $\delta (\delta < \delta^*)$ and a natural number n_0 such that

$$|f_k(x) - f_k(a)| < \varepsilon \text{ for } |x-a| < \delta \text{ and all } k \in \mathbb{N} \tag{3}$$

and

$$g^{-1}\Big(-\frac{1}{n\varepsilon}\Big), g^{-1}\Big(\frac{1}{n\varepsilon}\Big) \in [a-\delta, a+\delta] \; for \; n \ge n_0. \tag{4}$$

Hence for $n > n_0$ we have

$$a - \delta < b_n < a_n < a + \delta. \tag{5}$$

By Mean Value Theorem, for each k there exists $x_{nk} \in (b_n, a_n)$ such that

$$\frac{F_k(a_n) - F_k(b_n)}{a_n - b_n} = f_k(x_{nk}).$$

Hence

$$\Big|\frac{F_k(a_n) - F_k(b_n)}{a_n - b_n} - f_k(a)\Big| = |f_k(x_{nk}) - f_k(a)| < \varepsilon.$$

Step 3 We show that a sequence $(X_{nk})\, k = 1, 2, \ldots, l_n(q)\ ; n = 1, 2, \ldots$ is infinitesimal for every $q \in (0,1)$ i.e. $\lim\limits_{n\to\infty} \max\limits_{1\le k\le l_n(q)} P(|X_{nk}| \ge \varepsilon) = 0$.

We note that

$$\begin{aligned}0 \le P(|X_{nk}| \ge \varepsilon) &= P(X_{nk} \le -\varepsilon) + P(X_{nk} \ge \varepsilon)\\ &= P(X_{nk} \le -\varepsilon) + 1 - P(X_{nk} \le \varepsilon)\\ &= F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big)\Big) - F_k\Big(g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big).\end{aligned} \tag{6}$$

By step 2 we have $\lim\limits_{n\to\infty} \dfrac{F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big)\Big) - F_k\Big(g^{-1}\Big(\frac{-1}{l_n(q)\varepsilon}\Big)\Big)}{g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(\frac{-1}{l_n(q)\varepsilon}\Big)} = f_k(a)$ uniform on k.

Hence

$$\left|\frac{F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big)\Big) - F_k\Big(g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big)}{g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)} - f_k(a)\right| < \varepsilon$$

for every k and $n \geq n_0$, i.e.

$$\begin{aligned}
&F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big)\Big) - F_k\Big(g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big) \\
&\leq \Big[g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big](\varepsilon + f_k(a)) \\
&= \varepsilon\Big[g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big] + l_n(q)\Big[g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big]\frac{f_k(a)}{l_n(q)}
\end{aligned} \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}
&\max_{1\leq k\leq l_n(q)} P(|X_{nk}| \geq \varepsilon) \\
&\leq \Big[g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big]\varepsilon + l_n(q)\Big[g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big]\max_{1\leq k\leq l_n(q)}\frac{f_k(a)}{l_n(q)}
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\max_{1\leq k\leq l_n(q)} P(|X_{nk}| \geq \varepsilon) \leq \lim_{n\to\varepsilon}(l_n(q))\Big[g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big]\lim_{n\to\infty}\max_{1\leq k\leq l_n(q)}\frac{f_k(a)}{l_n(q)}.$$

By the fact that $\lim\limits_{n\to\infty}(l_n(q))\Big(g^{-1}\Big(\frac{1}{l_n(q)\varepsilon}\Big) - g^{-1}\Big(-\frac{1}{l_n(q)\varepsilon}\Big)\Big)$ exists and (ii),

we can show that $\lim\limits_{n\to\infty}\max\limits_{1\leq k\leq l_n(q)}\dfrac{f_k(a)}{l_n(q)} = 0$ so we have $\lim\limits_{n\to\infty}\max\limits_{1\leq k\leq l_n(q)} P(|X_{nk}| \geq \varepsilon) = 0$.

Hence (X_{nk}) , $k = 1, 2, ..., n, n = 1, 2, ...$ is infinitesimal.

