Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1677–1699

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

New Exponential Passivity Analysis of Integro-Differential Neural Networks with Time-Varying Delays

Ninrat Promdee[1], Kanit Mukdasai[1,*] and Prem Junsawang[2]

[1] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*
e-mail : Ninrat_P@kkumail.com (N. Promdee); kanit@kku.ac.th (K. Mukdasai)

[2] *Department of Statistics, Faculty of Science, Khon Kaen University, Khon Kaen 40000, Thailand*
e-mail : prem@kku.ac.th (P.J unsawang)

Abstract This paper aims to deal with the problems of exponential stability and exponential passivity analysis for integro-differential neural networks with time-varying delays, based on the mixed model transformation approach. In this work, we investigate both discrete and distributed time-varying delays for which the upper bounds are available. By constructing augments Lyapunov-Krasovskii functional and various inequalities, the new delay-dependent criterion is established and is mathematically expressed in terms of linear matrix inequalities (LMIs) to guarantee the exponential stability of the considered system. Furthermore, depended on the proof for the exponential stability of the system, the constructed delay-dependent method was derived from the exponential passivity for neural networks with mixed time-varying delays. Also, numerical examples are given to illustrate the effectiveness of the findings.

MSC: 37B25; 37C75

Keywords: exponential passivity; neural networks; linear matrix inequality (LMIs); distributed time-varying delay; model transformation

Submission date: 28.02.2020 / Acceptance date: 12.04.2020

1. Introduction

Nowadays, driven by computing advanced technology, a large number of neural networks applications have been widely applied in various areas such as signal processing, pattern recognition, associate memories, control, in references [6, 13, 35, 39] and so forth [3, 10, 15]. Among these applications, one of the most challenging problems in the network design is how to construct the system with stable equilibrium points.

Several types of stability criteria derived by different methods for neural networks

*Corresponding author.

Published by The Mathematical Association of Thailand.

have been proposed such as asymptotical stability [46], complete stability [8], absolute stability [29] and exponential stability [1]. In favour of the faster convergence rate to the equilibrium point, exponential stability has usually been applied instead of asymptotic stability. The property of exponential stability is particularly important when the exponential convergence rate is used to determine the speed of neural computations [28].

In reality, time-delay systems have been frequently encountered in neural networks. When time delay occurred in neural network processes, it is a source of instability and oscillations. Recently, for both delay-independent and delay-dependent systems, various sufficient conditions have been proposed to verify the asymptotic or exponential stability of delay neural networks by applying Lyapunov-Krasovskii functional (LKF) and several model transformation [28, 45, 47, 48, 50], and the references cited therein. In [22, 53], the authors investigated the exponential stability problem of neural networks with time-varying delay by using LKF and various approaches.

The passivity theory has been a significant impact on the analysis of the stability of the dynamical system, complexity, signal processing, chaos control and synchronization, fuzzy control [33]. Firstly, many systems require being passive to alleviate noise effects. Secondly, the robustness measure, such as robust stability or robust performance, of a system often reduces to a subsystem or a modified system called passivity analysis. Passivity analysis plays an essential role in studying the stability of uncertain or nonlinear systems, especially for high-order systems. So, in [9] the passivity analysis has been applied to tackle the control problems for stability robustness in uncertain systems. The essence of the passivity theory is that the passive properties of a system can keep the system internal stability. Therefore, many researchers have emphasized the criteria for the passivity of neural networks with time delay [5, 40, 49, 52, 54]. The exponential passivity problem for neural networks with time-varying delay by several approaches was addressed in [24, 44]. The authors studied exponential passivity criteria for neural networks with discrete and distributed delays, such as [9]. Furthermore, the researchers have widely investigated the issue of exponential passivity analysis for neural networks with interval time-varying delays in [9, 24, 37, 44, 49, 57]. However, from our point of view, only a few authors have been considering and studying the exponential passivity conditions for integro-differential neural networks with mixed interval time-varying delays.

A forementioned, in this paper, the exponential stability and passivity condition for delays neural networks are obtained. Based on constructing new LKF, utilization of zero equation, decomposition techniques. Consequently, delay-dependent exponential passivity conditions are derived. A unified linear matrix inequality (LMI) approach is developed to establish sufficient conditions for neural networks to meet the exponential stability and passivity. Noting that LMI could be easily solved by using the Matlab LMI toolbox, and no parameter tuning is required. It is worth mentioning that the stability and passivity criteria of the neural networks with Markovian switching include the passivity criteria of neural networks without Markovian switching as special cases [33]. Numerical examples are also provided to illustrate the usefulness and effectiveness of the proposed delay-dependent exponential passivity conditions.

2. Problem Formulation and Preliminaries

Consider the following continuous time neural networks with time-varying delays

$$\left.\begin{array}{rcl} \dot{\xi}(t) & = & -C\xi(t) + Af(\xi(t)) + Bf(\xi(t-d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds, \quad t \in R^+ \\ \xi(t) & = & \phi(t), \quad t \in [-d_M, 0], \end{array}\right\} \tag{2.1}$$

where $\xi(t) = [\xi_1(t), \xi_2(t), ..., \xi_n(t)]^T \in R^n$ is the neural state vector, $d(t)$ and $\rho(t)$ are the discrete and distributed time-varying delays, respectively. Equation (2.1) satisfies the following conditions

$$0 \le d(t) \le d_M, \quad 0 \le \dot{d}(t) \le d_d, \tag{2.2}$$

$$0 \le \rho(t) \le \rho_M, \tag{2.3}$$

where d_M, d_d and ρ_M are positive real constants. The diagonal matrix

$$C = \mathbf{diag}\{c_1, c_2, ..., c_n\},$$

with $c_i > 0$, $i = 1, 2, ..., n$, A, B and D are the connection weight matrices between neurons with appropriate dimensions, $\phi(t)$ denotes continuous vector-valued initial function on interval $[-d_M, 0]$. In addition, the neural activation functions $f(\cdot) = [f_1(\cdot), f_2(\cdot), ..., f_n(\cdot)]^T$ $\in R^n$ are assumed to satisfy the following conditions.

Assumption. The activation function f is continuous and the exist constants F_i^- and F_i^+ such that

$$F_i^- \quad \le \quad \frac{f_i(x) - f_i(y)}{x - y} \le F_i^+, \tag{2.4}$$

for all $x \neq y$, and $f = [f_1, f_2, ..., f_n]^T$ and for any $i \in \{1, 2, ..., n\}$, $f_i(0) = 0$. For ease of presentation, we denote
$F_i^- = \mathbf{diag}(F_1^- F_1^+, F_2^- F_2^+, ..., F_n^- F_n^+)$ and $F_i^+ = \mathbf{diag}(\frac{F_1^- + F_1^+}{2}, \frac{F_2^- + F_2^+}{2}, ..., \frac{F_n^- + F_n^+}{2})$.

Definition 2.1. [43] The system defined by (2.1) is said to be exponentially stable, if there exist the positive constant α and N such that the solution $\xi(t, \phi)$ of the system (2.1) satisfies

$$||\xi(t, \phi)|| \quad \le \quad N \sup_{-d_M \le \theta \le 0} ||\phi(\theta)|| e^{-\alpha t}, \qquad \forall t \ge 0. \tag{2.5}$$

Futhermore, α is called the exponential convergence rate.

which would be used in the proof of exponential stability and passivity, respectively.

Lemma 2.2. *(Jensen's Inequality)* [25] *For any symmetric positive definite matrix Q, positive real constant d_M, and vector function $\dot{\xi} : [-d_M, 0] \to R^n$ such that the following integral is well defined, then*

$$-d_M \int_{-d_M}^{0} \dot{\xi}^T(s+t) Q \dot{\xi}(s+t) ds \le -\Big(\int_{-d_M}^{0} \dot{\xi}(s+t) ds\Big)^T Q \Big(\int_{-d_M}^{0} \dot{\xi}(s+t) ds\Big).$$

Lemma 2.3. *(Wirtinger-based integral inequality)* [36] *For any matrix $Z > 0$, the following inequality holds for all continuously differentiable function $\dot{\xi} : [\alpha, \beta] \to R^n$*

$$-(\beta - \alpha) \int_{\alpha}^{\beta} \dot{\xi}^T(s) Z \dot{\xi}(s) ds \quad \le \quad \begin{bmatrix} \xi(\beta) \\ \xi(\alpha) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \xi(s) ds \end{bmatrix}^T \phi \begin{bmatrix} \xi(\beta) \\ \xi(\alpha) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \xi(s) ds \end{bmatrix},$$

where $$\Phi = \begin{bmatrix} -4Z & -2Z & 6Z \\ * & -4Z & 6Z \\ * & * & -12Z \end{bmatrix}.$$

