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The Algebraic Structure of a Semigroup of Sets of Transformations with Restricted Range

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Abstract We study a semigroup which represents a semigroup of sets of Boolean functions on a finite set using the concept of transformations with restricted range. For this semigroup, we determine the algebraic structure. In particular, we characterize the (left, right, and two-sided) ideals and the Green's relations. Moreover, for each of the Green's relations, we provide the greatest included congruence.

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1. Introduction

In algebra and also in other fields of Mathematics, associative operations on a given set of mathematical objects are defined and the resulting structures, which are called semigroups, are studied. Let us mention for example the following sets of mathematical objects: transformations on a given set (transformation semigroups); partitions (diagram monoids); matrices of given type (matrices semigroups); Boolean operations on a given set; but also linear transformations in a Hilbert space. In particular, the set on which the associative operation is defined can be the power set of a given set. For example, such semigroups were already studied in the case of tree languages [1] and in the case of Boolean operations on a finite set [2, 3]. In the present paper, we consider a semigroup whose universe consists of sets of full transformations on a finite set, where the image is in a fixed two-element subset of this set. This semigroup can be regarded as a representation of a semigroup of sets of Boolean operations. This justifies that we restrict us to transformations with image in a two-element set. The purpose of this paper is a description of the algebraic structure of this semigroup. First results are published in [4]. Let X be a finite set, let $Y := \{y_1, y_2\}$ be a two-element subset of X, and denote by $T(X, Y)$ the

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semigroup (under composition) of all full transformations on the set X with image in Y. This semigroup is called semigroup of transformations with restricted range Y [5] and is studied by several authors [6–10]. It is interesting to note that several subsets of $T(X, Y)$ have an important interpretation in the automata theory and thus in the wider sense also in the Theoretical Computer Sciences, namely as non-deterministic transformations. A non-deterministic transformation from X in Y is a mapping α^{nd} from X in the set $\{Y, \{y_1\}, \{y_2\}\}$ of all non-empty subsets of Y [11]. In an algebraic setting, α^{nd} can be regarded as a set of transformations $\{\alpha \in T(X, Y) : x\alpha \in x\alpha^{nd}$ for all $x \in X\}$, i.e. as an element of the set $T_P(X, Y)$ of all non-empty subsets of $T(X, Y)$.
In a canonical way, one can define an associative operation $\cdot$ on the set $T_P(X, Y)$ by

$$A \cdot B := \{\alpha\beta \ : \ \alpha \in A, \beta \in B\},$$

i.e. $T_P(X, Y)$ forms a semigroup under the operation $\cdot$. We will write AB rather than $A \cdot B$. For sets $K, L \subseteq T_P(X, Y)$, we put $KL := \{AB \mid A \in K, B \in L\}$. If K is a singleton set $K = \{A\}$ then we write AL rather than $\{A\}L$. Dually, we write KB rather than $K\{B\}$, whenever L is a singleton set $\{B\}$. It is interesting to note that the set of non-deterministic transformations from X to Y forms a subsemigroup of $T_P(X, Y)$. On the other hand, each semigroup of binary relations on X with codomain in Y is a subsemigroup of $T_P(X, Y)$. Note that semigroups of binary relations on X are studied intensively. In particular, R. J. Plemmons and M. T. West characterized the Green's relations [12].
In 2016, the authors of this paper determined the maximal idempotent subsemigroups and the maximal regular subsemigroups of $T_P(X, Y)$ [13]. The semigroup $T_P(X, Y)$ had been already studied by Y. Susanti several years ago, but from a different point of view, namely as the multiplicative reduct of the semiring associated to the semigroup of Boolean operations on a finite set. In particular, she determined the k-regular elements of $T_P(X, Y)$ in [3].
For the study of $T_P(X, Y)$, the structure of the monoid $T(X, Y)$ is central. Therefore, we summarize several important facts about $T(X, Y)$. Since $|Y| = 2$, it is obvious to define a unary operation $*$ on $T(X, Y)$ by

$$x\alpha^* := \begin{cases} y_1 & \text{if} \quad x\alpha = y_2; \\ y_2 & \text{if} \quad x\alpha = y_1. \end{cases}$$

Clearly, $(\alpha^*)^* = \alpha$ for any $\alpha \in T(X, Y)$ and we put $A^* := \{\alpha^* \mid \alpha \in A\}$ for $A \in T_P(X, Y)$. Moreover, α restricted to Y is one of the four full transformations on Y: $\begin{pmatrix} y_1 & y_2 \\ y_1 & y_2 \end{pmatrix}$, $\begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix}$, $\begin{pmatrix} \overline{y_1 y_2} \\ y_1 \end{pmatrix}$, and $\begin{pmatrix} \overline{y_1 y_2} \\ y_2 \end{pmatrix}$. We can decompose $T(X, Y)$ in four sets, namely in

$$\begin{aligned} &\mathbf{T}_1 := \{\alpha \in T(X, Y) \ : \ y_i\alpha = y_i \text{ for } i = 1, 2\}, \\ &\mathbf{T}_2 := \mathbf{T}_1^*, \\ &\mathbf{T}_3 := \{\alpha \in T(X, Y) : y_1\alpha = y_2\alpha = y_1\}, \quad \text{and} \\ &\mathbf{T}_4 := \mathbf{T}_3^* = \{\alpha \in T(X, Y) : y_1\alpha = y_2\alpha = y_2\}. \end{aligned}$$

Recall that $T(X, Y)$ is a so-called 4-part semigroup [2]. This concept of 4-part semigroup is essential in the present paper and can be used only in the case $|Y| = 2$. For solving similar problems in the case $|Y| > 2$, one has to search for other tools. Analogously, one can decompose any $A \in T_P(X, Y)$ in four (also possible empty) sets:

$$A_i := A \cap \mathbf{T}_i \text{ for } i = 1, 2, 3, 4.$$

It is easy to verify that $\alpha\beta = \alpha$ and $\alpha\beta^* = \alpha^*$ for any $\alpha \in T(X,Y)$ and $\beta \in \mathbf{T}_1$. In particular, $\mathbf{T}_1$ operates as a right-identity element, more precisely, for any $A, B \in T_P(X,Y)$ with $B_1 \neq \emptyset$, it holds $A \subseteq AB$ and $A = AB$, whenever $B_1 = B$. This becomes clear by the fact that $\mathbf{T}_1$ consists of all idempotents in $T(X,Y)$, except of the constant mappings. So, there are exactly $2^{2^{n-2}} - 1$ right-identity elements in $T_P(X,Y)$. Note that $T(X,Y)$ contains exactly two constant mappings, denoted by c_1 and c_2, with the image y_1 and y_2, respectively. Clearly, $c_1^* = c_2$ and $AB = \{c_i\}$ for any $A \in T_P(X,Y)$ and $B \subseteq \mathbf{T}_{2+i}$, $i \in \{1,2\}$. This shows that $T_P(X,Y)$ has no identity element and it is not a monoid, but becomes a monoid $T_P(X,Y)^1$ adding an identity element, denoted by $\mathbf{1}$.

The structure of a semigroup can be described very well by its Green's relations as well as by the ideals. S. Mendes-Gonçalves and R. P. Sullivan [8] described the ideal structure of the semigroups with restricted range. The Green's relations on these semigroups were described by J. Sanwong and W. Sommanee [9]. In particular, a characterization of all minimal and maximal congruences on a semigroup of transformations with restricted range is given in [14]. Here, we will characterize the Green's relations and their greatest included congruences as well as the (left, right, and two-sided) ideals of the semigroup $T_P(X,Y)$. Recall that a subset $I \subseteq T_P(X,Y)$ is called left (right) ideal if $T_P(X,Y)I \subseteq I$ (and $IT_P(X,Y) \subseteq I$, respectively). If I is both a left ideal and a right ideal then it is called two-sided ideal (for short: ideal). In particular, we have $T_P(X,Y)IT_P(X,Y) \subseteq I$, whenever I is an ideal. Let $\mathcal{L}, \mathcal{R}$, $\mathcal{H}$, and $\mathcal{J}$ $(= \mathcal{D})$ be the Green's relations on $T_P(X,Y)$, with other words

$A\mathcal{L}B \Leftrightarrow \exists P, Q \in T_P(X,Y)^1\big(PA = B, QB = A\big)$,

$A\mathcal{R}B \Leftrightarrow \exists P, Q \in T_P(X,Y)^1\big(AP = B, BQ = A\big)$,

$A\mathcal{H}B \Leftrightarrow A\mathcal{L}B$ and $A\mathcal{R}B$, and

$A\mathcal{J}B \Leftrightarrow \exists P_1, P_2, Q_1, Q_2 \in T_P(X,Y)^1\big(P_1AP_2 = B, Q_1BQ_2 = A\big)$.

