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Fixed Points and Quadratic ρ-Functional Inequalities in Banach Spaces

Gang Lu[1], **Gwang Hui Kim**[2] **and Choonkil Park**[3,*]

[1] *Division of Foundational Teaching, Guangzhou College of Technology and Business, Guangzhou 510850, P.R. China*
e-mail : lvgang1234@hanmail.net (G. Lu)

[2] *Department of Mathematics, Kangnam University, Gyeonggi 16979, Korea*
e-mail : ghkim@kangnam.ac.kr (G. H. Kim)

[3] *Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea*
e-mail : baak@hanyang.ac.kr (C. Park)

Abstract In this paper, we solve the quadratic ρ-functional inequalities

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\|,$$

where ρ is a fixed number with $|\rho|<1$, and

$$\left\|4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right\| \le \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|,$$

where ρ is a fixed number with $|\rho|<\frac{1}{2}$. Using the fixed point method, we prove the Hyers-Ulam stability of the above quadratic ρ-functional inequalities in complex Banach spaces.

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1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved

*Corresponding author.

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by Skof [6] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

The functional equation $4f\left(\frac{x+y}{2}\right) + (x-y) = 2f(x) + 2f(y)$ is called a *Jensen type quadratic equation*. Park [8, 9] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [10–16]).

We recall a fundamental result in fixed point theory.

Theorem 1.1 ([17, 18])**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y^*) \le \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [20–24]).

In Section 2, we solve the quadratic ρ-functional inequality

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \le \left\|\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\|, \end{aligned} \tag{1.1}$$

where ρ is a fixed complex number with $|\rho| < 1$, and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (1.1) in Banach spaces by using the fixed point method. In Section 3, we solve the quadratic ρ-functional inequality

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\quad \le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \end{aligned} \tag{1.2}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$, and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (1.2) in Banach spaces by using the fixed point method.

Throughout this paper, let G be a 2-divisible abelian group. Assume that X is a real or complex normed space with norm $\|\cdot\|$ and that Y is a complex Banach space with norm $\|\cdot\|$.

2. Quadratic ρ-Functional Inequality (1.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$. In this section, we solve and investigate the quadratic ρ-functional inequality (1.1) in complex Banach spaces.

Lemma 2.1. *If a mapping $f : G \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad \leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\| \end{aligned} \tag{2.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (2.1). Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$. Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in G$. It follows from (2.1) and (2.2) that

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad \leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\| \\ &\quad = |\rho|\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \end{aligned}$$

and so

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all $x, y \in G$. ■

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4}\varphi\left(x, y\right) \tag{2.3}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\| + \varphi(x,y) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{4(1-L)}\varphi\left(x, x\right)$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x) \tag{2.5}$$

for all $x \in X$. Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi\left(x, x\right), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [25]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leq 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\leq 4\varepsilon\frac{L}{4}\varphi(x,x) = L\varepsilon\varphi(x,x) \end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$. It follows from (2.5) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{4}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{4}$. By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \leq \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{4(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \\
&= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 4^n |\rho| \left\| 4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= |\rho| \left\| 4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \\
&\quad \le \left\| \rho\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. ■

Corollary 2.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\| \rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r)
\end{aligned} \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $L = 2^{2-r}$ and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. ■

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.8}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (2.5) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x,x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. ■

Corollary 2.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4-2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $L = 2^{r-2}$ and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. ■

Remark 2.6. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Quadratic ρ-Functional Inequality (1.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we solve and investigate the quadratic ρ-functional inequality (1.2) in complex Banach spaces.

Lemma 3.1. *If a mapping $f : G \to Y$ satisfies*

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\quad \le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \end{aligned} \tag{3.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (3.1). Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \le |\rho|\|2f(0)\|$. So $f(0) = 0$. Letting $y = 0$ in (3.1), we get $\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le 0$ and so

$$4f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in G$. It follows from (3.1) and (3.2) that

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &= \left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\le |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. ■

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying (2.3). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right\| \tag{3.3}$$
$$\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \varphi(x,y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L}\varphi(x,x)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \varphi(x,0) \tag{3.4}$$

for all $x \in X$. Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x,0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [25]). We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. ∎

Corollary 3.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right\| \tag{3.5}$$
$$\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $L = 2^{2-r}$ and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. ∎

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying (2.8). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{L}{1-L}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$. It follows from (3.4) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. ■

Corollary 3.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{4 - 2^r}\|x\|^r \tag{3.6}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $L = 2^{2-r}$ and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. ■

Remark 3.6. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

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