Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1661–1675

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Split Variational Inclusion and Fixed Point Problem for Asymptotically Nonexpansive Semigroup in Hilbert Spaces

Issara Inchan

Department of Mathematics, Uttaradit Rajabhat University, Uttaradit, Thailand
e-mail : peissara@uru.ac.th

Abstract The main objective of this paper is to introduced the iterative method for the split variational inclusion and asymptotically nonexpansive semigroups in Hilbert space. We study the iterative scheme and prove the strong convergence to a common solution of the split variational inclusion and the set of common fixed points of one-parameter asymptotically nonexpansive semigroups. The results presented in this paper improved the result of Wen and Chen [D.J. Wen, Y.A. Chen, Iterative methods for split variational inclusion and fixed point problem of nonexpansive semigroup in Hilbert spaces, Journal of Inequalities and Applications (2015)], and many authors.

MSC: 46C05; 47D03; 47H09; 47H10; 47H20

Keywords: asymptotically nonexpansive semigroup; fixed point; split variational inclusion

Submission date: 12.03.2020 / Acceptance date: 27.08.2020

1. Introduction

Let H be a real Hilbert Space, C a nonempty closed convex subset of H. Recall that a self-mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and a self-mapping T of C is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and T is an asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. Let $A : H_1 \to H_2$ be a mapping then $A^* : H_2 \to H_1$ is an adjoint operator of A if and only if $\langle A^*y, x\rangle = \langle y, Ax\rangle$ for $x \in H_1, y \in H_2$.

Recall also that a one-parameter family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see [2]) if the following conditions are satisfied:

Published by The Mathematical Association of Thailand.

(i) $T(0)x = x, x \in C$
(ii) $T(s+t)(x) = T(s)T(t), s,t \geq 0, x \in C$
(iii) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0,\infty)$
(iv) there exists a bounded measurable function $L : [0,\infty) \to [0,\infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\|x-y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called contraction semigroup if $L_t < 1$, $\mathcal{T}$ is called nonexpansive semigroup if $L_t = 1$ for all $t > 0$ and $\mathcal{T}$ is an asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities(see [3–7]). However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space(see [7, 8]).

The theory of variational inequalities, which was introduce by Stampacchia [9] is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [10–12] and the references cited therein.

The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems.

In 2006, Marino and Xu [13], introduced the approximate a fixed point of a nonexpansive mapping for the following general iterative methods:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n B)Tx_n, \tag{1.1}$$

where $\{\alpha_n\} \subseteq [0,1]$, f is a contraction of H into itself, and B is a strongly positive bounded linear operator on H. They prove that the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(T)$ and the unique solution of the following variational inequality:

$$\langle (B - \gamma f)x^*, x^* - w\rangle \leq 0, \forall w \in Fix(T), \tag{1.2}$$

which is also the optimality condition of the minimization problem.

Recall also that a multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if, for all $x, y \in H_1, u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0. \tag{1.3}$$

A monotone mapping M is maximal if the Graph(M) is not property contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $(x,u) \in H_1 \times H_1, \langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in Graph(M)$ implies that $u \in Mx$.

From a monotone mapping M the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \forall x \in H_1, \tag{1.4}$$

for some $\lambda > 0$, where I is the identity mapping on H_1. Note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive and firmly nonexpansive mapping.

In 2012, Byrne et. al., [14] introduce the **split variational inclusion problem** : find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1(x^*), \\ y^* = Ax^* \in H_2 : 0 \in B_2(y^*). \end{cases} \tag{1.5}$$

The solution set of problem (1.5) is denote by $\Im = \{x^* \in H_1 : 0 \in B_1(x^*), y^* = Ax^* \in H_2 : 0 \in B_2(y^*)\}$.