Step 4

Let $M : (-\infty, 0) \to \mathbb{R}$ and $N : (0, \infty) \to \mathbb{R}$ be defined by $M(x) = -[g^{-1}(0)]'\dfrac{L}{x}$ and $N(x) = -[g^{-1}(0)]'\dfrac{L}{x}$ where $[g^{-1}(0)]'$ is derivative of g^{-1} at 0. We shall show that conditions (3-1) and (3-2) of Theorem 3 are satisfied. Note that for $x < 0$ and sufficient large n

$$\Big|\sum_{k=1}^{l_n(q)}(F_{nk}(x))+[g^{-1}(0)]'\frac{L}{x}\Big|$$
$$=\Big|\sum_{k=1}^{l_n(q)}\Big(F_k(a)-F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)+[g^{-1}(0)]'\frac{L}{x}\Big|$$

$$\leq\Big|l_n(q)\Big(a-g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)\frac{1}{l_n(q)}\Big|\Big|\sum_{k=1}^{l_n(q)}\Big(\frac{F_k(a)-F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)}{a-g^{-1}\Big(\frac{1}{l_n(q)x}\Big)}-f_k(a)\Big)\Big|$$
$$+\Big|l_n(q)\Big(a-g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)\Big|\Big|\frac{1}{l_n(q)}\sum_{k=1}^{l_n(q)}f_k(a)-L\Big|$$
$$+\Big|l_n(q)\Big(a-g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)L+[g^{-1}(0)]'\frac{L}{x}\Big| \qquad (8)$$

Since

$$\lim_{n\to\infty}\Big[\frac{F_k(a)-F_k\Big(g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)}{a-g^{-1}\Big(\frac{1}{l_n(q)x}\Big)}-f_k(a)\Big]=0 \text{ uniform on } k, \qquad (9)$$

$\lim_{n\to\infty}\frac{1}{l_n(q)}\sum_{k=1}^{l_n(q)}f_k(a)=L$ and $\lim_{n\to\infty}l_n(q)\Big(a-g^{-1}\Big(\frac{1}{l_n(q)x}\Big)\Big)=\frac{-[g^{-1}(0)]'}{x}$ we have $\lim_{n\to\infty}\sum_{k=1}^{l_n(q)}F_{nk}(a)=-[g^{-1}(0)]'\frac{L}{x}$.

i.e.,condition (2.3-1) of Theorem 2.3 is satisfied. In the same way we can show that the condition (2.3-2) of Theorem 2.3 is satisfied for $N(x)=-[g^{-1}(0)]'\frac{L}{x}$.

Step 5

We will show that the condition (2.3-3) of Theorem 2.3 is also satisfied for $\sigma=0$. Since domain of g is an interval and g is continuous, Img is also an interval. In this case we shall show the condition (2.3-3) of Theorem 2.3 is satisfied where Im $g=(-\infty,\infty)$. In the other cases we can show in the same argument. Let δ

and n_0 defined as in step 2. Hence for $n \geq n_0$

$$\begin{aligned}
0 &\leq \sum_{k=1}^{l_n(q)} \Big\{ \int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x \, dF_{nk}(x) \Big)^2 \Big\} \\
&\leq \sum_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 \, dF_{nk}(x) \\
&= - \sum_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} x^2 \, d\Big(F_k \Big(g^{-1} \Big(\frac{1}{l_n(q)x} \Big) \Big) \Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{l_n(q)} \sum_{k=1}^{l_n(q)} \int_{|x|<\varepsilon} f_k \Big(g^{-1} \Big(\frac{1}{l_n(q)x} \Big) \Big) \Big[g^{-1} \Big(\frac{1}{l_n(q)x} \Big) \Big]' \, dx \\
&= \frac{1}{(l_n(q))^2} \sum_{k=1}^{l_n(q)} \Big[\lim_{t \to 0^-} \int_{g^{-1}\left(\frac{1}{l_n(q)t}\right)}^{g^{-1}\left(-\frac{1}{l_n(q)\varepsilon}\right)} \frac{f_k(y)}{(g(y))^2} \, dy + \lim_{t \to 0^+} \int_{g^{-1}\left(\frac{1}{l_n(q)\varepsilon}\right)}^{g^{-1}\left(\frac{1}{l_n(q)t}\right)} \frac{f_k(y)}{(g(y))^2} \, dy \Big] \\
&\quad \Big(y = g^{-1} \Big(\frac{1}{l_n(q)x} \Big) \Big) \\
&\leq \frac{1}{(l_n(q))^2} \sum_{k=1}^{l_n(q)} \Big[\int_{\inf A}^{g^{-1}\left(-\frac{1}{l_n(q)\varepsilon}\right)} \frac{f_k(y)}{(g(y))^2} \, dy + \int_{g^{-1}\left(\frac{1}{l_n(q)\varepsilon}\right)}^{\sup A} \frac{f_k(y)}{(g(y))^2} \, dy \Big] \\
&= \frac{1}{(l_n(q))^2} \sum_{k=1}^{l_n(q)} \Big[\int_{\inf A}^{a-\delta} \frac{f_k(y)}{(g(y))^2} \, dy + \int_{a-\delta}^{g^{-1}\left(-\frac{1}{l_n(q)\varepsilon}\right)} \frac{f_k(y)}{(g(y))^2} \, dy \\
&\quad + \int_{g^{-1}\left(\frac{1}{l_n(q)\varepsilon}\right)}^{a+\delta} \frac{f_k(y)}{(g(y))^2} \, dy + \int_{a+\delta}^{\sup A} \frac{f_k(y)}{(g(y))^2} \, dy \Big]. \qquad (10)
\end{aligned}$$