Lemma 2.4. *(Peng-Park's integral inequality)* [33, 34] *For any matrix* $\begin{bmatrix} Z & S \\ * & Z \end{bmatrix} \geq 0$, *positive constants* d_M *and* $d(t)$ *satisfying* $0 < d(t) < d_M$, *vector function* $\dot{\xi} : [-d_M, 0] \to R^n$ *such that the concerned integrations are well defined, then*

$$-d_M \int_{t-d_M}^{t} \dot{\xi}^T(s) Z \dot{\xi}(s) ds \leq \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix}^T \Theta \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix},$$

where $$\Theta = \begin{bmatrix} -Z & Z-S & S \\ * & -2Z+S+S^T & Z-S \\ * & * & -Z \end{bmatrix}.$$

Lemma 2.5. [26] *For a positive matrix* M, *the following inequality holds:*

$$-\frac{(\alpha-\beta)^2}{2} \int_{\beta}^{\alpha} \int_{s}^{\alpha} \xi^T(u) M \xi(u) du ds \leq -\Big(\int_{\beta}^{\alpha} \int_{s}^{\alpha} \xi(u) du ds\Big)^T M \Big(\int_{\beta}^{\alpha} \int_{s}^{\alpha} \xi(u) du ds\Big).$$

Lemma 2.6. [38] *For any constant symmetric positive definite matrix* $Q \in R^{n\times n}$, $d(t)$ *is discrete time-varying delays with* (2.3), *vector function* $\omega : [-d_M, 0] \to R^n$ *such that the integrations concerned are well defined, then*

$$\begin{aligned} -d_M \int_{-d_M}^{0} \omega^T(s) Q \omega(s) ds \leq & -\int_{-d(t)}^{0} \omega^T(s) ds Q \int_{-d(t)}^{0} \omega(s) ds \\ & -\int_{-d_M}^{-d(t)} \omega^T(s) ds Q \int_{-d_M}^{-d(t)} \omega(s) ds. \end{aligned}$$

Lemma 2.7. [38] *For any constant matrices* $Q_1, Q_2, Q_3 \in R^{n\times n}$, $Q_1 \geq 0, Q_3 > 0$, $\begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \geq 0$, $d(t)$ *is discrete time-varying delays with* (2.3) *and vector function* $\dot{x} : [-d_M, 0] \to R^n$ *such that the following integration is well defined, then*

$$-d_M \int_{t-d_M}^{t} \begin{bmatrix} \xi(s) \\ \dot{\xi}(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \dot{\xi}(s) \end{bmatrix} ds \leq \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \\ \int_{t-d(t)}^{t} \xi(s) ds \\ \int_{t-d_M}^{t-d(t)} \xi(s) ds \end{bmatrix}^T \Delta \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \\ \int_{t-d(t)}^{t} \xi(s) ds \\ \int_{t-d_M}^{t-d(t)} \xi(s) ds \end{bmatrix}.$$

where $$\Delta = \begin{bmatrix} -Q_3 & Q_3 & 0 & -Q_2^T & 0 \\ * & -Q_3 - Q_3^T & Q_3 & Q_2^T & -Q_2^T \\ * & * & -Q_3 & 0 & Q_2^T \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_1 \end{bmatrix}.$$

Lemma 2.8. [38] *Let* $\xi(t) \in R^n$ *be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any constant matrices*

$X, M_i \in R^{n\times n}, i = 1, 2, \ldots, 5$ *and* $d(t)$ *is discrete time-varying delays with* (2.1),

$$-\int_{t-h_M}^{t} \dot{\xi}^T(s)X\dot{\xi}(s)ds \le \begin{bmatrix}\xi(t)\\ \xi(t-d(t))\\ \xi(t-d_M)\end{bmatrix}^T \Xi \begin{bmatrix}\xi(t)\\ \xi(t-d(t))\\ \xi(t-d_M)\end{bmatrix} + h_M \begin{bmatrix}\xi(t)\\ \xi(t-d(t))\\ \xi(t-d_M)\end{bmatrix}^T \begin{bmatrix}M_3 & M_4 & 0\\ * & M_3+M_5 & M_4\\ * & * & M_5\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-d(t))\\ \xi(t-d_M)\end{bmatrix},$$

$$where\ \Xi = \begin{bmatrix} M_1+M_1^T & -M_1^T+M_2 & 0\\ * & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2\\ * & * & -M_2-M_2^T\end{bmatrix}, \begin{bmatrix}X & M_1 & M_2\\ * & M_3 & M_4\\ * & * & M_5\end{bmatrix} \ge 0.$$

3. Exponential Stability

In this section, we present our exponential stability analysis for neural networks and introduce the following notations used throughout this work.

$$\begin{aligned}\zeta(t) &= \Big[\xi(t),\ \dot{\xi}(t),\ y(t),\ \xi(t-d(t)),\ \xi(t-d_M),\ \int_{t-d(t)}^{t}\xi(s)ds,\ \int_{t-d_M}^{t-d(t)}\xi(s)ds,\\ &\quad \frac{1}{d_M}\int_{t-d_M}^{t}\xi(s)ds,\ \int_{t-d(t)}^{t}y(s)ds,\ \int_{t-d_M}^{t-d(t)}y(s)ds,\ f(\xi(t)),\ f(\xi(t-d(t))),\\ &\quad f(\xi(t-d_M)),\ \int_{t-\rho_M}^{t}f(\xi(s))ds,\ \int_{-d_M}^{0}\int_{t+s}^{t}\xi(s)dsd\lambda,\ \int_{-d_M}^{0}\int_{t+s}^{t}\dot{\xi}(s)dsd\lambda\Big],\end{aligned}$$

$$\sum = \left[\Omega_{(i,j)}\right]_{16\times16}, \tag{3.1}$$

$$\begin{aligned}
\Omega_{(1,1)} &= 2\alpha P_1 - Q_3^TC - C^TQ_3 + Q_4^T + Q_4 + 2\alpha P_2 + P_3 + R_1 + R_4 + d_M^2P_4 + M_1\\
&\quad + M_1^T + M_3 - 4e^{-2\alpha d_M}P_6 - e^{-2\alpha d_M}P_7 + d_M^2R_7 - e^{-2\alpha d_M}R_9 - \frac{d_M^4}{4}P_9\\
&\quad -2\epsilon_1H_1,\ \ \Omega_{(1,2)} = P_1,\ \ \Omega_{(1,3)} = P_2 - Q_3^T - C^TQ_{15} + Q_{16} + d_M^2R_8,\\
\Omega_{(1,4)} &= -C^TQ_6 - Q_4^T + Q_7 + Q_5^T - M_1^T + M_2 + M_4 + e^{-2\alpha d_M}P_7 - e^{-2\alpha d_M}S\\
&\quad + e^{-2\alpha d_M}R_9,\ \ \Omega_{(1,5)} = -C^TQ_{12} + Q_{13} - Q_5^T - 2e^{-2\alpha d_M}P_6 + e^{-2\alpha d_M}S,\\
\Omega_{(1,6)} &= -C^TQ_9 + Q_{10} - e^{-2\alpha d_M}R_9,\ \ \Omega_{(1,8)} = 6e^{-2\alpha d_M}P_6,\ \ \Omega_{(1,9)} = -Q_4^T,\\
\Omega_{(1,10)} &= -Q_5^T,\ \ \Omega_{(1,11)} = 4\alpha K^T + Q_3^TA + R_2 + R_5 - \epsilon_1H_2,\ \ \ \Omega_{(1,12)} = Q_3^TB,\\
\Omega_{(1,14)} &= Q_3^TD,\ \ \Omega_{(2,1)} = P_1^T,\ \ \Omega_{(2,2)} = -Q_1 - Q_1^T,\ \ \Omega_{(2,3)} = Q_1 - Q_2^T,\\
\Omega_{(2,11)} &= K^T,\ \ \Omega_{(3,1)} = P_2^T - Q_3 - Q_{15}^TC + Q_{16}^T + d_M^2R_8,\ \ \ \Omega_{(3,2)} = Q_1^T - Q_2,
\end{aligned}$$