In 2015 Wen and Chen [15] introduce a modified general iterative method for a split variational inclusion and nonexpansive semigroups, which is defined sequence $\{x_n\}$ the following way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds, \tag{1.6}$$

where $\gamma \in [0,1]$ and $\{\alpha_n\} \subseteq [0,1]$, then they prove the strong convergence of $\{x_n\}$ to $q \in Fix(\mathcal{T}) \cap \Im$.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.1. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x - y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 1.2. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x - y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \not< 1$, then $\mathcal{T}$ is not necessary nonexpansive semigroup.

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

The motivation of this work we study the iterative scheme of Wen and Chen [15] for $\mathcal{T}$ is an asymptotically nonexpansive semigroup then we peove the strong convergence theorem of $\{x_n\}$ generated by (1.6).

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space, then the following holds:*
(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y,(x+y)\rangle, \forall x,y \in H$;
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, t \in [0,1], \forall x,y \in H$.

Lemma 2.2 ([8]). *Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, If $\{x_n\}$ is a sequence in C satisfying the properties:*
(i) $x_n \rightharpoonup z$; *and*
(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,
then $z \in Fix(\mathcal{T})$.

Lemma 2.3 ([8]). *Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,*

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(u)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.4 ([13]). *Let B be a strongly positive linear bounded operator on a Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \varrho < \|B\|^{-1}$. Then $\|I - \varrho B\| \leq 1 - \varrho\overline{\gamma}$.*

Lemma 2.5 ([13]). *Let C be a nonempty closed convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0,1)$ and B is a strongly positive bounded linear operator with a coefficient $\overline{\gamma} > 0$. Then for $0 < \gamma < \frac{\overline{\gamma}}{\rho}$,*

$$\langle x-y, (B-\gamma f)x - (B-\gamma f)y\rangle \geq (\overline{\gamma} - \gamma\rho)\|x-y\|^2, \forall x,y \in H.$$

That is $B - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\rho$.

Lemma 2.6 ([16]). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n b_n + \sigma_n,$$

where $\{\gamma_n\}_{n=1}^{\infty} \subseteq (0,1)$ and $\{b_n\}_{n=1}^{\infty}, \{\sigma_n\}_{n=1}^{\infty}$ are sequence in $\mathbb{R}$ such that
(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \leq 0$;
(iii) $\sigma_n \geq 0$ *and* $\Sigma_{n=1}^{\infty}\sigma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([17, 18]). *Let $S : H \to H$ be averaged and $T : H \to H$ be nonexpansive have:*
(i) $W = (1-\alpha)S + \alpha T$ *is averaged, where* $\alpha \in (0,1)$.
(ii) The composite of finitely many averaged mapping is averaged.

Theorem 2.8 ([19]). *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0,1)$ and $T : H_1 \to H_1$ be a nonexpansive mapping such that $\Omega = Fix(\mathcal{T}) \cap \Im \neq \emptyset$. For a given $x_0 \in H_1$ arbitrary, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n], \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \end{cases} \tag{2.1}$$

where $\lambda > 0$ *and* $\epsilon \in (0, 1/L)$*,* L *is the spectral radius of the operator* A^*A*, and* A^* *is the adjoint of* A*;* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$*, and* $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$*. Then the sequens* $\{u_n\}$ *and* $\{x_n\}$ *both convergence strongly to* $z \in \Omega$*, where* $z = P_\Omega(z)$*.*

Lemma 2.9 ([19])**.** *The split variational inclusion problem* (2.1) *is equivalent to finding* $x^* \in H_1$ *such that* $y^* = Ax^* \in H_2 : x^* = J_\lambda^{B_1}$ *and* $y^* = J_\lambda^{B_2}(y^*)$ *for some* $\lambda > 0$*.*

3. Main Results

In the first Theorem in this section we prove the unique fixed point by Banach contraction principle of Φ. The second Theorem we prove the strong convergence of modified general iterative method for a split variational inclusion and asymptotically nonexpansive semigroups to $q \in \Omega$ which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega.$$