Since $\mathcal{J} = \mathcal{D}$, we have $\mathcal{J} = \mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L} = \{(A,B) :$ there is $D \in T_P(X,Y)$ with $(A,D) \in \mathcal{L}$ and $(D,B) \in \mathcal{R}\}$. One of the purposes of this paper is the characterization of the Green's relations. Sets containing only constant mappings have particular properties. It is routine to verify that the set of the non-empty subsets of the set $\mathbb{C} := \{c_1, c_2\}$ of constant mappings forms an $\mathcal{R}$-class as well as a $\mathcal{J}$-class. But each non-empty subset $A \subseteq \mathbb{C}$ forms a singleton $\mathcal{L}$-class $\{A\}$ but not an $\mathcal{R}$-class. Therefore, the main work will be the description of the ρ-classes, for $\rho \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}\}$, whose representatives A do not be contained in $\mathbb{C}$, i.e. $A_{\mathbb{C}} := A \cap \mathbb{C} \neq A$.

Notice that neither the right-congruence $\mathcal{L}$ nor the left-congruence $\mathcal{R}$ nor one of the relations $\mathcal{H}$ and $\mathcal{J}$ is a congruence in general. But we determine the greatest congruence $\rho^{\bigcirc}$ contained in ρ, i.e.

$$\rho^{\bigcirc} := \big\{(A,B) \in \rho \;:\; \forall P, Q \in T_P(X,Y)^1\big((PAQ, PBQ) \in \rho\big)\big\}$$

will be described, for $\rho \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}\}$. The given description will base on the characterization of the respective Green's relation.

In the next section, we characterize the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$, and $\mathcal{J}$, respectively. Section 3 is devoted to the ideal structure of $T_P(X,Y)$. We characterize the left ideals as well as the right ideals. In particular, we are able to provide a constructive description of the ideals and the principal ideals.

2. Green's Relations

We begin with the characterization of the relation $\mathcal{R}$ and of the greatest congruence $\mathcal{R}^{\bigcirc}$ contained in $\mathcal{R}$. As already mentioned, $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$ is an $\mathcal{R}$-class. The characterization of the relation $\mathcal{R}$ will show that the remaining $\mathcal{R}$-classes contain at most two elements.

Lemma 2.1. *Let $A, B \in T_P(X, Y)$. If $A\mathcal{R}B$ then $A, B \subseteq \mathbb{C}$ or $|A_{\mathbb{C}}| = |B_{\mathbb{C}}|$.*

Proof. Suppose that $A\mathcal{R}B$ with $|A_{\mathbb{C}}| \neq |B_{\mathbb{C}}|$, say $|A_{\mathbb{C}}| < |B_{\mathbb{C}}|$. Then there is a set $V \in T_P(X, Y)$ with $A = BV$. Since $|(B_{\mathbb{C}})\alpha| = |B_{\mathbb{C}}|$ for any $\alpha \in \mathbf{T}_1 \cup \mathbf{T}_2$, from $A = BV$ and $|A_{\mathbb{C}}| < |B_{\mathbb{C}}|$, we conclude $V \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$ and thus $A = BV \subseteq \mathbb{C}$. Since $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$ forms an $\mathcal{R}$-class, we have $B \subseteq \mathbb{C}$, too. ∎

Proposition 2.2. *Let $A, B \in T_P(X, Y)$ with $A \neq B$. Then $A\mathcal{R}B$ if and only if $A, B \subseteq \mathbb{C}$ or $B = A^*$.*

Proof. Suppose that $A\mathcal{R}B$. Then there is $V \in T_P(X, Y)$ with $B = AV$. If $V_1 \cup V_2 = \emptyset$ then $B = AV \subseteq A(\mathbf{T}_3 \cup \mathbf{T}_4) = \mathbb{C}$ and thus $A \subseteq \mathbb{C}$ since $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$ is an $\mathcal{R}$-class. If $V_1 \neq \emptyset$ and $V_2 = \emptyset$ then $B = AV = A \cup A(V_3 \cup V_4)$, which provides $A, B \subseteq \mathbb{C}$. Otherwise, we have $|A_{\mathbb{C}}| = |B_{\mathbb{C}}|$ by Lemma 2.1 and $B = A \cup A(V_3 \cup V_4)$ implies $A(V_3 \cup V_4) \subseteq A$, i.e. $A = B$, a contradiction. It remains to consider that either $V_1 = \emptyset$ and $V_2 \neq \emptyset$ or $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, i.e. $B = A^* \cup A(V_3 \cup V_4)$ or $B = A \cup A^* \cup A(V_3 \cup V_4)$, respectively. Suppose that $A, B \nsubseteq \mathbb{C}$. Lemma 2.1 implies $|A_{\mathbb{C}}| = |B_{\mathbb{C}}|$, i.e. $A(V_3 \cup V_4) \subseteq A^*$ and $A(V_3 \cup V_4) \subseteq A \cup A^*$, respectively. Thus, $B = A^*$ or $B = A \cup A^*$. Dually, we obtain $A = B^*$ (i.e. $B = A^*$) or $A = B \cup B^*$, whenever $A, B \nsubseteq \mathbb{C}$. The case $A = B \cup B^*$ and $B = A \cup A^*$ is not possible. Otherwise, we have $A = B \cup B^* = B \cup (A \cup A^*)^* = B \cup (A \cup A^*) = B \cup B = B$, a contradiction.
The converse direction is clear. ∎

An immediately consequence of Proposition 2.2 is the fact that the $\mathcal{R}$-classes are of the form $\{A, A^*\}$ for all $A \in T_P(X, Y)$ with $A \neq A_{\mathbb{C}}$. Obviously, $\mathcal{R}$ is not a congruence. But we can verify that $\mathcal{R}^{\bigcirc}$ consists of all non-singleton classes $\{A, A^*\}$ with $\mathbb{C} \subsetneq A \in T_P(X, Y)$, together with $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$, as the following proposition will show.

Proposition 2.3. $\mathcal{R}^{\bigcirc} = \{(A, A) : A \in T_P(X, Y)\} \cup \{(A, B) : \emptyset \neq A, B \subseteq \mathbb{C}\} \cup \{(A, A^*) : \mathbb{C} \subseteq A \in T_P(X, Y)\}$.

Proof. Clearly, $\{(A, A) : A \in T_P(X, Y)\} \subseteq \mathcal{R}^{\bigcirc}$. Let $U, V \in T_P(X, Y)^1$. Further, let $\emptyset \neq A, B \subseteq \mathbb{C}$. Then $(A, B) \in \mathcal{R}$ and $UAV, UBV \subseteq \mathbb{C}$ and $(UAV, UBV) \in \mathcal{R}$ by Proposition 2.2. Let now $\mathbb{C} \subseteq A \in T_P(X, Y)$. Then $(A, A^*) \in \mathcal{R}$. Since $A\mathbf{1} = A\mathbf{T}_1$, we can skip the case $V = \mathbf{1}$. So, we have $UAV = (UA)V_1 \cup (UA)V_2 \cup UA(V_3 \cup V_4)$ and $UBV = (UB)V_1 \cup (UB)V_2 \cup UB(V_3 \cup V_4)$ for $B := A^*$. If $V_1 \cup V_2 = \emptyset$ then $UAV, UBV \subseteq \mathbb{C}$, i.e. $(UAV, UBV) \in \mathcal{R}$ by Proposition 2.2. Admit that $V_1 \cup V_2 \neq \emptyset$. Then $UA(V_3 \cup V_4) \subseteq \mathbb{C} \subseteq A \subseteq UAV_1 \cup UAV_2$ as well as $UB(V_3 \cup V_4) \subseteq \mathbb{C} \subseteq B \subseteq UBV_1 \cup UBV_2$. If $V_1 \neq \emptyset$ and $V_2 = \emptyset$ then $AV = A$ and $BV = B$. If $V_1 = \emptyset$ and $V_2 \neq \emptyset$ then $AV = A^* = B$ and $BV = B^* = A$. In both cases, we have $(AV, BV) \in \mathcal{R}$ and thus $(UAV, UVB) \in \mathcal{R}$ since $\mathcal{R}$ is a left-congruence. Finally, if $V_1, V_2 \neq \emptyset$ then $UAV = UAV_1 \cup UAV_2 = UA \cup UA^* = UB^* \cup UB = UBV_2 \cup UBV_1 = UBV$, i.e. $(UAV, UBV) \in \mathcal{R}$.
Conversely, let $(A, B) \in \mathcal{R}^{\bigcirc}$. Then, $(A, B) \in \mathcal{R}$. By Proposition 2.2, it remains to consider the case $A = B^*$. Assume that $\mathbb{C} \nsubseteq A$ (and thus $B \neq A \neq A_{\mathbb{C}}$). Without loss of generality, we assume that $c_1 \notin A$. We choose $U = \mathbf{1}$ and $V = \mathbf{T}_1 \cup \{c_1\}$, i.e.

$UAV = A \cup \{c_1\}$ and $UBV = B \cup \{c_1\}$. Note that $c_1^* = c_2 \notin B = A^*$. This shows $(A \cup \{c_1\})^* = A^* \cup \{c_2\} = B \cup \{c_2\}$. Thus, $(A \cup \{c_1\}, B \cup \{c_1\}) \notin \mathcal{R}$ by Proposition 2.2, i.e. $(UAV, UBV) \notin \mathcal{R}$, a contradiction. ∎

On the other hand, we have eight $\mathcal{L}$-classes of size greater than one. It is a straightforward consequence of the following proposition characterizing the relation $\mathcal{L}$. Only the sets $A \in T_P(X,Y)$ with either $A \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$ or $A_1, A_2 \neq \emptyset$ and $A \setminus A_{\mathbb{C}} \neq (A \setminus A_{\mathbb{C}})^*$ form singleton $\mathcal{L}$-classes.