Since g is increasing and $g(y) < 0$ on $(\inf A, a - \delta)$,

$$\begin{aligned}
\int_{\inf A}^{a-\delta} \frac{f_k(y)}{(g(y))^2} dy &\leq \frac{1}{(g(a-\delta))^2} \int_{\inf A}^{a-\delta} f_k(y) dy \\
&\leq \frac{1}{(g(a-\delta))^2} \int_{-\infty}^{\infty} f_k(y) dy \\
&= \frac{1}{(g(a-\delta))^2}. \qquad (11)
\end{aligned}$$

Similarly we can show that

$$\int_{a+\delta}^{\sup A} \frac{f_k(y)}{(g(y))^2} dy \leq \frac{1}{(g(a+\delta))^2}. \qquad (12)$$

Note that there exist constants c_1 and c_2 such that

$$\begin{aligned}\int_{a-\delta}^{g^{-1}(-\frac{1}{l_n(q)\varepsilon})} \frac{f_k(y)}{(g(y))^2}dy &\le (\varepsilon + f_k(a)) \int_{a-\delta}^{g^{-1}(-\frac{1}{l_n(q)\varepsilon})} \frac{1}{(g(y))^2}dy \quad \text{(by(3)and(4))} \\ &= (\varepsilon + f_k(a)) \int_{g(a-\delta)}^{-\frac{1}{l_n(q)\varepsilon}} \frac{(g^{-1}(u))'}{u^2}du \quad (u = g(y)) \\ &= (\varepsilon + f_k(a))c_1 \int_{g(a-\delta)}^{-\frac{1}{l_n(q)\varepsilon}} \frac{1}{u^2}du \qquad \text{(by}(g\text{-3))} \\ &= (\varepsilon + f_k(a))c_1\Big((l_n(q))\varepsilon + \frac{1}{g(a-\delta)}\Big) \qquad (13)\end{aligned}$$

and

$$\int_{g^{-1}(\frac{1}{l_n(q)\varepsilon})}^{a+\delta} \frac{f_k(y)}{l_n(q)(g(y))^2}dy \le (\varepsilon + f_k(a))c_2\Big(l_n(q)\varepsilon - \frac{1}{g(a+\delta)}\Big). \qquad (14)$$

From (10)-(14) we have

$$\begin{aligned}0 &\le \sum_{k=1}^{l_n(q)} \Big(\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x dF_{nk}(x)\Big)^2\Big) \\ &\le \frac{1}{(l_n(q))^2} \sum_{k=1}^{l_n(q)} [c_3 + c_4(\varepsilon + f_k(a))(l_n(q))\varepsilon + c_5(\varepsilon + f_k(a))]\end{aligned}$$

for some constants c_3, c_4 and c_5. Hence, by (ii) we have

$$0 \le \limsup_{n\to\infty} \sum_{k=1}^{l_n(q)} \Big(\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x dF_{nk}(x)\Big)^2\Big) \le c_6\varepsilon$$

for some constant c_6 which implies that

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sum_{k=1}^{l_n(q)} \Big(\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x dF_{nk}(x)\Big)^2\Big) = 0.$$

Similarly we can show that

$$\lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} \sum_{k=1}^{l_n(q)} \Big(\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x dF_{nk}(x)\Big)^2\Big) = 0.$$

Hence the condition (2.3-3) of Theorem 2.3 is satisfied.
By Theorem 2.3 the distribution functions of the sums

$$\frac{1}{l_n(q)} \sum_{k=1}^{l_n(q)} \frac{1}{g(X_k)} - A_{l_n(q)}$$

converge to the distribution function which the logarithm of its characteristic function is defined by formula (1) where $\sigma = 0, M(x) = -[g^{-1}(0)]' \dfrac{L}{x}$ and $N(x) = -[g^{-1}(0)]' \dfrac{L}{x}$. From [Lukas, p.93] we know that the limit distribution function is Cauchy distribution function F which defined by

$$F(x) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan \frac{x}{\pi[g^{-1}(0)]'L}\Big).$$

Hence, by theorem 2.2, we have the main theorem. □

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