Theorem 3.1. *Let* H_1 *and* H_2 *be two Hilbert space, let* $A : H_1 \to H_2$ *be a bounded linear operator and* B *be a strongly positive bounded linear operator on* H_1 *with constant* $\overline{\gamma} > 0$*. Let* $B_1 : H_1 \to 2^{H_1}$ *be maximal monotone mapping and* $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ *be a one-operator asymptotically nonexpansive semigroup on* H_1 *such the* $\Omega = Fix(\mathcal{T}) \cap \Im \neq \emptyset$*. Assume that* $f : H_1 \to H_1$ *is a contraction mapping with constant* $\rho \in (0,1)$*. For any* $\alpha \in (0,1)$*, define the mapping* Φ *on* H_1 *by*

$$\Phi(x) = \alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds,$$

where $t > 0$*,* $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$*, and* $\epsilon \in (0, \frac{1}{L})$*,* L *is spectral radius of the operator* A^*A*, and* A^* *is the adjoint of* A *and* $1 < \frac{1}{t}\int_0^t L_s ds < a < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$*. Then the mapping* Φ *is a contraction and has a unique fixed point.*

Proof. Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are firmly nonexpansive, they are averaged. For $\epsilon \in (0, \frac{1}{L})$, the mapping $I + \epsilon A^*(J_\lambda^{B_2} - I)A$ is averaged; see e.g.[20]. It follows from Lemma 2.7 (ii) that the mapping $J_\lambda^{B_1}(I + \epsilon A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence nonexpansive. By Lemma 2.4, for any $x, y \in H_1$, we have

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\| &= \|\alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds \\
&\quad -\alpha\gamma f(y) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]ds\| \\
&\leq \alpha\gamma\|f(x) - f(y)\| \\
&\quad +(1 - \alpha\overline{\gamma})\|\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds \\
&\quad -\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]ds\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha\gamma\rho\|x-y\| + (1-\alpha\overline{\gamma})(\frac{1}{t}\int_0^t L_s ds)\|J_\lambda^{B_1}[x+\epsilon A^*(J_\lambda^{B_2}-I)Ax] \\
&\quad - J_\lambda^{B_1}[y+\epsilon A^*(J_\lambda^{B_2}-I)Ay]\| \\
&\leq \alpha\gamma\rho\|x-y\| + (1-\alpha\overline{\gamma})(\frac{1}{t}\int_0^t L_s ds)\|x-y\| \\
&= \alpha\gamma\rho\|x-y\| + (1-\alpha\overline{\gamma})a\|x-y\| \\
&\leq [a-\alpha(\overline{\gamma}a-\gamma\rho)]\|x-y\|.
\end{aligned}$$

From $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$ and $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$, we have $[a-\alpha(\overline{\gamma}a-\gamma\rho)] < 1$. It follows that Φ is a contraction mapping. By the Banach contraction principle, $\Phi(x)$ has a unique fixed point x_α, that is

$$x_\alpha = \alpha\gamma f(x_\alpha) + (I-\alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x_\alpha+\epsilon A^*(J_\lambda^{B_2}-I)Ax_\alpha]ds.$$

■

Next Theorem we study the general iterative scheme (1.6) for the split variational inclusion of asymptotically nonexpansive semigroups and prove the strong convergence of iterative to $q \in \Omega$ which is the unique solution of the following variational inequality: $\langle (B-\gamma f)q, q-w\rangle \leq 0, \forall w \in \Omega$.