Proposition 2.4. *Let $A, B \in T_P(X,Y)$ with $A \neq B$. Then $A\mathcal{L}B$ if and only if the following four statements are satisfied:*
(i) $A_{i+2} \neq \emptyset \iff B_{i+2} \neq \emptyset \iff c_i \in A \cap B$ for $i \in \{1,2\}$;
(ii) $A_1 \cup A_2 \neq \emptyset$ and $B_1 \cup B_2 \neq \emptyset$;
(iii) $A_1 = \emptyset$ or $A_2 = \emptyset$ if and only if $B_1 = \emptyset$ or $B_2 = \emptyset$;
(iv) if $A_1, A_2 \neq \emptyset$ then $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$.

Proof. Suppose that $A\mathcal{L}B$. Then there are $V, W \in T_P(X,Y)$ such that $A = VB$ and $B = WA$. This implies $A_{\mathbb{C}} = B_{\mathbb{C}}$.
Let $i \in \{1,2\}$ and admits that $A_{i+2} \neq \emptyset$. Then we obtain $c_i \in WA = B$, i.e. $c_i \in B_{\mathbb{C}} = A_{\mathbb{C}} \subseteq A \cap B$. Thus, $A_{i+2} \neq \emptyset$ if and only if $c_i \in A \cap B$. Dually, we can check that $B_{i+2} \neq \emptyset$ if and only if $c_i \in A \cap B$. This shows (i).
Assume that $A_1 \cup A_2 = \emptyset$. Then $B = WA \subseteq W(\mathbf{T}_3 \cup \mathbf{T}_4) \subseteq \mathbb{C}$ and thus $A = VB \subseteq V\mathbb{C} \subseteq \mathbb{C}$. This provides $A = A_{\mathbb{C}} = B_{\mathbb{C}} = B$, a contradiction. Hence, $A_1 \cup A_2 \neq \emptyset$ and we can show $B_1 \cup B_2 \neq \emptyset$ dually. So, (ii) is shown.
We observe that $W \subseteq WA = B$ if $A_1 \neq \emptyset$, $V \subseteq VB = A$ if $B_1 \neq \emptyset$, $W^* \subseteq WA = B$ if $A_2 \neq \emptyset$, and $V^* \subseteq VB = A$ if $B_2 \neq \emptyset$. Thus, $W \cup W^* \subseteq WA$ and $V \cup V^* \subseteq VB$, whenever $A_1, A_2 \neq \emptyset$ and $B_1, B_2 \neq \emptyset$, respectively. Let $A_1 = \emptyset$ or $A_2 = \emptyset$ and assume that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$. Then $V \cup V^* \subseteq VB = A$ implies $V \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$, i.e. $A \subseteq (\mathbf{T}_3 \cup \mathbf{T}_4)B \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$, a contradiction to (ii). Hence, $B_1 = \emptyset$ or $B_2 = \emptyset$. Dually, we can show the converse implication in (iii). Consequently, we have (iii).
Suppose that $A_1, A_2 \neq \emptyset$. Then $W(A_1 \cup A_2) = W \cup W^*$ and $B = WA = W(A_1 \cup A_2) \cup W(A_3 \cup A_4) = W \cup W^* \cup W(A_3 \cup A_4)$, where $W(A_3 \cup A_4) \subseteq B_{\mathbb{C}}$. Thus, $B \setminus B_{\mathbb{C}} = (W \cup W^*) \setminus \mathbb{C}$. If $(W \cup W^*) \cap \mathbb{C} = \emptyset$ then $B \setminus B_{\mathbb{C}} = (W \cup W^*) \setminus \mathbb{C} = W \cup W^* = (W \cup W^*)^* = ((W \cup W^*) \setminus \mathbb{C})^* = (B \setminus B_{\mathbb{C}})^*$. If $(W \cup W^*) \cap \mathbb{C} \neq \emptyset$ then $\mathbb{C} \subseteq W \cup W^*$, $B_{\mathbb{C}} \subseteq \mathbb{C}$, and it is easy to verify that $((W \cup W^*) \setminus \mathbb{C})^* = (W \cup W^*) \setminus \mathbb{C}$. Thus, $(B \setminus B_{\mathbb{C}})^* = B \setminus B_{\mathbb{C}}$. So, (iv) is shown.
For the converse direction, we assume that $(i), (ii), (iii)$, and (iv) hold. Suppose that $A_1 \neq \emptyset$ and $A_2 = \emptyset$. Then we have $BA = B \cup B(A_3 \cup A_4)$. By (i), we can calculate that $B(A_3 \cup A_4) \subseteq B$ and thus $BA = B$. We can verify that $B^*A = B$, whenever $A_1 = \emptyset$ and $A_2 \neq \emptyset$, similarly. Because of (ii) and (iii), it remains the case $A_1, A_2 \neq \emptyset$. Here, we have $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$ by (iv). Moreover, we obtain $(B \setminus B_{\mathbb{C}})(A_3 \cup A_4) = B_{\mathbb{C}}$ by (i). This provides $(B \setminus B_{\mathbb{C}})A = (B \setminus B_{\mathbb{C}}) \cup (B \setminus B_{\mathbb{C}})^* \cup (B \setminus B_{\mathbb{C}})(A_3 \cup A_4) = (B \setminus B_{\mathbb{C}}) \cup B_{\mathbb{C}} = B$. Daully, we can conclude that $AB = A$ or $A^*B = A$ or $B_1, B_2 \neq \emptyset$. In the later case, we can verify that $A \setminus A_{\mathbb{C}} = (A \setminus A_{\mathbb{C}})^*$ and thus $(A \setminus A_{\mathbb{C}})B = A$ by the same arguments as above. Consequently, $A\mathcal{L}B$. ∎

Splitting four of the eight non-singleton $\mathcal{L}$-classes, one obtains a congruence.

Proposition 2.5. *$\mathcal{L}^{\bigcirc}$ is the set of all $(A,B) \in T_P(X,Y)^2$ such that $A = B$ or the following four properties are satisfied:*

(i) $A_{i+2} \neq \emptyset \Longleftrightarrow B_{i+2} \neq \emptyset \Longleftrightarrow c_i \in A \cap B$ *for* $i \in \{1,2\}$*;*
(ii) $A_1 \cup A_2 \neq \emptyset$ *and* $B_1 \cup B_2 \neq \emptyset$*;*
(iii) $A_i = \emptyset$ *if and only if* $B_i = \emptyset$ *for* $i \in \{1,2\}$*;*
(iv) if $A_1, A_2 \neq \emptyset$ *then* $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$*.*

Proof. Suppose that $(A,B) \in \mathcal{L}^{\bigcirc}$. Then $(A,B) \in \mathcal{L}$ and we have $A = B$ or all the properties, except of (iii), are satisfied. So, we have to show (iii). Assume that $A \neq B$ and there is $i \in \{1,2\}$ with $A_i = \emptyset$ and $B_i \neq \emptyset$ or conversely. Without loss of generality, we can assume that $A_1 = \emptyset$ and $B_1 \neq \emptyset$. Then $A_2 \neq \emptyset$ by (ii) and we can see that $\mathbf{T}_3A\mathbf{1} = \mathbf{T}_3A = \mathbf{T}_3^* \cup \mathbb{C} = \mathbf{T}_4 \cup \mathbb{C}$. On the other hand, we have $\mathbf{T}_3 \subseteq \mathbf{T}_3B = \mathbf{T}_3B\mathbf{1}$. This shows $\mathbf{T}_3A\mathbf{1} \neq \mathbf{T}_3B\mathbf{1}$ but $(\mathbf{T}_3A\mathbf{1})_1 \cup (\mathbf{T}_3A\mathbf{1})_2 = \emptyset$, i.e. $(\mathbf{T}_3A\mathbf{1}, \mathbf{T}_3B\mathbf{1}) \notin \mathcal{L}$ by Proposition 2.4, a contradiction.
Conversely, suppose that $A = B$ or that $(i), (ii), (iii)$, and (iv) are valid. Since $A = B$ implies $(A,B) \in \mathcal{L}^{\bigcirc}$, we have only to consider the latter case. Let $U, V \in T_P(X,Y)^1$. First, we admit that U is not the identity element. Then we calculate

$$U(A_1 \cup A_2), U(B_1 \cup B_2) = \begin{cases} U & \text{if} \quad A_1, B_1 \neq \emptyset \quad \text{and} \quad A_2, B_2 = \emptyset \\ U^* & \text{if} \quad A_1, B_1 = \emptyset \quad \text{and} \quad A_2, B_2 \neq \emptyset \\ U \cup U^* & \text{if} \quad A_1, B_1 \neq \emptyset \quad \text{and} \quad A_2, B_2 \neq \emptyset. \end{cases}$$

By (ii) and (iii), we have shown $U(A_1 \cup A_2) = U(B_1 \cup B_2)$. Further, we have $U(A_3 \cup A_4) = \{c_i \ : \ i \in \{1,2\}, A_{i+2} \neq \emptyset\} = \{c_i \ : \ i \in \{1,2\}, B_{i+2} \neq \emptyset\} = U(B_3 \cup B_4)$ by (i). Consequently, $UA = UB$, $UAV = UBV$, and thus $(UAV, UBV) \in \mathcal{L}$. Finally, we have to check the case $U = \mathbf{1}$. It is enough to check that $(A,B) \in \mathcal{L}$ since $\mathcal{L}$ is a right- congruence. In fact, it remains to justify that $A_1 = \emptyset$ or $A_2 = \emptyset$ if and only if $B_1 = \emptyset$ or $B_2 = \emptyset$. But this is a straightforward consequence of (iii). ∎

Next, we will examine the Green's relation $\mathcal{H}$,which is the intersection between the relations $\mathcal{L}$ and $\mathcal{R}$.