$$
\begin{aligned}
\Omega_{(3,3)} &= Q_2 + Q_2^T - Q_{15} - Q_{15}^T + d_M P_5 + d_M^2 P_6 + d_M^2 P_7 + d_M^2 R_9 + d_M^2 P_8 + \frac{d_M^4}{4} P_{10},\\
\Omega_{(3,4)} &= -Q_6 - Q_6^T + Q_{17}^T, \quad \Omega_{(3,5)} = -Q_{12}^T - Q_{17}^T, \quad \Omega_{(3,6)} = -Q_9, \ \ \Omega_{(3,9)} = -Q_{16}^T,\\
\Omega_{(3,10)} &= -Q_{17}^T, \ \ \Omega_{(3,11)} = Q_{15}^T A, \quad \Omega_{(3,12)} = Q_{15}^T B, \quad \Omega_{(3,14)} = Q_{15}^T D,\\
\Omega_{(4,1)} &= -Q_6^T C - Q_4 + Q_7^T + Q_5 - M_1 + M_2^T + M_4^T + e^{-2\alpha d_M} P_7^T - e^{-2\alpha d_M} S^T\\
&\quad + e^{-2\alpha d_M} R_9^T, \quad \Omega_{(4,3)} = -Q_6^T - Q_{16} + Q_{17},\\
\Omega_{(4,4)} &= Q_8 + Q_8^T - Q_7 - Q_7^T + e^{-2\alpha d_M} R_1 - d_d e^{-2\alpha d_M} R_1 + M_1 + M_1^T - M_2 - M_2^T\\
&\quad + M_3 + M_5 - 2e^{-2\alpha d_M} P_7 + e^{-2\alpha d_M}(S + S^T) - e^{-2\alpha d_M}(R_9 + R_9^T) - 2H_2\epsilon_1\\
\Omega_{(4,5)} &= Q_{14} - Q_{13} - Q_8^T - M_1^T + M_2 + M_4 + e^{-2\alpha d_M} P_7 - e^{-2\alpha d_M} S + e^{-2\alpha d_M} R_9,\\
\Omega_{(4,6)} &= Q_{11} - Q_{10} + e^{-2\alpha d_M} R_8^T, \quad \Omega_{(4,7)} = -e^{-2\alpha d_M} R_8^T, \quad \Omega_{(4,9)} = -Q_7^T,\\
\Omega_{(4,10)} &= -Q_8^T, \quad \Omega_{(4,11)} = Q_6^T A \quad \Omega_{(4,12)} = Q_6^T B + e^{-2\alpha d_M} R_2 - d_d e^{-2\alpha d_M} R_2 + H_2\epsilon_2,\\
\Omega_{(4,14)} &= Q_6^T D, \quad \Omega_{(5,1)} = Q_{13}^T - Q_{12}^T C - Q_5 - 2e^{-2\alpha d_M} P_6 + e^{-2\alpha d_M} S^T,\\
\Omega_{(5,3)} &= -Q_{12} - Q_{17}, \quad \Omega_{(5,4)} = Q_{14}^T - Q_{13}^T - Q_8 - M_1 - M_2^T + M_4^T + e^{-2\alpha d_M} P_7^T\\
&\quad - e^{-2\alpha d_M} S - e^{-2\alpha d_M} R_9^T, \ \ \Omega_{(5,5)} = -Q_{14} - Q_{14}^T - e^{-2\alpha d_M} P_3 - e^{-2\alpha d_M} R_4\\
&\quad - M_2 - M_2^T + M_5 - 4e^{-2\alpha d_M} R_4 - M_2 - M_2^T + M_5 - 4e^{-2\alpha d_M} P_6\\
&\quad - e^{-2\alpha d_M} P_7 - e^{-2\alpha d_M} R_9, \Omega_{(5,6)} = -Q_{11}, \quad \Omega_{(5,7)} = e^{-2\alpha d_M} R_8,\\
\Omega_{(5,8)} &= 6e^{-2\alpha d_M} P_6, \quad \Omega_{(5,9)} = -Q_{13}^T, \quad \Omega_{(5,10)} = -Q_{14}^T, \quad \Omega_{(5,11)} = Q_{12}^T A,\\
\Omega_{(5,12)} &= Q_{12}^T B, \quad \Omega_{(5,13)} = -e^{-2\alpha d_M} R_5, \quad \Omega_{(5,14)} = Q_{12}^T D,\\
\Omega_{(6,1)} &= Q_{10}^T - Q_9^T C - e^{-2\alpha d_M} R_9, \quad \Omega_{(6,3)} = -Q_9^T,\\
\Omega_{(6,4)} &= Q_{11}^T - Q_{10}^T + e^{-2\alpha d_M} R_8, \quad \Omega_{(6,5)} = -Q_{11}^T,\\
\Omega_{(6,6)} &= -e^{-2\alpha d_M} P_4 - e^{-2\alpha d_M} R_7, \quad \Omega_{(6,9)} = -Q_{10}^T, \quad \Omega_{(6,10)} = -Q_{11}^T,\\
\Omega_{(6,11)} &= Q_9^T A, \quad \Omega_{(6,12)} = Q_9^T B, \quad \Omega_{(6,14)} = Q_9^T D, \quad \Omega_{(7,4)} = -e^{-2\alpha d_M} R_8,\\
\Omega_{(7,5)} &= e^{-2\alpha d_M} R_8^T, \quad \Omega_{(7,7)} = -e^{-2\alpha d_M} P_5 - e^{-2\alpha d_M} R_7, \quad \Omega_{(8,1)} = 6e^{-2\alpha d_M} P_6,\\
\Omega_{(8,5)} &= 6e^{-2\alpha d_M} P_6, \quad \Omega_{(8,8)} = -12e^{-2\alpha d_M} P_6, \quad \Omega_{(9,1)} = -Q_4, \quad \Omega_{(9,4)} = -Q_7,\\
\Omega_{(9,5)} &= -Q_{13}, \quad \Omega_{(9,6)} = -Q_{10}, \quad \Omega_{(9,9)} = -e^{-2\alpha d_M} P_8, \quad \Omega_{(9,10)} = -e^{-2\alpha d_M} P_8,\\
\Omega_{(10,1)} &= -Q_5, \quad \Omega_{(10,3)} = -Q_{17}, \quad \Omega_{(10,4)} = -Q_8, \quad \Omega_{(10,5)} = -Q_{14},\\
\Omega_{(10,6)} &= -Q_{11}, \quad \Omega_{(10,9)} = -e^{-2\alpha d_M} P_8, \quad \Omega_{(10,10)} = -e^{-2\alpha d_M} P_8,\\
\Omega_{(11,1)} &= 4\alpha K + A^T Q_3 + R_2^T + R_5^T - \epsilon_2 H_1^T, \quad \Omega_{(11,2)} = K, \quad \Omega_{(11,3)} = A^T Q_{15},\\
\Omega_{(11,4)} &= A^T Q_6, \quad \Omega_{(11,5)} = A^T Q_{12}, \quad \Omega_{(11,6)} = A^T Q_9,\\
\Omega_{(11,11)} &= R_3 + R_6 + \rho_M P_{11} + 2\epsilon_1, \quad \Omega_{(12,1)} = B^T Q_3, \quad \Omega_{(12,3)} = B^T Q_{15},\\
\Omega_{(12,4)} &= B^T Q_6 + e^{-2\alpha d_M} R_2^T - d_d e^{-2\alpha d_M} R_2^T + \epsilon_2^T H_2, \quad \Omega_{(12,5)} = B^T Q_{12},\\
\Omega_{(12,6)} &= B^T Q_9, \quad \Omega_{(12,12)} = e^{-2\alpha d_M} R_3 - d_d e^{-2\alpha d_M} R_3 - 2H_1,\\
\Omega_{(13,13)} &= -e^{-2\alpha d_M} R_6, \quad \Omega_{(14,1)} = D^T Q_3, \quad \Omega_{(14,3)} = D^T Q_{15}, \quad \Omega_{(14,4)} = D^T Q_6,\\
\Omega_{(14,5)} &= D^T Q_{12}, \quad \Omega_{(14,6)} = D^T Q_9, \quad \Omega_{(14,14)} = -e^{-2\alpha \rho_M} P_{11}, \quad \Omega_{(15,15)} = -P_9,\\
\Omega_{(16,16)} &= -P_{10},
\end{aligned}
$$

and other are equal zero.