Theorem 3.2. *Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \Im \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0,1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0,1), \{t_n\} \subseteq (0,\infty)$ and $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ satisfy:*
(i) $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^\infty \alpha_n = \infty$, *and* $\Sigma_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
(ii) $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty}\frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.
Then the sequence $\{x_n\}$ generated by (1.6) converge strongly to $q \in \Omega$, which is the unique solution of the following variational inequality:

$$\langle (B-\gamma f)q, q-w\rangle \leq 0, \forall w \in \Omega.$$

Proof. Let $p \in \Omega$, we have $p = J_\lambda^{B_1}p, J_\lambda^{B_2}(Ap) = Ap$ and $T(s)p = p$. From (1.6), let $u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2}-I)Ax_n]$, and Lemma 2.9, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|J_\lambda^{B_1}[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n] - J_\lambda^{B_1}p\|^2 \\
&\leq \|x_n + \epsilon A^*(J_\lambda^{B_2}-I)Ax_n - p\|^2 \\
&\leq \|x_n - p\|^2 \\
&\quad +2\epsilon\langle x_n - p, A^*(J_\lambda^{B_2}-I)Ax_n\rangle + \epsilon^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2. \qquad (3.1)
\end{aligned}$$

By the definition of A and A^*, we obtain

$$\begin{aligned}\epsilon^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2 &= \epsilon^2\langle A^*(J_\lambda^{B_2}-I)Ax_n, A^*(J_\lambda^{B_2}-I)Ax_n\rangle \\ &= \epsilon^2\langle (J_\lambda^{B_2}-I)Ax_n, AA^*(J_\lambda^{B_2}-I)Ax_n\rangle \\ &\leq L\epsilon^2\langle (J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= L\epsilon^2\|(J_\lambda^{B_2}-I)Ax_n\|^2. \end{aligned} \tag{3.2}$$

And we have

$$\begin{aligned}2\epsilon\langle x_n-p, A^*(J_\lambda^{B_2}-I)Ax_n\rangle &= 2\epsilon\langle A(x_n-p), (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= 2\epsilon\langle A(x_n-p)+(J_\lambda^{B_2}-I)Ax_n \\ &\quad -(J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= 2\epsilon\langle A(x_n-p)+(J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle \\ &\quad -\langle (J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= 2\epsilon[\langle J_\lambda^{B_2}Ax_n-Ap, (J_\lambda^{B_2}-I)Ax_n\rangle - \|(J_\lambda^{B_2}-I)Ax_n\|^2] \\ &\leq 2\epsilon[\frac{1}{2}\|(J_\lambda^{B_2}-I)Ax_n\| - \|(J_\lambda^{B_2}-I)Ax_n\|^2] \\ &= -\epsilon\|(J_\lambda^{B_2}-I)Ax_n\|^2. \end{aligned} \tag{3.3}$$

From (3.1), (3.2), (3.3) and $\epsilon \in (0, \frac{1}{L})$, it follows that

$$\|u_n-p\|^2 \leq \|x_n-p\|^2+\epsilon(L\epsilon-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2 \leq \|x_n-p\|^2. \tag{3.4}$$

Next, we set $w_n = \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds$ for $n \geq 0$, since $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ and from (3.4), we have

$$\begin{aligned}\|w_n-p\| &= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(s)p\| \\ &\leq \frac{1}{t_n}\int_0^{t_n} \|T(s)u_n - T(s)p\| ds \\ &\leq \frac{1}{t_n}\int_0^{t_n} L_s ds\|u_n-p\| \\ &\leq \frac{1}{t_n}\int_0^{t_n} L_s ds\|x_n-p\| \\ &\leq a\|x_n-p\|, \end{aligned} \tag{3.5}$$

where $a = \sup_{n\geq 1}\{a_n\}$. It follows from (1.6), (3.5) and Lemma 2.4, that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n) \\ &\quad +(I-\alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n]ds - p\| \\ &= \|\alpha_n(\gamma f(x_n)-Bp) \\ &\quad +(I-\alpha_n B)\frac{1}{t_n}\int_0^{t_n} (T(s)J_\lambda^{B_1}[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n]-T(s)p)ds\| \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|\gamma f(x_n) - Bp\| + (1-\alpha_n\overline{\gamma})\frac{1}{t_n}\int_0^{t_n}\|T(s)u_n - T(s)p\|ds\\
&\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + (1-\alpha_n\overline{\gamma})a\|x_n - p\|\\
&\leq \alpha_n\gamma\rho\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + (1-\alpha_n\overline{\gamma})a\|x_n - p\|\\
&= [1-\alpha_n(\overline{\gamma}a - \gamma p)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|.
\end{aligned}
$$