Proposition 2.6. *$\mathcal{H}$ is the set of all $(A,B) \in T_P(X,Y)^2$ such that $A = B$ or $A = B^*$ with the following conditions:*
(i) $A_3 \cup A_4 \neq \emptyset \Leftrightarrow \ \mathbb{C} \subseteq A$*;*
(ii) either $A_1 = \emptyset$ *or* $A_2 = \emptyset$*.*

Proof. Let $A\mathcal{H}B$. Then $A\mathcal{L}B$ and $A\mathcal{R}B$. Since $A\mathcal{R}B$ then by Proposition 2.2, $A = B$ or $A, B \subseteq \mathbb{C}$ or $A = B^*$. Suppose that $A, B \subseteq \mathbb{C}$. By Proposition 2.4(i) it follows that $A = B$. Suppose now that neither $A = B$ nor $A, B \subseteq \mathbb{C}$, i.e. $A = B^*$. Let $A_3 \cup A_4 \neq \emptyset$. Therefore $A_3 \neq \emptyset$ or $A_4 \neq \emptyset$. It follows that $c_1 \in A \cap B$ or $c_2 \in A \cap B$ by Proposition 2.4(i). Without loss of generality let $c_1 \in A$, then $c_2 \in B^*$ since $A = B^*$. By Proposition 2.4(i), $c_2 \in A$. Therefore, $\mathbb{C} \subseteq A$. The converse direction is clear. This shows (i).
Next, we want to show (ii). From Proposition 2.4(ii), it follows that $A_1 \neq \emptyset$ or $A_2 \neq \emptyset$. Assume that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. If $B_{\mathbb{C}} \neq \emptyset$ then $A_{\mathbb{C}} \neq \emptyset$ since $A = B^*$. It follows that $A_3 \cup A_4 \neq \emptyset$. Then $\mathbb{C} \subseteq A$ by (i). Therefore, $A_{\mathbb{C}} = \mathbb{C} = B_{\mathbb{C}}$. Hence, $B \setminus \mathbb{C} = B \setminus B_{\mathbb{C}}$. From this and from Proposition 2.4(iv), $B \setminus \mathbb{C} = B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^* = B^* \setminus B_{\mathbb{C}} = B^* \setminus \mathbb{C} = A \setminus \mathbb{C}$. Therefore, $A = B$, a contradiction. If $B_{\mathbb{C}} = \emptyset$ then $A_{\mathbb{C}} = \emptyset$ since $A = B^*$. Then $B = B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^* = B^* = A$, a contradiction. That means $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$ is not possible. Hence, either $A_1 \neq \emptyset$ or $A_2 \neq \emptyset$.
Conversely, let $(A,B) \in T_P(X,Y)^2$ such that $A = B$ or $A = B^*$ with conditions (i) and (ii). We have to show that $A\mathcal{L}B$ and $A\mathcal{R}B$. If $A = B$ all is clear. Suppose now that $A \neq B$ and $A = B^*$ with conditions (i) and (ii). Then Proposition 2.2 provides $A\mathcal{R}B$.

It remains to show that $A\mathcal{L}B$. Let $i \in \{1,2\}$ and $A_{i+2} \neq \emptyset$. Then $A_3 \cup A_4 \neq \emptyset$. By (i) we get $\mathbb{C} \subseteq A$ and thus, $\mathbb{C} \subseteq B$ (since $A = B^*$). So $\mathbb{C} \subseteq A \cap B$. If $B_{i+2} \neq \emptyset$ then $B_3 \cup B_4 \neq \emptyset$. It follows that $A_3 \cup A_4 \neq \emptyset$ since $A = B^*$. Then $\mathbb{C} \subseteq A$ by (i) and $\mathbb{C} \subseteq B$ since $A = B^*$. Therefore $c_i \in A \cap B$. This shows (i) of Proposition 2.4. By (ii), we conclude $A_1 \cup A_2 \neq \emptyset$. Then $B_1 \cup B_2 \neq \emptyset$ since $A = B^*$. This shows (ii) of Proposition 2.4. The condition (iii) of Proposition 2.4 is clear since $A = B^*$. Since $A_1, A_2 \neq \emptyset$ is not possible, condition (iv) of Proposition 2.4 is true, trivially. ∎

The relation $\mathcal{H}^{\bigcirc}$ is the intersection of $\mathcal{L}^{\bigcirc}$ and $\mathcal{R}^{\bigcirc}$ and it is described in the next proposition.

Proposition 2.7. *$\mathcal{H}^{\bigcirc}$ is the set of all $(A,B) \in T_P(X,Y)^2$ such that $A = B$.*

Proof. If $A = B$ then $(A,B) \in \mathcal{R}^{\bigcirc} \cap \mathcal{L}^{\bigcirc} = \mathcal{H}^{\bigcirc}$. If $(A,B) \in \mathcal{H}^{\bigcirc}$ then $(A,B) \in \mathcal{R}^{\bigcirc}$ and $(A,B) \in \mathcal{L}^{\bigcirc}$. From Proposition 2.3, we obtain $A = B$ or $A, B \subseteq \mathbb{C}$ or $A = B^*$ with $\mathbb{C} \subseteq A$. Assume now that $A \neq B$. Then $A, B \subseteq \mathbb{C}$ or $A = B^*$ with $\mathbb{C} \subseteq A$. The case $A, B \subseteq \mathbb{C}$ is not possible since Proposition 2.5(ii). In the remaining case, we obtain $\mathbb{C} \subseteq B = A^*$ and $A \setminus \mathbb{C} = B^* \setminus \mathbb{C} = (B \setminus \mathbb{C})^* = (B \setminus B_{\mathbb{C}})^* = B \setminus B_{\mathbb{C}} = B \setminus \mathbb{C}$ by Proposition 2.5(iv), i.e. $A = B$, a contradiction. ∎

We finish this section with the description of $\mathcal{J}$. Here, we have seven equivalence classes with more than two elements. One of them is the three element set $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$. In fact, if $A, B \subseteq \mathbb{C}$ then $\mathbf{1}AB = B$ and $\mathbf{1}BA = A$.

Proposition 2.8. *Let $A \neq B \in T_P(X,Y)$ with $A, B \nsubseteq \mathbb{C}$. Then $A\mathcal{J}B$ if and only if $A^* = B$ or the following four statements are satisfied:*
(i) $|A_{\mathbb{C}}| = |B_{\mathbb{C}}|$ and $A_{i+2} \neq \emptyset \Leftrightarrow c_i \in A$ and $B_{i+2} \neq \emptyset \Leftrightarrow c_i \in B$ for $i \in \{1,2\}$;
(ii) $A_1 \cup A_2 \neq \emptyset$ and $B_1 \cup B_2 \neq \emptyset$;
(iii) $A_1 = \emptyset$ or $A_2 = \emptyset$ if and only if $B_1 = \emptyset$ or $B_2 = \emptyset$;
(iv) if $A_1, A_2 \neq \emptyset$ then $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$.