Theorem 3.1. *For given positive real constants* d_M, ρ_M, d_d*,* K *and* N*, the system* (2.1) *is exponentially stable with a decay rate* α *if there exist positive definite symmetric matrices* P_i*,* R_j*,* R_9*, any appropriate dimensional matrices* S, R_7, R_8, Q_i*,* $R_7 \geq 0$ *where* $i = 1, 2, \dots, 11$*,* $j = 1, 2, \dots, 6$*, satisfying the following*

$$\begin{bmatrix} P_7 & S \\ * & P_7 \end{bmatrix} \geq 0, \tag{3.2}$$

$$\begin{bmatrix} R_7 & R_8 \\ * & R_7 \end{bmatrix} \geq 0, \tag{3.3}$$

$$\begin{bmatrix} P_4 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.4}$$

$$\sum < 0. \tag{3.5}$$

Proof. First, we show the exponential stability of the system (2.1). To this end, we consider the nominal system (2.1) satisfying

$$\dot{\xi}(t) = -C\xi(t) + Af(\xi(t)) + Bf(\xi(t - d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds. \tag{3.6}$$

From model transformation method, we rewrite the system (3.6) in the following system

$$\dot{\xi}(t) = y(t), \tag{3.7}$$

$$0 = -y(t) - C\xi(t) + Af(\xi(t)) + Bf(\xi(t - d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds. \tag{3.8}$$

For positive real numbers k_i and w_i, where $i = 1, 2, \dots, 10$, considering the Lyapunov-Krasovskii functional candidate for equation (2.1) constructs a Lyapunov-Krasovskii functional candidate for the system (3.6)-(3.8) of the form

$$V(t) = \sum_{i=1}^{9} V_i(t), \tag{3.9}$$

where

$$\begin{aligned}
V_1(t) &= \xi^T(t)P_1\xi(t) + 2\sum_{i=1}^{N} k_i \int_0^{\xi_i(t)} f(s)ds, \\
V_2(t) &= \zeta^T(t)EP_2\zeta(t) + 2\sum_{i=1}^{N} k_i \int_0^{\xi_i(t)} f(s)ds, \\
V_3(t) &= \int_{t-d_M}^{t} e^{2\alpha(s-t)}\xi^T(s)P_3\xi(s)ds \\
&\quad + \int_{t-d(t)}^{t} e^{2\alpha(s-t)} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix} ds \\
&\quad + \int_{t-d_M}^{t} e^{2\alpha(s-t)} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix} ds,
\end{aligned}$$

$$
\begin{aligned}
V_4(t) &= d_M \int_{-d_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\xi^T(\theta)P_4\xi(\theta)d\theta ds \\
&\quad + \int_{-d_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}y^T(\theta)P_5y(\theta)d\theta ds, \\
V_5(t) &= d_M \int_{-d_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}y^T(\theta)P_6y(\theta)d\theta ds \\
&\quad + d_M \int_{-d_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}y^T(\theta)P_7y(\theta)d\theta ds, \\
V_6(t) &= d_M \int_{-d_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)} \begin{bmatrix}\xi(\theta)\\ y(\theta)\end{bmatrix}^T \begin{bmatrix}R_7 & R_8\\ * & R_9\end{bmatrix} \begin{bmatrix}\xi(\theta)\\ y(\theta)\end{bmatrix} d\theta ds, \\
V_7(t) &= d_M \int_{-d_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}y^T(\theta)P_8y(\theta)d\theta ds, \\
V_8(t) &= \frac{(d_M)^2}{2}\int_{-d_M}^{0}\int_{\lambda}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\xi^T(\theta)P_9\xi(\theta)d\theta ds d\lambda \\
&\quad + \frac{(d_M)^2}{2}\int_{-d_M}^{0}\int_{\lambda}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}y^T(\theta)P_{10}y(\theta)d\theta ds d\lambda, \\
V_9(t) &= \rho_M \int_{-\rho_M}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}f(\xi(\theta))^T P_{11}f(\xi(\theta))d\theta ds,
\end{aligned}
$$

where $E = \begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The time derivative of $V(t)$ along the trajectory of system (3.6)-(3.8) is given by

$$
\dot{V}(t) = \sum_{i=1}^{9}\dot{V}_i(t). \tag{3.10}
$$

Then the time derivative of $V_1(t)$ is calculated as follows:

$$
\dot{V}_1(t) \leq 2\xi^T(t)P_1\dot{\xi}(t) + 2f^T(\xi(t))K\dot{\xi}(t) + 4\alpha f^T(\xi(t))K\xi(t) + 2\alpha\xi^T(t)P_1\xi(t) - 2\alpha V_1(t).
$$

Taking the deravative of $V_2(t)$ along any trajectory of solution of system (2.1), we have

$$
\begin{aligned}
\dot{V}_2(t) &= 2\xi^T(t)P_2\dot{\xi}(t) + 2\dot{\xi}^T(t)Q_1\left[-\dot{\xi}(t) + y(t)\right] + 4\alpha f^T(\xi(t))K\xi(t) + 2\alpha\xi^T(t)P_2\xi(t) \\
&\quad -2\alpha V_2(t)
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
= \quad & 2\zeta^T(t)P_2^T \begin{bmatrix} \dot{\xi}(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\dot{\xi}^T(t)Q_1\left[-\dot{\xi}(t)+y(t)\right] + 2y^T(t)Q_2\left[-\dot{\xi}(t)+y(t)\right] \\
& +4\alpha f^T(\xi(\theta))C\xi(t) + 2\alpha\xi^T(t)P_2\xi(t) - 2\alpha V_2(t) \\
= \quad & 2\begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \int_{t-d(t)}^{t}\xi(s)ds \\ \xi(t-d_M) \\ y(t) \end{bmatrix}^T \begin{bmatrix} P_2 & Q_3^T & Q_4^T & Q_5^T \\ 0 & Q_6^T & Q_7^T & Q_8^T \\ 0 & Q_9^T & Q_{10}^T & Q_{11}^T \\ 0 & Q_{12}^T & Q_{13}^T & Q_{14}^T \\ 0 & Q_{15}^T & Q_{16}^T & Q_{17}^T \end{bmatrix} \begin{bmatrix} \dot{\xi}(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\dot{\xi}^T(t)Q_1\left[-\dot{\xi}(t)+y(t)\right] \\
& +2y^T(t)Q_2\left[-\dot{\xi}(t)+y(t)\right] + 4\alpha f^T(\xi(\theta))C\xi(t) + 2\alpha\xi^T(t)P_2\xi(t) - 2\alpha V_1(t) \\
= \quad & 2\xi^T(t)P_2y(t) + 2\dot{\xi}^T(t)Q_1\left[-\dot{\xi}(t)+y(t)\right] + 2y^T(t)Q_2\left[-\dot{\xi}(t)+y(t)\right] \\
& +2\left[\xi^T(t)Q_3^T + \xi^T(t-d(t))Q_6^T + \int_{t-d(t)}^{t}\xi^T(s)dsQ_9^T + \xi^T(t-d_M)Q_{12}^T\right. \\
& \left.+y^T(t)Q_{15}^T\right]\left[-y(t) - C\xi(t) + Af(\xi(t)) + Bf(\xi(t-d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds\right] \\
& +2\left[\xi^T(t)Q_4^T + \xi^T(t-d(t))Q_7^T + \int_{t-d(t)}^{t}\xi^T(s)dsQ_{10}^T + \xi^T(t-d_M)Q_{13}^T + y^T(t)Q_{16}^T\right] \\
& \times\left[\xi(t) - \xi(t-d(t)) - \int_{t-d(t)}^{t} y(s)ds\right] + 2\left[\xi^T(t)Q_5^T + \xi^T(t-d(t))Q_8^T\right. \\
& \left.+\int_{t-d(t)}^{T}\xi^T(s)dsQ_{11}^T + \xi^T(t-d_M)Q_{14}^T + y^T(t)Q_{17}^T\right]\left[\xi(t-d(t)) - \xi(t-d_M)\right. \\
& \left.-\int_{t-d_M}^{t-d(t)} y(s)ds\right] + 4\alpha f^T(\xi(\theta))K\xi(t) + 2\alpha\xi^T(t)P_2\xi(t) - 2\alpha V_2(t).
\end{aligned}
$$

Since the scalars $e^{-2\alpha d_M} \leq e^{-2\alpha d(t)} \leq 1$. $V_3(t)$ and $\dot{d}(t) \leq d_d$, we have

$$
\begin{aligned}
\dot{V}_3(t) \quad = \quad & \xi^T(t)P_3\xi(t) - e^{-2\alpha d_M}\xi^T(t-d_M)P_3\xi(t-d_M) \\
& +\begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix} \\
& -e^{-2\alpha d_M}\begin{bmatrix} \xi(t-d_M) \\ f(\xi(t-d_M)) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ R_5^T & R_6 \end{bmatrix} \begin{bmatrix} \xi(t-d_M) \\ f(\xi(t-d_M)) \end{bmatrix} - 2\alpha V_3(t) \\
\leq \quad & \xi^T(t)P_3\xi(t) - e^{-2\alpha d_M}\xi^T(t-d_M)P_3\xi(t-d_M) \\
& +\begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix} - d_d\begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix} \\
& -e^{-2\alpha d_M}\begin{bmatrix} \xi(t-d(t)) \\ f(\xi(t-d(t))) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} \xi(t-d(t)) \\ f(\xi(t-d(t))) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ R_5^T & R_6 \end{bmatrix} \begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix} \\
&-e^{-2\alpha d_M} \begin{bmatrix} \xi(t-d_M) \\ f(\xi(t-d_M)) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ R_5^T & R_6 \end{bmatrix} \begin{bmatrix} \xi(t-d_M) \\ f(\xi(t-d_M)) \end{bmatrix} - 2\alpha V_3(t).
\end{aligned}
$$