Since $1 < a \leq \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ and $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, we have $\overline{\gamma}a - \gamma\rho > 0$. By a simple induction, we have

$$
\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{\overline{\gamma}a - \gamma\rho}\|\gamma f(p) - Bp\|\}.
$$

Therefor, $\{x_n\}$ is bounded, and so are $\{u_n\}$ and $\{w_n\}$. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (1.6), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_nB)w_n - \alpha_n\gamma f(x_{n-1}) + (I-\alpha_{n-1}B)w_{n-1}\|\\
&= \|\alpha_n\gamma[f(x_n) - f(x_{-1})] + (\alpha_n - \alpha_{n-1})\gamma f(x_{n-1})\\
&\quad +(I-\alpha_nB)(w_n - w_{n-1}) - (\alpha_n - \alpha_{n-1})Bw_{n-1}\|\\
&\leq \alpha_n\gamma\|f(x_n) - f(x_{-1})\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1})\|\\
&\quad +(1-\alpha_n\overline{\gamma})\|w_n - w_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Bw_{n-1}\|\\
&\leq \alpha_n\gamma\rho\|x_n - x_{-1}\| + (1-\alpha_n\overline{\gamma})\|w_n - w_{n-1}\|\\
&\quad +|\alpha_n - \alpha_{n-1}|[\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|]. \qquad (3.6)
\end{aligned}
$$

Since $\frac{1}{t_n}\int_0^{t_n} T(s)pds = \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds$, we consider

$$
\begin{aligned}
\|w_n - w_{n-1}\| &= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_nds - \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)u_{n-1}ds\|\\
&= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_nds - \frac{1}{t_n}\int_0^{t_n} T(s)u_{n-1}ds + \frac{1}{t_n}\int_0^{t_n} T(s)u_{n-1}ds\\
&\quad +\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds\\
&\quad -\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds - \frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds\\
&\quad +\frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds\\
&\quad -\frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds + \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds\\
&\quad -\frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)u_{n-1}ds\|
\end{aligned}
$$

$$\begin{aligned}
&= \|(\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)u_{n-1}ds) \\
&\quad +(\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds - \frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds \\
&\quad -\frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)u_{n-1}ds + \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds) \\
&\quad +(\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds + \frac{1}{t_n}\int_{t_{n-1}}^{0} T(s)u_{n-1}ds) \\
&\quad -(\frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds + \frac{1}{t_n}\int_{t_{n-1}}^{0} T(s)pds)\| \\
&= \|\frac{1}{t_n}\int_0^{t_n} [T(s)u_n - T(s)u_{n-1}]ds \\
&\quad +(\frac{1}{t_n} - \frac{1}{t_{n-1}})\int_0^{t_{n-1}} [T(s)u_{n-1} - T(s)p]ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n} T(s)u_{n-1}ds - \frac{1}{t_n}\int_{t_{n-1}}^{t_n} T(s)pds\| \\
&= \|\frac{1}{t_n}\int_0^{t_n} [T(s)u_n - T(s)u_{n-1}]ds \\
&\quad +(\frac{1}{t_n} - \frac{1}{t_{n-1}})\int_0^{t_{n-1}} [T(s)u_{n-1} - T(s)p]ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n} [T(s)u_{n-1} - T(s)p]ds\| \\
&\leq \frac{1}{t_n}\int_0^{t_n} \|T(s)u_n - T(s)u_{n-1}\|ds \\
&\quad +|\frac{1}{t_n} - \frac{1}{t_{n-1}}|\int_0^{t_{n-1}} \|T(s)u_{n-1} - T(s)p\|ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n} \|T(s)u_{n-1} - T(s)p\|ds\| \\
&\leq \frac{1}{t_n}\int_0^{t_n} L_s ds\|u_n - u_{n-1}\| + \frac{|t_{n-1} - t_n|}{t_n}\frac{1}{t_{n-1}}\int_0^{t_{n-1}} L_s ds\|u_{n-1} - p\| \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n} L_s ds\|u_{n-1} - p\|.
\end{aligned}$$