Proof. Suppose $A\mathcal{J}B$. Then there is $Q \in T_P(X,Y)$ such that $(A,Q) \in \mathcal{L}$ and $(Q,B) \in \mathcal{R}$, i.e. $B^* = Q$ or $B = Q$ by Proposition 2.2. If $B = Q$ then $(A,B) \in \mathcal{L}$ implies $(i) - (iv)$, obviously. So, we suppose that $B \neq Q$, i.e. $(A, B^*) \in \mathcal{L}$. Additional, we suppose that $A^* \neq B$, i.e. $A \neq B^*$. Note that $(D_1)^* = (D^*)_2$, $(D_2)^* = (D^*)_1$, and $(D_1)^* \cup (D_2)^* = (D_1 \cup D_2)^*$ for all $D \in T_P(X,Y)$ (we will call it "Fact 1"). Using Fact 1 and Proposition 2.4 $(ii) - (iv)$, from $(A, B^*) \in \mathcal{L}$, we obtain $(ii) - (iv)$ by simple calculations. It remains to show (i). From Proposition 2.4 (i), we have $A_{\mathbb{C}} = (B^*)_{\mathbb{C}}$, and thus $|A_{\mathbb{C}}| = |(B^*)_{\mathbb{C}}| = |B_{\mathbb{C}}|$. Let $k \in \{1,2\}$ and $i \in \{1,2\}$ such that $i + k = 3$. From Proposition 2.4 (i), we have $(B^*)_{i+2} \neq \emptyset \Leftrightarrow c_i \in B^*$ and $A_{i+2} \neq \emptyset \Leftrightarrow c_i \in A$. Note that $(B^*)_{i+2} \neq \emptyset \Leftrightarrow (B_{k+2})^* \neq \emptyset \Leftrightarrow B_{k+2} \neq \emptyset$ and $c_i \in (B^*)_{i+2} \Leftrightarrow c_i \in (B_{k+2})^* \Leftrightarrow (c_i)^* = c_k \in B_{k+2}$. Thus, we get $c_j \in A \Leftrightarrow A_{j+2} \neq \emptyset$ and $c_j \in B \Leftrightarrow B_{j+2} \neq \emptyset$ for all $j \in \{1,2\}$. Altogether, we have shown (i).
Conversely, suppose that $A^* = B$ or $(i) - (iv)$ are satisfied. If $A^* = B$ then $(A, A^*) \in \mathcal{R}$ and $(B,B) \in \mathcal{L}$ implies $A\mathcal{J}B$.
Suppose now that $A^* \neq B$ and $(i) - (iv)$ are satisfied. Suppose $A_{\mathbb{C}} = B_{\mathbb{C}}$. Then from (i), we get $B_{i+2} \neq \emptyset \Leftrightarrow c_i \in A \cap B \Leftrightarrow A_{i+2} \neq \emptyset$. This satisfies (i) in Proposition 2.4. It is easy to see that $(ii) - (iv)$ imply the remaining items in Proposition 2.4. Altogether we get $(A,B) \in \mathcal{L}$. Therefore, $(A,B) \in \mathcal{J}$ since $(A,B) \in \mathcal{L}$ and $(B,B) \in \mathcal{R}$.
Require now $A_{\mathbb{C}} \neq B_{\mathbb{C}}$. This implies $|A_{\mathbb{C}}| = |B_{\mathbb{C}}| = 1$. Without loss of generality, let

$c_1 \in A_{\mathbb{C}} \setminus B_{\mathbb{C}}$ and $c_2 \in B_{\mathbb{C}} \setminus A_{\mathbb{C}}$. Then $(c_2)^* = c_1 \in (B^*)_{\mathbb{C}}$. By the condition (i), we have $B_3 = \emptyset$ and $A_4 = \emptyset$ since $c_1 \notin B$ and $c_2 \notin A$, respectively. Using Fact 1, we obtain $(B^*)_4 = \emptyset$, where $c_2 = c_1^* \notin (B_3)^* = (B^*)_4$ and $c_1 = c_2^* \in (B_4)^* = (B^*)_3$. This shows that $A_{i+2} \neq \emptyset \Leftrightarrow (B^*)_{i+2} \neq \emptyset \Leftrightarrow c_i \in A \cap B^*$, for $i = 1, 2$, i.e. (i) in Proposition 2.4 is valid for A and B^*. The items $(ii) - (iv)$ can be shown for A and B^* using Fact 1 again by simple calculations. Altogether, we get $(A, B^*) \in \mathcal{L}$. Therefore, $(A, B) \in \mathcal{J}$ since $(B^*, B) \in \mathcal{R}$. ■

Notice that representatives of the equivalence classes with more than two elements are $\mathbf{T}_1$, $\mathbf{T}_1 \cup \mathbf{T}_2$, $\mathbf{T}_1 \cup \mathbf{T}_3$, $\mathbf{T}_1 \cup \mathbf{T}_2 \cup \mathbf{T}_3$, $\mathbf{T}_1 \cup \mathbf{T}_3 \cup \mathbf{T}_4$, $T(X, Y)$ and $\mathbb{C}$. Finally, we determine the largest congruence in $\mathcal{J}$.

Proposition 2.9. *$\mathcal{J}^{\bigcirc}$ is the set of all $(A, B) \in \mathcal{J}$ such that $A = B$ or $A_{\mathbb{C}} = A$ or the following both statements are true*
(i) $\mathbb{C} \subseteq A, B$, whenever $A = B^* \neq B$ or either $A_1 = \emptyset$ or $B_1 = \emptyset$;
(ii) $A_{\mathbb{C}} = B_{\mathbb{C}}$.

Proof. Let $(A, B) \in \mathcal{J}^{\bigcirc}$ with $A \neq B$ and $A_{\mathbb{C}} \neq A$. Assume that $A_{\mathbb{C}} \neq B_{\mathbb{C}}$. Then $B_{\mathbb{C}} \neq B$ and without loss of generality, we can assume that $c_1 \in B \setminus A$. Let

$$\begin{aligned} \overline{A} \ &: = \mathbf{1}A(\mathbf{T}_1 \cup \mathbf{T}_4) = A \cup \{c_2\} \text{ and} \\ \overline{B} \ &: = \mathbf{1}B(\mathbf{T}_1 \cup \mathbf{T}_4) = B \cup \{c_2\}. \end{aligned}$$

Because of $\overline{A}, \overline{B} \nsubseteq \mathbb{C}$ and $\overline{B}_{\mathbb{C}} = \mathbb{C} \neq \{c_2\} = \overline{A}_{\mathbb{C}}$, we conclude $|\overline{A}_{\mathbb{C}}| \neq |\overline{B}_{\mathbb{C}}|$ and $\overline{A} \neq \overline{B}$. Thus, $(\overline{A}, \overline{B}) \notin \mathcal{J}$, a contradiction. So, we have still to show (i) using $A_{\mathbb{C}} = B_{\mathbb{C}}$.
Assume that $\mathbb{C} \nsubseteq A, B$ and either $A_1 = \emptyset$ or $B_1 = \emptyset$. Without loss of generality, we can assume that $c_1 \notin A$, $A_1 = \emptyset$, and $B_1 \neq \emptyset$. Then $c_1 \notin B$ (since $A_{\mathbb{C}} = B_{\mathbb{C}}$) and $B_2 = \emptyset$ and $A_2 \neq \emptyset$ (by Proposition 2.8 $(ii), (iii)$). We put

$$\begin{aligned} \widehat{A} \ &: = (\mathbf{T}_4 \setminus \{c_2\})A(\mathbf{T}_1 \cup \{c_2\}) = (\mathbf{T}_3 \setminus \{c_1\}) \cup \{c_2\} \ \text{ and} \\ \widehat{B} \ &: = (\mathbf{T}_4 \setminus \{c_2\})B(\mathbf{T}_1 \cup \{c_2\}) = \mathbf{T}_4. \end{aligned}$$

Note that $\widehat{A}, \widehat{B} \nsubseteq \mathbb{C}$ but $\widehat{A}, \widehat{B} \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$. Since $(\mathbf{T}_3 \setminus \{c_1\}) \cup \{c_2\}$ is different to $\mathbf{T}_4$ as well as to $(\mathbf{T}_4)^* = \mathbf{T}_3$, we conclude that $(\widehat{A}, \widehat{B}) \notin \mathcal{J}$, a contradiction. Next, assume that $A = B^* \neq B$ and $\mathbb{C} \nsubseteq A, B$. Since $B_{\mathbb{C}} = A_{\mathbb{C}} = (B^*)_{\mathbb{C}}$, we obtain $A_{\mathbb{C}} = B_{\mathbb{C}} = \emptyset$, i.e. $A, B \subseteq \mathbf{T}_1 \cup \mathbf{T}_2$ by Proposition 2.8 (i). If $A_1, A_2 \neq \emptyset$ then let us consider the sets

$$\begin{aligned} \tilde{A} \ &: = \mathbf{1}A(\mathbf{T}_1 \cup \mathbf{T}_4) = A \cup \{c_2\} \text{ and} \\ \tilde{B} \ &: = \mathbf{1}B(\mathbf{T}_1 \cup \mathbf{T}_4) = B \cup \{c_2\}. \end{aligned}$$

Since $\tilde{A}, \tilde{B} \nsubseteq \mathbb{C}$, $\tilde{A} \neq \tilde{B}$, $\tilde{A} \neq (\tilde{B})^* = A \cup \{c_1\}$, and $\tilde{A} \setminus \mathbb{C} = A \neq B = A^* = (\tilde{A} \setminus \mathbb{C})^*$ (where $\tilde{A}_1, \tilde{A}_2 \neq \emptyset$), we can conclude that $(\tilde{A}, \tilde{B}) \notin \mathcal{J}$ (by Proposition 2.8 (iv)), a contradiction. Finally, we suppose that $A_1 = \emptyset$ or $A_2 = \emptyset$. Without loss of generality, we assume that $A_1 = \emptyset$ and thus $B_2 = \emptyset$ (since $A = B^*$). Here, we have $A_2 \neq \emptyset$ and $B_1 \neq \emptyset$ by Proposition 2.8 (ii). Let us consider the sets

$$\begin{aligned} \check{A} \ &: = \mathbf{T}_1 A(\mathbf{T}_1 \cup \{c_1\}) = \mathbf{T}_4 \cup \{c_2\} \text{ and} \\ \check{B} \ &: = \mathbf{T}_1 B(\mathbf{T}_1 \cup \{c_1\} = \mathbf{T}_3. \end{aligned}$$