Using Lemma 2.6 and Lemma 2.8, $V_4(t)$ is calculated as

$$
\begin{aligned}
\dot{V}_4(t) &= d_M^2 \xi^T(t) P_4 \xi(t) - d_M e^{-2\alpha d_M} \int_{t-d_M}^{t} \xi^T(s) P_4 \xi(s) ds \\
&\quad + d_M y^T(t) P_5 y(t) - e^{-2\alpha d_M} \int_{t-d_M}^{t} \dot{\xi}^T(s) P_5 \dot{\xi}(s) ds - 2\alpha V_4(t) \\
&\leq d_M^2 \xi^T(t) P_4 \xi(t) + d_M y^T(t) P_5 y(t) - e^{-2\alpha d_M} \int_{t-d(t)}^{t} \xi^T(s) ds P_4 \int_{t-d(t)}^{t} \xi(s) ds \\
&\quad - e^{-2\alpha d_M} \int_{t-d_M}^{t-d(t)} \xi^T(s) ds P_5 \int_{t-d_M}^{t-d(t)} x(s) ds + \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} M_1 + M1^T & -M1^T + M_2 & 0 \\ * & M_1 + M_1^T - M_2 - M_2^T & -M_1^T + M_2 \\ * & * - M_2 - M_2^T & \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix} \\
&\quad + h_M \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 & 0 \\ * & M_3 + M_5 & M_4 \\ * & * & M_5 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix} - 2\alpha V_4(t).
\end{aligned}
$$

Using Lemma 2.3 (Wirtinger-base integral inequality) and Lemma 2.4 (Peng-Park's integral inequality), an upper bound of $V_5(t)$ can be obtained as

$$
\begin{aligned}
\dot{V}_5(t) &\leq d_M^2 y^T(t) P_6 y(t) + d_M^2 y^T(t) P_7 y(t) \\
&+ e^{-2\alpha d_M} \begin{bmatrix} \xi(\beta) \\ \xi(\alpha) \\ \frac{1}{\beta-\alpha}\int_\alpha^\beta \xi(s) ds \end{bmatrix}^T \begin{bmatrix} -4P_6 & -2P_6 & 6P_6 \\ -2P_6^T & -4P_6 & 6Q_4 \\ 6P_6^T & 6P_6^T & -12P_6 \end{bmatrix} \begin{bmatrix} \xi(\beta) \\ \xi(\alpha) \\ \frac{1}{\beta-\alpha}\int_\alpha^\beta \xi(s) ds \end{bmatrix} \\
&\quad + e^{-2\alpha d_M} \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix}^T \begin{bmatrix} -P_7 & P_7 - S & S \\ P_7^T - S^T & -2P_7 + S + S^T & P_7 - S \\ S^T & P_7^T - S^T & -P_7 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \end{bmatrix} \\
&- 2\alpha V_5(t).
\end{aligned}
$$

It is from Lemma 2.7 that we have

$$
\begin{aligned}
\dot{V}_6(t) &= d_M^2 \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ R_8^T & R_9 \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} - d_M \int_{t-d_M}^{t} e^{2\alpha(s-t)} \begin{bmatrix} \xi(s) \\ y(s) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ R_8^T & R_9 \end{bmatrix} \begin{bmatrix} \xi(s) \\ y(s) \end{bmatrix} ds \\
&\quad - 2\alpha V_6(t)
\end{aligned}
$$

$$\begin{aligned}
&\leq \; d_M^2 \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ R_8^T & R_9 \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} + e^{-2\alpha d_M} \\
&\times \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \\ \int_{t-d(t)}^{t} \xi(s)ds \\ \int_{t-d_M}^{t-d(t)} \xi(s)ds \end{bmatrix}^T \begin{bmatrix} -R_9 & R_9 & 0 & -R_9^T & 0 \\ R_9^T & -R_9 - R_9^T & R_9 & R_8^T & -R_8^T \\ 0 & R_9^T & -R_9 & 0 & R_8^T \\ -R_9 & R_8 & 0 & -R_7 & 0 \\ 0 & -R_8 & R_8 & 0 & -R_7 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-d(t)) \\ \xi(t-d_M) \\ \int_{t-d(t)}^{t} \xi(s)ds \\ \int_{t-d_M}^{t-d(t)} \xi(s)ds \end{bmatrix} \\
&- \; 2\alpha V_6(t).
\end{aligned}$$

Using Lemma 2.2 (Jensen's Inequality) that we have

$$\begin{aligned}
\dot{V}_7(t) \;\leq\; & d_M^2 y^T(t) P_8 y(t) - e^{-2\alpha d_M} \left[\int_{t-d(t)}^{t} y^T(s)ds + \int_{t-d_M}^{t-d(t)} y^T(s)ds \right] P_8 \\
& \times \left[\int_{t-d(t)}^{t} y(s)ds + \int_{t-d_M}^{t-d(t)} y(s)ds \right] - 2\alpha V_7(t).
\end{aligned}$$

By Lemma 2.5, we can obtain $\dot{V}_8(t)$ as follows

$$\begin{aligned}
\dot{V}_8(t) \;\leq\; & \frac{d_M^4}{4} \xi^T(t) Q_6 \xi(t) - \int_{t-d_M}^{t} \int_{u}^{t} \xi^T(\lambda) d\lambda du Q_6 \int_{t-h_M}^{t} \int_{u}^{t} \xi(\lambda) d\lambda du \\
& + \frac{d_M^4}{2} y^T(t) Q_7 y(t) - 2 \int_{t-d_M}^{t} \int_{u}^{t} \dot{\xi}^T(\lambda) d\lambda du Q_7 \int_{t-h_M}^{t} \int_{u}^{t} \dot{\xi}(\lambda) d\lambda du - 2\alpha V_8(t).
\end{aligned}$$

By Lemma 2.2 (Jensen's Inequality) and calculating $\dot{V}_9(t)$, we have

$$\begin{aligned}
\dot{V}_9(t) \;=\; & \rho_M f^T(\xi(t)) P_{11} f(\xi(t)) - \rho_M \int_{t-\rho_M}^{t} e^{2\alpha(s-t)} f^T(\xi(s)) P_{11} f(\xi(s)) ds - 2\alpha V_9(t) \\
\leq\; & \rho_M f^T(\xi(t)) P_{11} f(\xi(t)) - e^{-2\alpha \rho_M} \int_{t-\rho_M}^{t} f^T(\xi(s)) ds P_{11} \int_{t-\rho_M}^{t} f(\xi(s)) ds - 2\alpha V_9(t).
\end{aligned}$$

From (2.4), we obtain for any positive real constants ϵ_1 and ϵ_2,

$$\begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix}^T \begin{bmatrix} -2H_1\epsilon_1 & H_1\epsilon_2 \\ \epsilon_2^T H_1^T & -2H_1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ f(\xi(t)) \end{bmatrix} \geq 0, \tag{3.12}$$

$$\begin{bmatrix} \xi(t-d(t)) \\ f(\xi(t-d(t))) \end{bmatrix}^T \begin{bmatrix} -2H_2\epsilon_1 & H_2\epsilon_2 \\ \epsilon_2^T H_2^T & -2H_2 \end{bmatrix} \begin{bmatrix} \xi(t-d(t)) \\ f(\xi(t-d(t))) \end{bmatrix} \geq 0. \tag{3.13}$$

According to (3.10)-(3.13), it is straightforward to see that

$$\dot{V}(t) + 2\alpha V(t) \leq \zeta^T(t) \sum \zeta(t), \tag{3.14}$$

where
$\zeta(t) = \Big[\xi(t), \dot{\xi}(t), y(t), \xi(t-d(t)), \xi(t-d_M), \int_{t-d(t)}^{t} \xi(s)ds, \int_{t-d_M}^{t-d(t)} \xi(s)ds, \frac{1}{d_M}\int_{t-d_M}^{t} \xi(s)ds,$
$\int_{t-d(t)}^{t} y(s)ds, \int_{t-d_M}^{t-d(t)} y(s)ds, f(\xi(t)), f(\xi(t-d(t))), f(\xi(t-d_M)), \int_{t-\rho_M}^{t} f(\xi(s))ds,$
$\int_{-d_M}^{0}\int_{t+s}^{t} \xi(s)dsd\lambda, \int_{-d_M}^{0}\int_{t+s}^{t} \dot{\xi}(s)dsd\lambda\Big]$.
and $\sum$ is define in (3.1). It is true that if conditions (3.5)-(3.14) hold, then

$$\dot{V}(t) + 2\alpha V(t) \leq 0, \qquad \forall t \in R^+.$$

Hence, we get

$$\dot{V}(t) \le -2\alpha V(t) \le -2\alpha\lambda_{min}(P_1)||\xi(t)||^2, \qquad \forall t \in R^+.$$

If we choose $\lambda_2 = 2\alpha\lambda_{min}(P_1)$, then

$$\dot{V}(t) \le -\lambda_2 2||\xi(t)||^2, \qquad \forall t \in R^+. \tag{3.15}$$

Integrating both sides of (3.15) from 0 to t, we get

$$V(t) \le V(0)e^{-2\alpha t}, \qquad \forall t \in R^+, \tag{3.16}$$

where $V(0) = \sum_{i=1}^{9} V_i(0)$.