Now, we taking $\lim_{s\to\infty} L_s = 1$, it follows that $\frac{1}{t_n}\int_0^{t_n} L_s ds \to \frac{1}{t_n}\int_0^{t_n} ds$, and hence

$$\|w_n - w_{n-1}\| \leq \|u_n - u_{n-1}\| + \frac{2|t_{n-1} - t_n|}{t_n}\|u_{n-1} - p\|. \tag{3.7}$$

From $\epsilon \in (0, \frac{1}{L})$ and mapping $J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]$ is averaged and hence nonexpansive, then we have

$$\begin{aligned} \|u_n - u_{n-1}\| &= \|J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]x_n - J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\|. \end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \gamma\rho \|x_n - x_{-1}\| \\ &\quad + (1 - \alpha_n \overline{\gamma})[\|x_n - x_{n-1}\| + \frac{2|t_{n-1} - t_n|}{t_n}\|u_{n-1} - p\|] \\ &\quad + |\alpha_n - \alpha_{n-1}|[\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|] \\ &\leq [1 - \alpha_n(\overline{\gamma} - \gamma\rho)]\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \frac{2|t_{n-1} - t_n|}{t_n})M, \end{aligned} \tag{3.9}$$

where $m = \max\{sup_{n\in\mathbb{N}}[\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|], sup_{n\in\mathbb{N}}\|u_{n-1} - p\|\}$. It follows from condition $(i) - (ii)$ and Lemma 2.6, hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Consider,

$$\begin{aligned} \|x_n - w_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + (I - \alpha_n B)w_n - w_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) + Bw_n\|. \end{aligned}$$

From condition (i) and (3.10), we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0,$$

and that

$$\lim_{n\to\infty} \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| = 0. \tag{3.11}$$

For any $u \geq 0$, we have

$$\begin{aligned} \|x_n - T(u)x_n\| &\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| + \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\ &\quad - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \|T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)x_n\| \\ &\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\ &\quad + \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\ &\quad + L_u\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - x_n\|. \end{aligned}$$

From (3.11), Lemma 2.2 and $\limsup_{u\to\infty} L_u \leq 1$, we have

$$\lim_{n\to\infty} \|x_n - T(u)x_n\| = 0. \tag{3.12}$$

By the definition of x_n, (3.3), (3.4) and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|\alpha_n\gamma f(x_n)+(1-\alpha_n B)w_n-p\|^2 \\
&= \|(w_n-p)+\alpha_n(\gamma f(x_n)-Bw_n)\|^2 \\
&\leq \|w_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Bw_n, x_{n+1}-p\rangle \\
&\leq \|u_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Bw_n, x_{n+1}-p\rangle \\
&\leq [\|x_n-p\|^2+\epsilon(L\epsilon-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2] \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bw_n, x_{n+1}-p\rangle \\
&\leq \|x_n-p\|^2-\epsilon(1-L\epsilon)\|(J_\lambda^{B_2}-I)Ax_n\|^2+2\alpha_n M_2^2, \qquad (3.13)
\end{aligned}$$

where $M_2=\max\{\sup_{n\in\mathbb{N}}\|\gamma f(x_n)-Bw_n\|, \sup_{n\in\mathbb{N}}\|x_{n+1}-p\|\}$ and $\epsilon\in(0,\frac{1}{L})$, it implies that

$$\begin{aligned}
\epsilon(1-L\epsilon)\|(J_\lambda^{B_2}-I)Ax_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n M_2^2 \\
&\leq \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\alpha_n M_2^2.
\end{aligned}$$