It is easy to verify (by Proposition 2.8) that $(\mathbf{T}_4 \cup \{c_1\}, \mathbf{T}_3) \notin \mathcal{J}$, i.e. $(\check{A}, \check{B}) \notin \mathcal{J}$, a contradiction.
Conversely, let $(A, B) \in \mathcal{J}$ with $A = B$ or $A = A_{\mathbb{C}}$ or such that (i) and (ii) are satisfied. If

$A = B$ then $(A, B) \in \mathcal{J}$, obviously. Further let $U, V \in T_P(X,Y)^1$. If $A = A_{\mathbb{C}}$, i.e. $A, B \subseteq \mathbb{C}$, then $UAV, UBV \subseteq \mathbb{C}$, i.e. $(UAV, UBV) \in \mathcal{J}$. Suppose now that $A_{\mathbb{C}} \neq A \neq B$ but (i) and (ii) are satisfied. Admit that $A = B^*$. Then $\mathbb{C} \subseteq A, B$ by (i). If either $V_1 \neq \emptyset$ and $V_2 = \emptyset$ or $V = \mathbf{1}$ then $UAV = UA = UB^* = U(B\mathbf{T}_2) = (UB)\mathbf{T}_2 = (UB)^* = (UBV)^*$. If $V_1 = \emptyset$ and $V_2 \neq \emptyset$ then $UAV = UA\mathbf{T}_2 = UB\mathbf{T}_2\mathbf{T}_2 = (UB\mathbf{T}_2)\mathbf{T}_2 = (UBV)^*$. In both cases, we get $(UAV, UBV) \in \mathcal{J}$. If $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ then $UAV = U(A \cup B) = UBV$. If $V_1, V_2 = \emptyset$ then $V \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$ and $UAV = U\mathbb{C} = UBV$.
Finally, we admit that $A \neq B^*$. Further, we suppose that $U \neq \mathbf{1}$. First, we consider the case that $A_i = \emptyset$ and $A_j \neq \emptyset$ for $i, j \in \{1, 2\}$ with $i + j = 3$. Then, either $B_i = \emptyset$ and $B_j \neq \emptyset$ or $B_j = \emptyset$ and $B_i \neq \emptyset$ by Proposition 2.8 (iii). Suppose that $B_j = \emptyset$ and $B_i \neq \emptyset$. Since $\mathbb{C} \subseteq A, B$ (by (i)), we conclude that either $UA = U \cup \mathbb{C}$ and $UB = U^* \cup \mathbb{C}$ or $UA = U^* \cup \mathbb{C}$ and $UB = U \cup \mathbb{C}$, i.e. $UA = (UB)^*$. We have already shown that $(\mathbf{1}(UB)^*V, \mathbf{1}(UB)V) \in \mathcal{J}$, i.e. $(UAV, UBV) \in \mathcal{J}$. Suppose that $B_i = \emptyset$ and $B_j \neq \emptyset$, then $UA = UB$ since $A_{\mathbb{C}} = B_{\mathbb{C}}$ (by (ii)), i.e. $UAV = UBV$. By Proposition 2.8 $(ii), (iii)$, we have still to consider the case that $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $B_1 \neq \emptyset$, and $B_2 \neq \emptyset$. Then $A_{\mathbb{C}} \cup U \cup U^* \subseteq UA$. On the other hand, we have $UA = U \cup U^* \cup U(A_3 \cup A_4)$. Because of Proposition 2.8 (i), we can conclude that $A_{\mathbb{C}} = U(A_3 \cup A_4)$ and thus $UA = A_{\mathbb{C}} \cup U \cup U^*$. Dually, we can show that $UB = B_{\mathbb{C}} \cup U \cup U^*$. Since $A_{\mathbb{C}} = B_{\mathbb{C}}$, we obtain $UA = UB$, i.e. $UAV = UBV$. Let $U = \mathbf{1}$. Here, we have to verify that $(AV, BV) \in \mathcal{J}$. If $(A, B) \in \mathcal{L}$, all is done since $\mathcal{L}$ is a right-congruence. Assume that $(A, B) \notin \mathcal{L}$. It is routine to check that $(i) - (iv)$ in Proposition 2.4 are consequences of $A_{\mathbb{C}} = B_{\mathbb{C}}$ and the items $(i) - (iv)$ in Proposition 2.8. Since $(A, B) \in \mathcal{J} \setminus \mathcal{L}$ and $A, B \nsubseteq \mathbb{C}$, Proposition 2.8 provides $A = B^*$, a contradiction. This finishes the proof. ∎

3. The Ideals Structure

This section is devoted to the ideal structure of the semigroup $T_P(X,Y)$. First, we characterize the left ideals. In order to do it we need some more or less technical notations. We denote the mapping from $P_4T := \{\bigcup_{i \in M} \mathbf{T}_i : M \subseteq \{1,2,3,4\}\}$ to $\mathcal{P}(T_P(X,Y)) := \{K : K \subseteq T_P(X,Y)\}$ (where $\bigcup_{i \in \emptyset} \mathbf{T}_i := \emptyset$) assigning any $A \in P_4T$ to $T_P(X,Y)A$ by χ. Notice that $T_P(X,Y)\emptyset := \emptyset$. We observe that $\chi(\mathbf{T}_1) = \chi(\mathbf{T}_2) = T_P(X,Y)$, $\chi(\mathbf{T}_1 \cup \mathbf{T}_2) = \{A \cup A^* : A \in T_P(X,Y)\}$, $\chi(\mathbf{T}_3 \cup \mathbf{T}_4 \cup R) = \{\mathbb{C} \cup A : A \in \chi(R)\}$ for $R \in \{\emptyset, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_1 \cup \mathbf{T}_2\}$ and $\chi(\mathbf{T}_{i+2} \cup R) = \{\{c_i\} \cup A : A \in \chi(R)\}$ for $i = 1, 2$ and $R \in \{\emptyset, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_1 \cup \mathbf{T}_2\}$. We consider the following both subsets $\mathbf{Q}_1$ and $\mathbf{Q}_2$ of P_4T:
$\mathbf{Q}_1 := \{\emptyset, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_4, \mathbf{T}_1 \cup \mathbf{T}_2, \mathbf{T}_1 \cup \mathbf{T}_4, \mathbf{T}_2 \cup \mathbf{T}_4, \mathbf{T}_1 \cup \mathbf{T}_2 \cup \mathbf{T}_4\}$ and
$\mathbf{Q}_2 := \{\emptyset, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_1 \cup \mathbf{T}_2, \mathbf{T}_1 \cup \mathbf{T}_3, \mathbf{T}_2 \cup \mathbf{T}_3, \mathbf{T}_1 \cup \mathbf{T}_2 \cup \mathbf{T}_3\}$.
It is easy to verify that $\{\mathbf{T}_3 \cup J : J \in \mathbf{Q}_1\} \cup \{\mathbf{T}_4 \cup J : J \in \mathbf{Q}_2\} \cup \{\emptyset, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_1 \cup \mathbf{T}_2\} = P_4T$. Moreover for any non-empty set $M \subseteq \{1,2,3,4\}$, let $P(\bigcup_{i \in M} \mathbf{T}_i) := (P(M) :=) \{A \subseteq \bigcup_{i \in M} \mathbf{T}_i : A_i \neq \emptyset \text{ for all } i \in M\}$. In particular, we put $P_0 := P(\mathbf{T}_1 \cup \mathbf{T}_2)$ $(= P(\{1,2\}))$.
Now we define still three sets
$\widehat{Q}_i := \{\{\{c_i\} \cup A : A \in \chi(J)\} \cup K : K \subseteq P(\mathbf{T}_{i+2} \cup J), J \in \mathbf{Q}_i\}$ for $i = 1, 2$ and
$\widehat{Q}_3 := \{\chi(\mathbf{T}_1 \cup \mathbf{T}_2) \cup K : K \subseteq P_0\}$. Note that $\widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3 \subseteq \mathcal{P}(T_P(X,Y))$. We will show that $\bigcup \mathbf{K}$ is a left ideal for any set $\mathbf{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ and conversely. So, we obtain an explicit description of the left ideals of $T_P(X,Y)$.