Therefore, we conclude

$$\begin{aligned} \lambda_1(P_1)||\xi(t)||^2 &\le V(0)e^{-2\alpha t} \\ &\le N \sup_{-d_M \le \theta \le 0} ||\phi(\theta)||^2 e^{-2\alpha t}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} N &= \lambda_{max}(P_1 + 2\epsilon_3 K + EP_2) + d_M\lambda_{max}(R_1 + 1 + \epsilon_1 R_2^T R_2 + \epsilon_1 R_3) \\ &\quad + d_M^2\lambda_{max}(P_5 + P_7) + d_M^3\lambda_{max}(P_4 + P_6 + R_7 + 1 + R_8^T R_8 + R_9 + P_8) \\ &\quad + d_M^5\lambda_{max}(P_9 + P_{10}) + \rho_M^2\lambda_{max}(\epsilon_1 P_{11}), \\ \epsilon_3 &= \text{diag}\{F_1^+, F_2^+, ..., F_n^+\}. \end{aligned}$$

From (3.1), we obtain

$$||\xi(t,\phi)|| \le \sqrt{\frac{N}{\lambda_{min}(P_1)}} \sup_{-d_M \le \theta \le 0} ||\phi(\theta)||e^{-\alpha t}, \quad \forall t \in R^+.$$

Now we can conclude that (2.1) is exponentially stable with the exponential convergenc rate α. ∎

4. Exponential Passivity

In this section, we analyze the exponential passivity for neural networks with time-varying delays, interested in continuous neural networks, by the following

$$\left.\begin{aligned} \dot{\xi}(t) &= -C\xi(t) + Af(\xi(t)) + Bf(\xi(t - d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds \\ &\quad + u(t), \quad t \in R^+ \\ \xi(t) &= \phi(t), \quad t \in [-d_M, 0], \\ z(t) &= f(\xi(t)) + f(\xi(t - d(t))) + u(t), \quad t \in R^+ \end{aligned}\right\} \tag{4.1}$$

where $u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T \in R^n$ is an external input vector to neurons, $z(t) = [z_1(t), z_2(t), ..., z_n(t)]^T \in R^n$ is the output vector of neuron networks.

Definition 4.1. [57] The neural networks are said to be exponential passive from input $u(t)$ to output $z(t)$, if there exist an exponential Lyapunov function (or, called the exponential storage function) V defined on R^n, and a constant $\alpha > 0$ such that for all $u(t)$, all initial conditions $\xi(0)$, all $t \ge t_0$, the following inequality holds:

$$\dot{V}(t) + \alpha V(t) \le 2z^T(t)u(t), \qquad t \ge t_0, \tag{4.2}$$

where $\dot{V}(t)$ denotes the total derivative of $V(t)$ along the state trajectories $\xi(t)$, $t \geq t_0$, of (4.1).

By (3.5), we give the additional notations exponential passivity for (4.1),

$$\Pi = \begin{bmatrix}\Omega_{i,j}\end{bmatrix}_{17\times 17}. \tag{4.3}$$

We obtain that $\Omega_{(1,1)} - \Omega_{(16,16)}$, are the same as in Theorem 3.1 except

$$\begin{aligned}
\Omega_{(1,17)} &= Q_3^T,\ \Omega_{(3,17)} = Q_{15}^T,\ \Omega_{(4,17)} = Q_6^T,\\
\Omega_{(5,17)} &= Q_{12}^T,\ \Omega_{(6,17)} = Q_9^T,\ \Omega_{(17,1)} = Q_3,\\
\Omega_{(17,3)} &= Q_{15},\ \Omega_{(17,4)} = Q_6,\ \Omega_{(17,5)} = Q_{12},\\
\Omega_{(17,6)} &= Q_9,\\
\Omega_{(11,17)} &= \Omega_{(17,11)} = \Omega_{(12,17)} = \Omega_{(17,12)} = -I,\\
\Omega_{(17,17)} &= -2I,\\
\Omega_{(1,17)} &= \Omega_{(2,17)} = \cdots = \Omega_{(17,16)} = 0,
\end{aligned}$$

$\psi(t) = \Big[\xi(t), \dot{\xi}, y(t), \xi(t-d(t)), \xi(t-d_M), \int_{t-d(t)}^{t}\xi(s)ds, \int_{t-d_M}^{t-d(t)}\xi(s)ds, \frac{1}{d_M}\int_{t-d(t)}^{t}\xi(s)ds,$ $\int_{t-d(t)}^{t} y(s)ds, \int_{t-d_M}^{t-d(t)} y(s)ds,\ f(\xi(t)),\ f(\xi(t-d(t))), \int_{t-\rho_M}^{t} f(\xi(s))ds, \int_{-d_M}^{0}\int_{t+s}^{t}\xi(s)dsd\lambda,$ $\int_{-d_M}^{0}\int_{t+s}^{t}\dot{\xi}(s)dsd\lambda, u(t)\Big].$

Then, we construct a new theorem as follow,

Theorem 4.2. *For given positive real constants d_M, ρ_M, d_d, K and N, the system (4.1) is exponential passivity with a decay rate α if there exist positive definite symmetric matrices P_i, R_j, R_9, any appropriate dimensional matrices S, R_7, R_8, Q_i, $R_7 \geq 0$ where $i = 1, 2, \ldots, 11$, $j = 1, 2, \ldots, 6$, satisfying the following*

$$\begin{bmatrix} P_7 & S \\ * & P_7 \end{bmatrix} \geq 0, \tag{4.4}$$

$$\begin{bmatrix} R_7 & R_8 \\ * & R_7 \end{bmatrix} \geq 0, \tag{4.5}$$

$$\begin{bmatrix} P_4 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{4.6}$$

$$\Pi < 0. \tag{4.7}$$

Proof. Under the condition of the theorem, we focus on the exponential passivity of the neural networks (4.1) as the following

$$\dot{\xi}(t) = -C\xi(t) + Af(\xi(t)) + Bf(\xi(t-d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t). \tag{4.8}$$

By model transformation method, we have

$$\dot{\xi}(t) = y(t), \tag{4.9}$$

$$\begin{aligned} 0 &= -y(t) - C\xi(t) + Af(\xi(t)) + Bf(\xi(t-d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds \\ &\quad + u(t). \end{aligned} \tag{4.10}$$

Similary, Lyapunov function of neural networks is activated by replacing $\dot{V}_2(t)$ into (3.10).

$$\begin{aligned}
\dot{V}_2(t) \quad = \quad & 2\xi^T(t)P_2 y(t) + 2\dot{\xi}^T(t)Q_1\left[-\dot{\xi}(t)+y(t)\right] + 2y^T(t)Q_2\left[-\dot{\xi}(t)+y(t)\right] \\
& +2\left[\xi^T(t)Q_3^T + \xi^T(t-d(t))Q_6^T + \int_{t-d(t)}^{t}\xi^T(s)dsQ_9^T + \xi^T(t-d_M)Q_{12}^T\right. \\
& +\left[-y(t) - C\xi(t) + Af(\xi(t)) + Bf(\xi(t-d(t))) + D\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t)\right] \\
& +2\left[\xi^T(t)Q_4^T + \xi^T(t-d(t))Q_7^T + \int_{t-d(t)}^{t}\xi^T(s)dsQ_{10}^T + \xi^T(t-d_M)Q_{13}^T\right. \\
& \left.+y^T(t)Q_{16}^T\right]\left[\xi(t) - \xi(t-d(t)) - \int_{t-d(t)}^{t} y(s)ds\right] + 2\left[\xi^T(t)Q_5^T\right. \\
& \left.+\xi^T(t-d(t))Q_8^T + \int_{t-d(t)}^{T}\xi^T(s)dsQ_{11}^T + \xi^T(t-d_M)Q_{14}^T + y^T(t)Q_{17}^T\right] \\
& \left[\xi(t-d(t)) - \xi(t-d_M) - \int_{t-d_M}^{t-d(t)} y(s)ds\right] + 4\alpha f^T(\xi(\theta))C\xi(t) \\
& +2\alpha\xi^T(t)P_2\xi(t) - 2\alpha V_2(t).
\end{aligned}$$