From (3.10), we obtain

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Ax_n\|=0. \qquad (3.14)$$

From (3.1), (3.3) and $\epsilon\in(0,\frac{1}{L})$, that

$$\begin{aligned}
\|u_n-p\|^2 &= \|J_\lambda^{B_1}[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n]-J_\lambda^{B_1}p\|^2 \\
&\leq \langle u_n-p, x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n-p\rangle \\
&= \frac{1}{2}\{\|u_n-p\|^2+\|x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n-p\|^2 \\
&\quad -\|u_n-p-[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n-p]\|^2\} \\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2+\epsilon(L\epsilon-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2 \\
&\quad -\|u_n-x_n-\epsilon A^*(J_\lambda^{B_2}-I)Ax_n\|^2\} \\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-[\|u_n-x_n\|^2+\epsilon^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2 \\
&\quad -2\epsilon\langle u_n-x_n, A^*(J_\lambda^{B_2}-I)Ax_n\rangle]\} \\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2 \\
&\quad +2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\},
\end{aligned}$$

which implies that

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|u_n-x_n\|^2+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|. \qquad (3.15)$$

It follows from (3.13) and (3.15) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|u_n-p\|^2+2\alpha_n M_2^2 \\
&\leq \|x_n-p\|^2-\|u_n-x_n\|^2+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\| \\
&\quad +2\alpha_n M_2^2,
\end{aligned}$$

that is

$$\begin{aligned}\|u_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\ &\quad +2\alpha_n M_2^2 \\ &\leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\ &\quad +2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M_2^2.\end{aligned}$$

From (3.10), (3.15) and condition (i), it follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.16}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded, there exists weak limit w of $\{x_n\}$. Without loss of generality, we may assume that subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which is $x_{n_j} \rightharpoonup w$. From (3.16), we have subsequence $\{u_{n_j}\}$ of $\{u_n\}$, which is $u_{n_j} \rightharpoonup w$. Moreover, $u_{n_j} = J_\lambda^{B_1}[x_{n_j} + \epsilon A^*(J_\lambda^{B_2} - I)Ax_{n_j}]$ with

$$\frac{(x_{n_j} - u_{n_j}) + \epsilon A^*(J_\lambda^{B_2} - I)Ax_{n_j}}{\lambda} \in B_1 u_{n_j}.$$

By taking limit $j \to \infty$, and (3.14), (3.16) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $Ax_{n_j} \rightharpoonup Aw$. From (3.14) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, we obtain $Aw \in B_2(Aw)$. It follows from Lemma 2.9 that $w \in \Im$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \leq 0$, where $q = P_\Omega(I - B + \gamma f)q$. From the sequence $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \lim_{j\to\infty}\langle \gamma f(q) - Bq, x_{n_j} - q\rangle.$$

Assume that $w \neq T(u)w$. From (3.11) and Opial's property, we have

$$\begin{aligned}\liminf_{j\to\infty}\|x_{n_j} - w\| &< \liminf_{j\to\infty}\|x_{n_j} - T(u)w\| \\ &\leq \liminf_{j\to\infty}(\|x_{n_j} - T(u)x_{n_j}\| + \|T(u)x_{n_j} - T(u)w\|) \\ &\leq \liminf_{j\to\infty}(\|x_{n_j} - T(u)x_{n_j}\| + L_u\|x_{n_j} - w\|) \\ &\leq \liminf_{j\to\infty} L_u\|x_{n_j} - w\|.\end{aligned}$$

If we letting $u \to \infty$, we have $\limsup_{u\to\infty} L_u \leq 1$, it follows that

$$\liminf_{j\to\infty}\|x_{n_j} - w\| < \liminf_{j\to\infty}\|x_{n_j} - w\|.$$

This is a contradiction. Then $w \in Fix(\mathcal{T})$. Consequently, $w \in \Omega$. It follows from (**??**) that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \langle \gamma f(q) - Bq, w - q\rangle \leq 0. \tag{3.17}$$