Proposition 3.1. *Let $\emptyset \neq I \subseteq T_p(X,Y)$. Then I is a non-trivial left ideal of $T_P(X,Y)$ if and only if there is a set $\mathbf{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ such that $I = \bigcup \mathbf{K}$.*

Proof. Suppose that I is a non-trivial left ideal. Let $A \in I$ with $A_3 \neq \emptyset$. Then there is $J \in \mathbf{Q}_1$ such that $A \in P(\mathbf{T}_3 \cup J)$. We have $T_P(X,Y)A = T_P(X,Y)(\mathbf{T}_3 \cup J) = \{B\mathbf{T}_3 \cup BJ : B \in T_P(X,Y)\} = \{\{c_1\} \cup B : B \in \chi(J)\}$ since $T_P(X,Y)J = \chi(J)$, i.e. $\{\{c_1\} \cup B : B \in \chi(J)\} \subseteq I$. Let $K_J := I \cap P(\mathbf{T}_3 \cup J)$. Then $\{\{c_1\} \cup B : B \in \chi(J)\} \cup K_J \in \widehat{Q}_1$. Let $A \in I$ with $A_4 \neq \emptyset$. Then there is $L \in Q_2$ with $A \in P(\mathbf{T}_4 \cup L)$ such that $\{\{c_2\} \cup B : B \in \chi(L)\} \cup K_L \subseteq I$ with $K_L := I \cap P(\mathbf{T}_4 \cup L)$ and $\{\{c_2\} \cup B : B \in \chi(L)\} \cup K_L \in \widehat{Q}_2$. This shows that there are sets $\mathbf{K}_1 \subseteq \widehat{Q}_1$ and $\mathbf{K}_2 \subseteq \widehat{Q}_2$ such that $\bigcup \mathbf{K}_1$ is the set of all $B \in I$ with $B_3 \neq \emptyset$ and $\bigcup \mathbf{K}_2$ is the set of all $B \in I$ with $B_4 \neq \emptyset$.
Let now $A \in I$ with $A_3 = A_4 = \emptyset$. Assume that $A \subseteq \mathbf{T}_1$. Then we have $T_P(X,Y) \supseteq T_P(X,Y)I \supseteq T_P(X,Y)A = T_P(X,Y)\mathbf{T}_1 = T_P(X,Y)$, i.e. $I = T_P(X,Y)$, a contradiction. Assume that $A \subseteq \mathbf{T}_2$. Then we have $T_P(X,Y)AA = T_P(X,Y)\mathbf{T}_1 = T_P(X,Y)$ and thus $T_P(X,Y) = T_P(X,Y)AA \subseteq T_P(X,Y)A \subseteq I \subseteq T_P(X,Y)$, i.e. $I = T_P(X,Y)$, a contradiction. Hence, $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$, i.e. $A \in P_0$. Now we have $I \supseteq T_P(X,Y)I \supseteq T_P(X,Y)A = T_P(X,Y)(\mathbf{T}_1 \cup \mathbf{T}_2) = \{B \cup B^* : B \in T_P(X,Y)\} = \chi(\mathbf{T}_1 \cup \mathbf{T}_2)$, i.e. $\chi(\mathbf{T}_1 \cup \mathbf{T}_2) \subseteq I$. Let $K := P_0 \cap I$. Then $\widetilde{K} := \chi(\mathbf{T}_1 \cup \mathbf{T}_2) \cup K \in \widehat{Q}_3$ is the set of all $B \in I$ with $B_3 = B_4 = \emptyset$. Altogether, we have $I = \bigcup \mathbf{K}$ with $\mathbf{K} := \mathbf{K}_1 \cup \mathbf{K}_2 \cup \{\widetilde{K}\} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$.

Conversely, let $I = \bigcup \mathbf{K}$ for a set $\mathbf{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ and let $A \in I$. Suppose that $A \in \bigcup \widehat{Q}_1$. Then there are sets $J \in \mathbf{Q}_1$ and $L \in P(\mathbf{T}_3 \cup J)$ such that $A \in \{\{c_1\} \cup B : B \in \chi(J)\} \cup L \in \mathbf{K}$. In particular, we have $\{\{c_1\} \cup B : B \in \chi(J)\} \subseteq I$ and $A = \{c_1\} \cup B$ for some $B \in \chi(J)$ or $A \in L$. In the latter case, we obtain $T_P(X,Y)A = \{B\mathbf{T}_3 \cup BJ : B \in T_P(X,Y)\} = \{\{c_1\} \cup B : B \in \chi(J)\} \subseteq I$. Consider now the case $A = \{c_1\} \cup B$ for some $B \in \chi(J)$. Since $T_P(X,Y)J = \chi(J)$, there is $\widetilde{D} \in T_P(X,Y)$ such that $B = \widetilde{D}J$. So, we can calculate $T_P(X,Y)A = \{D\{c_1\} \cup DB : D \in T_P(X,Y)\} = \{\{c_1\} \cup D\widetilde{D}J : D \in T_P(X,Y)\} \subseteq \{\{c_1\} \cup DJ : D \in T_P(X,Y)\} = \{\{c_1\} \cup D : D \in \chi(J)\} \subseteq I$. If $A \in \bigcup \widehat{Q}_2$ then we can argue similarly that $T_P(X,Y)A \subseteq I$. Finally, if $A \in \bigcup \widehat{Q}_3$ then there is a set $L_0 \subseteq P_0$ such that $A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2) \cup L_0 \in \mathbf{K}$, i.e. $\chi(\mathbf{T}_1 \cup \mathbf{T}_2) \cup L_0 \subseteq I$. In particular, there is $B \in T_P(X,Y)$ with $A = B \cup B^*$ or $A \in L_0 \subseteq P_0$. In the latter case, we conclude $T_P(X,Y)A = T_P(X,Y)(\mathbf{T}_1 \cup \mathbf{T}_2) = \chi(\mathbf{T}_1 \cup \mathbf{T}_2) \subseteq I$. Now let $A = B \cup B^*$ for some $B \in T_P(X,Y)$. Here, we have $T_P(X,Y)A = \{DA : D \in T_P(X,Y)\} = \{DB \cup DB^* : D \in T_P(X,Y)\} = \{DB \cup DB\mathbf{T}_2 : D \in T_P(X,Y)\} \subseteq \{D \cup D\mathbf{T}_2 : D \in T_P(X,Y)\} = \{D \cup D^* : D \in T_P(X,Y)\} = \chi(\mathbf{T}_1 \cup \mathbf{T}_2) \subseteq I$. Altogether, we have shown that $T_P(X,Y)I \subseteq I$, i.e. I is a left ideal. ∎

Next, we characterize the right ideals. For a set $A \in T_P(X,Y)$, we define

$$Q_A := \{B \cup D : B \in \{A, A^*, A \cup A^*\}, D \in \mathcal{P}(\mathbb{C})\} \cup \{\{c_1\}, \{c_2\}, \mathbb{C}\}$$

and for $K \subseteq T_P(X,Y)$, let

$$Q_K := \bigcup_{A \in K} Q_A,$$

where $\mathcal{P}(\mathbb{C})$ is the power set of $\mathbb{C}$, i.e. $\mathcal{P}(\mathbb{C}) = \{\emptyset, \{c_1\}, \{c_2\}, \mathbb{C}\}$. Let $A \in T_P(X,Y)$. It is easy to verify that $AT_P(X,Y) = \{AB : B \in T_P(X,Y)\} = \{A\,(\bigcup_{i\in M} \mathbf{T}_i) : \emptyset \neq M \subseteq \{1,2,3,4\}\} = Q_A$. So, we can conclude that $KT_P(X,Y) = Q_K$ for any set $K \subseteq T_P(X,Y)$. This fact provides the following characterization of the right ideals:

Proposition 3.2. *Let $\emptyset \neq I \subseteq T_P(X,Y)$. Then I is a right ideal of $T_P(X,Y)$ if and only if there is a set $K \subseteq T_P(X,Y)$ such that $I = Q_K$.*

Proof. Suppose that I is a right ideal. Then we have $I \supseteq IT_P(X,Y) = Q_I$. Since $A = A \cup \emptyset \in Q_A \subseteq Q_I$ for all $A \in I$, we have the equality $I = Q_I$.
Conversely, we suppose that $I = Q_K$ for some set $K \subseteq T_P(X,Y)$. Let $A \in I$. Then there is $\widetilde{A} \in K$ such that $A \in Q_{\widetilde{A}}$. So, there is a non-empty set $M \subseteq \{1,2,3,4\}$ with $A = \widetilde{A}\,(\bigcup_{i\in M}\mathbf{T}_i)$. Let $B \in T_P(X,Y)$ and let $M_B := \{i \in \{1,2,3,4\} \,:\, B_i \neq \emptyset\}$, i.e. $B \in P(M_B)$. Then, $AB = \widetilde{A}(\bigcup_{i\in M}\mathbf{T}_i)(\bigcup_{i\in M_B}\mathbf{T}_i)$. It is easy to verify that $\widetilde{A}(\bigcup_{i\in M}\mathbf{T}_i)(\bigcup_{i\in M_B}\mathbf{T}_i) = \widetilde{A}(\bigcup_{i\in R}\mathbf{T}_i)$ with $R = (\{3,4\} \cap M_B) \cup M^+ \cup M^-$, where

$$M^+ := \begin{cases} M & \text{if} \quad 1 \in M_B \\ \emptyset & \quad\text{otherwise} \end{cases}$$

and

$$M^- := \begin{cases} \{i+1 : i \in (M \cap \{1,3\})\} \cup \{i-1 : i \in (M \cap \{2,4\})\} & \text{if} \quad 2 \in M_B \\ \emptyset & \quad\text{otherwise.} \end{cases}$$