Now, the exponential passivity analysis is presented. By (3.5), (3.12) and (3.13), it can be seen that

$$\begin{aligned}
& \dot{V}(t) - 2Z^T(t)u(t) \\
& \quad \leq \quad 2\xi^T(t)P_1\dot{\xi}(t) + 2f^T(\xi(t))K\dot{\xi}(t) + 4\alpha f^T(\xi(t))K\xi(t) + 2\alpha\xi^T(t)P_1\xi(t) \\
& \qquad +2\xi^T(t)P_2 y(t) + 2\dot{\xi}^T(t)Q_1\left[-\dot{\xi}(t)+y(t)\right] + 2y^T(t)Q_2\left[-\dot{\xi}(t)+y(t)\right] \\
& \qquad +2\left[\xi^T(t)Q_3^T + \xi^T(t-d(t))Q_6^T + \int_{t-d(t)}^{t}\xi^T(s)dsQ_9^T + \xi^T(t-d_M)Q_{12}^T\right. \\
& \qquad \left.+y^T(t)Q_{15}^T\right]\left[-y(t) - C\xi(t) + Af(\xi(t)) + Bf(\xi(t-d(t)))\right. \\
& \qquad \left.+D\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t)\right] + 2\left[\xi^T(t)Q_4^T + \xi^T(t-d(t))Q_7^T\right. \\
& \qquad \left.+\int_{t-d(t)}^{t}\xi^T(s)dsQ_{10}^T + \xi^T(t-d_M)Q_{13}^T + y^T(t)Q_{16}^T\right] \\
& \qquad \times\left[\xi(t) - \xi(t-d(t)) - \int_{t-d(t)}^{t} y(s)ds\right] + 2\left[\xi^T(t)Q_5^T + \xi^T(t-d(t))Q_8^T\right. \\
& \qquad \left.+\int_{t-d(t)}^{T}\xi^T(s)dsQ_{11}^T + \xi^T(t-d_M)Q_{14}^T + y^T(t)Q_{17}^T\right] \\
& \qquad \left[\xi(t-d(t)) - \xi(t-d_M) - \int_{t-d_M}^{t-d(t)} y(s)ds\right] + 4\alpha f(\xi(\theta))^T C\xi(t)
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha\xi^T(t)P_2\xi(t)+\xi^T(t)P_3\xi(t)\\
&-e^{-2\alpha d_M}\xi^T(t-d_M)P_3\xi(t-d_M)+\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}\\
&-e^{-2\alpha d_M}(1-d_d)\begin{bmatrix}\xi(t-d(t))\\ f(\xi(t-d(t)))\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}\xi(t-d(t))\\ f(\xi(t-d(t)))\end{bmatrix}\\
&+\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}-e^{-2\alpha d_M}\begin{bmatrix}\xi(t-d_M)\\ f(\xi(t-d_M))\end{bmatrix}^T\\
&\times\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t-d_M)\\ f(\xi(t-d_M))\end{bmatrix}+d_M^2\xi^T(t)P_4\xi(t)+d_My^T(t)P_5y(t)\\
&-e^{-2\alpha d_M}\int_{t-d(t)}^t\xi^T(s)dsP_4\int_{t-d(t)}^t\xi(s)ds+d_M^2y^T(t)P_8y(t)\\
&-e^{-2\alpha d_M}\int_{t-d_M}^{t-d(t)}\xi^T(s)dsP_5\int_{t-d_M}^{t-d(t)}x(s)ds+\frac{d_M^4}{4}\xi^T(t)Q_6\xi(t)\\
&+\rho_Mf^T(\xi(t))P_{11}f(\xi(t))+\rho_M^2f^T(\xi(t))P_{11}f(\xi(t))\\
&-e^{-2\alpha d_M}\left[\int_{t-d(t)}^t y^T(s)ds+\int_{t-d_M}^{t-d(t)}y^T(s)ds\right]P_8\\
&\times\left[\int_{t-d(t)}^t y(s)ds+\int_{t-d_M}^{t-d(t)}y(s)ds\right]+e^{-2\alpha d_M}\\
&\times\begin{bmatrix}\xi(t)\\ \xi(t-d(t))\\ \xi(t-d_M)\\ \int_{t-d(t)}^t\xi(s)ds\\ \int_{t-d_M}^{t-d(t)}\xi(s)ds\end{bmatrix}^T\begin{bmatrix}-R_9 & R_9 & 0 & -R_9^T & 0\\ R_9^T & -R_9-R_9^T & R_9 & R_8^T & -R_8^T\\ 0 & R_9^T & -R_9 & 0 & R_8^T\\ -R_9 & R_8 & 0 & -R_7 & 0\\ 0 & -R_8 & R_8 & 0 & -R_7\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-d(t))\\ \xi(t-d_M)\\ \int_{t-d(t)}^t\xi(s)ds\\ \int_{t-d_M}^{t-d(t)}\xi(s)ds\end{bmatrix}\\
&-\int_{t-d_M}^t\int_u^t\xi^T(\lambda)d\lambda duQ_6\int_{t-d_M}^t\int_u^t\xi(\lambda)d\lambda du+\frac{d_M^4}{2}y^T(t)Q_7y(t)\\
&-2\int_{t-d_M}^t\int_u^t\dot{\xi}^T(\lambda)d\lambda duQ_7\int_{t-d_M}^t\int_u^t\dot{\xi}(\lambda)d\lambda duduQ_6\int_{t-d_M}^t\int_u^t\xi(\lambda)d\lambda du\\
&+\rho_M^2f^T(\xi(t))P_{11}f(\xi(t))-e^{-2\alpha\rho_M}\int_{t-\rho_M}^t f^T(\xi(s))P_{11}f(\xi(s))ds-2\alpha V_9(t)\\
&-e^{-2\alpha\rho_M}\int_{t-\rho_M}^t f^T(\xi(s))dsP_{11}\int_{t-\rho_M}^t f(\xi(s))ds-2Z^T(t)u(t).
\end{aligned}\tag{4.11}
$$

By (4.11), we conclude that

$$\dot{V}(t)-2Z^T(t)u(t)\le\psi^T(t)\Pi\psi(t).$$

From the Schur complement in aspect of LMI (3.5), we have the fact of $\Pi<0$. Noticing that $|\xi(t)|\le|\psi(t)|$, so

$$\dot{V}(t)-2Z^T(t)u(t)\le\lambda_{\max}\Pi|\xi(t)|^2.$$

On the other hand, it is easy to detect that

$$\begin{aligned} V(t) \leq & \big(\|P\|+2\|K\|+\|E\|\|P\|)|\xi(t)|^2+(\|P\|+\|R\|\big)\int_{t-d_M}^{t}|\xi(s)|^2ds \\ & +\|R\|\int_{t-d(t)}^{t}|\xi(s)|^2ds+2d_M\|P\|\int_{t+s}^{t}|\xi(\theta)|^2d\theta+(3d_M\|P\|+\|P\|+d_M\|R\|) \\ & \times\int_{t+s}^{t}y(\theta)|^2d\theta+d_M^2\|P\|\int_{\lambda}^{0}\int_{t+s}^{t}|y(\theta)|^2d\theta ds+\rho_M\|P\|\int_{t+s}^{t}|\xi(\theta)|^2d\theta. \end{aligned}$$

Let α be sufficiently small such that

$$\begin{aligned} \alpha(3\|P\|+\|E\|\|P\|+5d_M\|P\|+d_M^2\|P\|+\rho_M\|P\|)-\lambda_{\max}(\Pi) &< 0, \\ \alpha(2\|R\|+d_M\|R\|)-\lambda_{\min}(R) &\leq 0, \\ \alpha(2\|K\|)-\lambda_{\min}(R) &\leq 0. \end{aligned} \tag{4.12}$$

Hence,

$$\dot{V}+\alpha V(t)-2Z^T(t)u(t)\leq 0. \tag{4.13}$$

By definition, the delayed neural networks are exponentail passive. The proof of the theorem is complete. ■

5. Numerical Examples

In this section, four numerical examples are given to illustrate the effectiveness of the method developed in this work.

Example 5.1. Consider the delayed neural networks system (2.1) with parameters

$$\begin{aligned} C &= \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix},\ A=\begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix},\ B=\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \\ \epsilon_1^- &= \epsilon_2^-=0,\quad \epsilon_1^+=\epsilon_2^+=1. \end{aligned}$$

First, we assumed that the upper bound d_m is fixed as 1. The exponential convergence rates with various d_d are obtained from Theorem 3.1, [20] and [43] as shown in Table 1. d_d can be an arbitrary value, even d_d is very largej or $d(t)$ is not differentiable. These are called unknown d_d. Theorem 3.1 in this paper can also provides significantly better results than those in other literature.

Table 1. Allowable exponential convergence rate α for various d_d and $d_m = 1$ of Example 5.1.

Method	$d_d = 0.8$	$d_d = 0.9$	Unknown d_d
Wu (2008) [43]	0.8643	0.8344	0.8169
Ji (2014) [20]	0.8696	0.8354	0.8169
Ji (2015) [21]	0.8784	0.8484	0.8217
He (2016) [18]	0.8841	0.8570	0.8260
Theorem 3.1	4.1450	4.1450	4.1420

Second, if the exponential convergence rate of α is fixed as 0.8, the upper bounds of d_M for various d_d's from Theorem 3.1, [19], [43], and [53] are listed in Table 2. On the other hand, when d_M is zero, the upper bound delay was investigated in [31] and [23] under the different rate of convergence α. The results of the delay bound are listed in Table 3. From Table 4, it is implied that when $d(t)$ is time-varying, Theorem 3.1 gives less conservative results than the ones in [12, 23, 27].

TABLE 2. Allowable upper bound of d_M for various d_d and $\alpha = 1$ of Example 5.1.

Method	$d_d = 0.5$	$d_d = 0.8$	Unknown d_d
Xu (2005) [53]	-	-	-
He (2006) [19]	1.2606	0.9442	0.8310
Wu (2008) [43]	1.2787	1.0819	1.0366
Theorem 3.1	4.3842	4.3841	4.3752

TABLE 3. Allowable upper bound of d_M for various α of Example 5.1.

Method	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$
Park (2007) [32]	2.5900	0.9700	0.3500
Gau (2007) [12]	2.8200	1.1800	0.5400
Mou (2008) [31]	2.9000	1.3200	0.7200
Kwon (2009) [23]	2.9400	1.3500	0.7002
Theorem 3.1	6.6021	3.6051	2.5198

TABLE 4. Allowable upper bound of d_M for various d_d and $\alpha = 0.25$ in Example 5.1.

Method	$d_d = 0$	$d_d = 0.8$	Unknown d_d
Gau (2007) [12]	5.9000	2.8000	1.0400
Kwon (2008) [27]	6.0000	2.9000	1.4000
Kwon (2009) [23]	6.0000	3.5000	2.5300
Theorem 3.1	11.9997	11.9991	11.9214

Example 5.2. Consider exponential stability for neural networks system (2.1) with parameters

$$\begin{aligned} C &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \\ \epsilon_1^- &= \epsilon_2^- = 0, \quad \epsilon_1^+ = 0.4, \quad \epsilon_2^+ = 0.8 \end{aligned}$$

We choose $d_d = 0.8$, $d_d = 0.9$ and unknown d_d. Table 5 provides the comparisons of allowed upper bound time delay d_M. It is clear that, for this example, the delayed stability condition in this paper has not less conservatism than those in [17, 41, 56].

In [11, 17, 18, 51, 55], the authors also studied this example with different d_d and fixed α. As presented in Table 6, our developed method even provides satisfying results.

TABLE 5. The upper bound delay d_M for $\alpha = 0$ of Example 5.2.

Method	$d_d = 0.8$	$d_d = 0.9$	Unnkown d_d
He (2007) [17]	2.2552	1.4769	1.3606
Zhang (2009) [56]	2.8335	1.9342	1.7532
Wang (2012) [41]	3.0385	2.0250	1.8573
Theorem 3.1	2802.3	2794.6	2677.3

TABLE 6. Allowable upper bound d_m for various d_d and $\alpha = 0$ of Example 5.2.

Method	$d_d = 0.8$	$d_d = 0.9$
He (2007) [17]	2.3534	1.6050
Zeng (2011)(m=3) [51]	3.2160	2.1995
Ge (2014)(m=2) [11]	2.8980	1.9562
Zhang (2013) [55]	3.1409	1.6375
He (2016) [18]	3.7756	2.2201
Theorem 3.1	5540.4	5480.3

Example 5.3. We focus on exponential passivity for neural networks system (4.1) with the following parameters

$$\begin{aligned} C &= \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \ A = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \\ \epsilon_1^- &= \epsilon_2^- = -0.1, \quad \epsilon_1^+ = \epsilon_2^+ = 0.5, \\ \text{for } d(t) &= 0.2 + \frac{\cos^2(t)}{2}, \quad \rho(t) = 0.2 + \frac{|\sin(t)|}{2}. \end{aligned}$$

In this example, we interest in the exponential passivity for neural networks with discrete time-varying delay. Table 7 provides the calculated allowable upper bound d_M by using linear matrix inequalities (4.3).

TABLE 7. Calculated delay upper bound d_M for fixed $\rho_M = 0.7$ and different d_d and α of Example 5.3.

d_d	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0	3.0111	1.2311	0.3210	0.1130
0.1	2.0110	1.1020	0.3110	0.1031
0.3	1.3011	1.0322	0.2120	0.1020
0.5	1.0120	1.0002	0.1302	0.1020

The maximum convergence rate α, that guarantees the exponentially passive of this paper with various values of d_d and ρ_M for fixed $d_M = 0.7$, are obtained and represented in Table 8.

TABLE 8. Calculated convergence's rate α for fixed $d_M = 0.7$ and different d_d and ρ_M of Example 5.3.

d_d	$\rho_M = 0.7$	$\rho_M = 1$	$\rho_M = 1.25$
0.2	4.0101	3.5020	1.0102
0.7	3.1001	1.0202	0.0120
0.85	1.1010	0.4002	0.0103

Example 5.4. Consider exponential passivity for neural networks system (4.1) with parameters

$$\begin{aligned} C &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \\ \epsilon_1^- &= \epsilon_2^- = -0.1, \quad \epsilon_1^+ = \epsilon_2^+ = 0.5. \end{aligned}$$

Now, our purpose is to find the allowable maximum time delay d_M under different d_d and α of the passive system (4.1). Table 9 gives the results on the maximum d_M allowed via d_d and α.

TABLE 9. Calculated delay upper bound d_M for different d_d and α of Example 5.4.

d_d	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$
0	2857.0	6.1603	3.3811	2.3665	1.8293
2	2619.6	6.1601	3.3810	2.3665	1.8293
4	2618.2	6.1599	3.3810	2.3664	1.8293
6	2614.3	6.1593	3.3809	2.3664	1.8293

Now, let us compare the passivity condition in Theorem 4.1 to [44] and [57] by using $\epsilon_1^- = \epsilon_2^- = 0$, $\epsilon_1^+ = 0.4$, $\epsilon_2^+ = 0.8$. The corresponding upper bounds of d_M for various d_d derived by Theorem 4.1 are listed in Table 10, it is clear that the proposed passivity criterion has considerably less conservative than that in [44, 57].

TABLE 10. Calculated delay upper bound d_M for $\alpha = 0$ and different d_d of Example 5.4.

Method	$d_d = 0.8$	$d_d = 0.85$	$d_d = 0.9$	$d_d = 0.95$
Zhu (2013) [57]	1.0638	0.8532	0.7856	0.7608
Wu (2012) [44]	2.1346	1.7173	1.5592	1.5043
Theorem 4.1	1871.8	1871.7	1871.6	1871.6

6. Conclusions

In this paper, exponential stability and exponential passivity analysis problems of integro-differential neural network systems with time-varying delays are solved by using Lyapunov means and LMI term. By constructing the augmented Lyapunov-Krasovskii functional and utilizing the model transformation approach, sufficient conditions for exponential stability of the system are achieved as expressed in Theorem 3.1 and provide less conservative than those for exponential stability in the existing literature. Moreover, based on the results of exponential stability, we conducted the proof of integro - differential of the exponential passivity in Theorem 4.1 with discrete and distributed time-varying delays. Moreover, we evaluate the developed method through the four numerical examples conducted in previous works. The numerical results verify the improvement and effectiveness of the proposed exponential passivity criteria.

Acknowledgements

This work is supported by Development and Promotion of Science and Technology Talents Project (DPST), Research and Academic Affairs Promotion Fund, Faculty of Science, Khon Kaen University, Fiscal year 2020 and National Research Council of Thailand and Khon Kaen University, Thailand (6200069).

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