This provides $AB = \widetilde{A}(\bigcup_{i\in R}\mathbf{T}_i) \in Q_{\widetilde{A}} \subseteq Q_K = I$. So, we have shown that $IT_P(X,Y) \subseteq I$, i.e. I is a right ideal. ∎

In particular, the proof of Proposition 3.2 shows that $I = Q_I$, whenever I is a right ideal. It is interesting to note that Proposition 3.2 justifies that a left ideal I is also a right ideal if it is "closed" under Q_I. In order to proof this, we need two technical lemmas:

Lemma 3.3. *Let $L \in \widehat{Q}_3$. Then there is a set $\boldsymbol{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ such that $\bigcup \boldsymbol{K} = Q_L$.*

Proof. There is $K \subseteq P_0$ such that $L = \chi(\mathbf{T}_1 \cup \mathbf{T}_2) \cup K$. First, we observe that $A^*, A\cup A^* \in P_0$ for any $A \in K$. Thus, $K^* := \{A^* : A \in K\} \subseteq P_0$ and $K^{\cup *} := \{A \cup A^* : A \in K\} \subseteq P_0$. Moreover, we observe that $(A \cup A^*) = (A \cup A^*)^* = (A \cup A^*) \cup (A \cup A^*)^*$ for any $A \in T_P(X,Y)$. This implies $\{A^* : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} = \{A \cup A^* : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} = \chi(\mathbf{T}_1 \cup \mathbf{T}_2)$. Using these facts, we obtain
$M_i := \{\{c_i\} \cup A : A \in L\} = \{\{c_i\} \cup A : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} \cup \{\{c_i\} \cup A : A \in K\} \in Q_i$,
$M_i^* := \{\{c_i\} \cup A^* : A \in L\} = \{\{c_i\} \cup A : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} \cup \{\{c_i\} \cup A : A \in K^*\} \in \widehat{Q}_i$,
and
$M_i^{\cup *} := \{\{c_i\} \cup A \cup A^* : A \in L\} = \{\{c_i\} \cup A : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} \cup \{\{c_i\} \cup A : A \in K^{\cup *}\} \in \widehat{Q}_i$
for $i = 1, 2$. Further we have
$M_C := \{\mathbb{C} \cup A : A \in L\} = \{\mathbb{C} \cup A : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} \cup \{\mathbb{C} \cup A : A \in K\} \in Q_1$,
$M_C^* := \{\mathbb{C} \cup A^* : A \in L\} = \{\mathbb{C} \cup A : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} \cup \{C \cup A : A \in K^*\} \in \widehat{Q}_1$, and
$M_C^{\cup *} := \{\mathbb{C} \cup A \cup A^* : A \in L\} = \{\mathbb{C} \cup A : A \in \chi(\mathbf{T}_1 \cup \mathbf{T}_2)\} \cup \{\mathbb{C} \cup A : A \in K^{\cup *}\} \in \widehat{Q}_1$.
On the other hand, we observe that
$M := \{A, A^*, A \cup A^* \,:\, A \in L\} = \chi(T_1 \cup \mathbf{T}_2) \cup K \cup K^* \cup K^{\cup *} \in \widehat{Q}_3$. Clearly,

$Q_L = M \cup M_1 \cup M_2 \cup M_{\mathbb{C}} \cup M_1^* \cup M_2^* \cup M_C^* \cup M_1^{\cup *} \cup M_2^{\cup *} \cup M_C^{\cup *}$. So, the set $\{M, M_1, M_2, M_{\mathbb{C}}, M_1^*, M_2^*, M_C^*, M_1^{\cup *}, M_2^{\cup *}, M_C^{\cup *}\}$ is the required set $\mathbf{K}$. ∎

Similary, one can prove the next lemma.

Lemma 3.4. *Let $L \in \widehat{Q}_1 \cup \widehat{Q}_2$. Then there is a set $\boldsymbol{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2$ such that $\bigcup \boldsymbol{K} = Q_L$.*

Proof. Without loss of generality, let $L \in \widehat{Q}_1$. Then we put: $\mathbf{K}_1 := \{\{A \cup B : A \in L\} : B \in \mathcal{P}(\mathbb{C})\}$ and $\mathbf{K}_2 := \{\{A^* \cup B : A \in L\}, \{A^* \cup A \cup B : A \in L\} : B \in \mathcal{P}(\mathbb{C})\}$. It is easy to verify that $\mathbf{K}_1 \subseteq \widehat{Q}_1$ and $\mathbf{K}_2 \subseteq \widehat{Q}_2$. Moreover, we observe that $\bigcup(\mathbf{K}_1 \cup \mathbf{K}_2) = Q_L$. ∎

Using these both lemmas, Proposition 3.1 and Proposition 3.2 provide a description of the ideals:

Corollary 3.5. *Let $\emptyset \neq I \subseteq T_P(X,Y)$. Then I is an ideal of $T_P(X,Y)$ if and only if there is a set $\boldsymbol{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ such that $I = Q_{\bigcup \boldsymbol{K}}$.*

Proof. Let I be an ideal. Then I is a left ideal and by Proposition 3.1 there is a set $\mathbf{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ such that $I = \bigcup \mathbf{K}$. This implies $Q_I = Q_{\bigcup \mathbf{K}}$, where $I = Q_I$ since I is a right ideal.
Suppose now that $I = Q_{\bigcup \mathbf{K}}$ for some set $\mathbf{K} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$. Then I is a right ideal by Proposition 3.2 . By Lemma 3.3 and Lemma 3.4, we conclude that there is a set $\mathbf{M} \subseteq \widehat{Q}_1 \cup \widehat{Q}_2 \cup \widehat{Q}_3$ such that $Q_{\bigcup \mathbf{K}} = \bigcup \mathbf{M}$. This equality justifies that $I = Q_{\bigcup \mathbf{K}}$ is a left idea, too, by Proposition 3.1. ∎

Finally, we will indicate the principal ideals among the ideals. Recall, that for $A \in T_P(X,Y)$, the set $T_P(X,Y)^1 A T_P(X,Y)^1$ forms an ideal, which is called principal ideal, i.e. a principial ideal is an ideal generating by one element of the semigroup $T_P(X,Y)$. Note that $T_P(X,Y)$ has right-identity elements (e.g. $\mathbf{T}_1$) and no left-dentity element. Therefore, the principal ideals are of the form $T_P(X,Y)^1 A T_P(X,Y)$, for $A \in T_P(X,Y)$.

Proposition 3.6. *A set $I \subseteq T_P(X,Y)$ is a principal ideal of $T_P(X,Y)$ if and only if $I = T_P(X,Y)$ or there are a non-empty set $J \in P_4T \setminus \{\boldsymbol{T}_1, \boldsymbol{T}_2\}$ and a set $A \in P(J)$ such that $I = Q_{\chi(J) \cup \{A\}}$.*

Proof. Suppose that $I = T_P(X,Y)$. It is easy to verify that $T_P(X,Y)^1 \mathbf{T}_1 T_P(X,Y) = T_P(X,Y)$. Thus, I is a principal ideal. Let $\emptyset \neq J \in P_4T \setminus \{\mathbf{T}_1, \mathbf{T}_2\}$ and let $A \in P(J)$ such that $I = Q_{\chi(J) \cup \{A\}}$. Then there is a non-empty set $M \subseteq \{1,2,3,4\}$ with $M \neq \{1\}, \{2\}$ such that $J = \bigcup\limits_{i \in M} \mathbf{T}_i$. Because of $T_P(X,Y)A = T_P(X,Y)J$, we can conclude that $T_P(X,Y)^1 A T_P(X,Y) = (T_P(X,Y)J \cup \{A\}) T_P(X,Y) = (\chi(J) \cup \{A\}) T_P(X,Y) = Q_{\chi(J) \cup \{A\}}$.
Conversely, let $A \in T_P(X,Y)$. Then there is a non-empty set $J \in P_4T$ such that $A \in P(J)$. Then either $J \in \{\mathbf{T}_1, \mathbf{T}_2\}$ or $J \in P_4T \setminus \{\mathbf{T}_1, \mathbf{T}_2\}$. In the later case, we have $T_P(X,Y)^1 A T_P(X,Y) = Q_{\chi(J) \cup \{A\}}$ as already shown. If $J = \mathbf{T}_1$ then $A \subseteq \mathbf{T}_1$. Thus, $T_P(X,Y)^1 A T_P(X,Y) = T_P(X,Y)^1 \mathbf{T}_1 T_P(X,Y) = T_P(X,Y) T_P(X,Y) = T_P(X,Y)$. If $J = \mathbf{T}_2$ then $A \subseteq \mathbf{T}_2$ and we can conclude $T_P(X,Y)^1 A T_P(X,Y) = T_P(X,Y)^1 \mathbf{T}_2 T_P(X,Y) \supseteq T_P(X,Y)^1 \mathbf{T}_2 \mathbf{T}_2 = T_P(X,Y)^1 \mathbf{T}_1 = T_P(X,Y)$. This provides $T_P(X,Y)^1 A T_P(X,Y) = T_P(X,Y)$. Hence, all the principal ideals from $T_P(X,Y)$ are of a form given in the assertion of the proposition. ∎

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