On the other hand, we shall show that the uniqueness of a solution of the variational inequality

$$\langle (B - \gamma f)x, x - w\rangle \leq 0, \forall w \in \Omega. \tag{3.18}$$

Suppose that $q, \widehat{q} \in \Omega$ are solution to (3.18), then

$$\langle (B - \gamma f)q, q - \widehat{q}\rangle \leq 0, \tag{3.19}$$

and

$$\langle (B-\gamma f)\widehat{q}, \widehat{q}-q\rangle \leq 0. \tag{3.20}$$

From (3.19) and (3.20), we have

$$\langle (B-\gamma f)q-(B-\gamma f)\widehat{q}, q-\widehat{q}\rangle \leq 0.$$

By Lemma 2.5, the strong monotone of $B-\gamma f$, we obtain $q=\widehat{q}$. Finally, we show that $\{x_n\}$ converges strongly to q as $n\to\infty$. From (1.6), (3.4) and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \langle \alpha_n\gamma f(x_n)+(I-\alpha_n B)w_n-q, x_{n+1}-q\rangle \\
&= \alpha_n\langle \gamma f(x_n)-Bq, x_{n+1}-q\rangle+\langle (I-\alpha_n B)(w_n-q), x_{n+1}-q\rangle \\
&\leq \alpha_n\langle \gamma f(x_n-f(q), x_{n+1}-q\rangle+\alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\quad +(1-\alpha_n\overline{\gamma})\|w_n-q\|\|x_{n+1}-q\| \\
&\leq \alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\|+\alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\quad +(1-\alpha_n\overline{\gamma})\|w_n-q\|\|x_{n+1}-q\| \\
&= [1-\alpha_n(\overline{\gamma}-\gamma\rho)]\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +\alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\leq \frac{1-\alpha_n(\overline{\gamma}-\gamma\rho)}{2}(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
&\quad +\alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\leq \frac{1-\alpha_n(\overline{\gamma}-\gamma\rho)}{2}\|x_n-q\|^2+\frac{1}{2}\|x_{n+1}-q\|^2 \\
&\quad +\alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle,
\end{aligned}$$

it follows that

$$\|x_{n+1}-q\|^2 \leq \frac{1-\alpha_n(\overline{\gamma}-\gamma\rho)}{2}\|x_n-q\|^2+\alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle.$$

From $0<\gamma<\frac{\overline{\gamma}}{\rho}$, condition (i) and (3.17), from Lemma 2.6, we obtain that $\lim_{n\to\infty}\|x_n-q\|=0$ and then $\{x_n\}$ converges strongly to q, which is the unique solution of the following variational inequality $\langle (B-\gamma f)q, q-w\rangle \leq 0$ for all $w\in\Omega$. This completes the proof. ■

Theorem 3.3 ([15]). *Let H_1 and H_2 be two Hilbert space, let $A: H_1\to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma}>0$. Let $B_1: H_1\to 2^{H_1}$, $B_2: H_2\to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T}:=\{T(s): 0\leq s<\infty\}$ be a one-operator nonexpansive semigroup on H_1 such the $\Omega=Fix(\mathcal{T})\cap\Im\neq\emptyset$. Assume that $f: H_1\to H_1$ is a contraction mapping with constant $\rho\in(0,1)$, $\gamma\in(0,\frac{\overline{\gamma}}{\rho})$, and $\epsilon\in(0,\frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1\in H_1$, and suppose that the sequence $\{\alpha_n\}\subseteq(0,1)$, $\{t_n\}\subseteq(0,\infty)$ satisfy:*

(i) $\lim_{n\to\infty}\alpha_n=0, \Sigma_{n=1}^{\infty}\alpha_n=\infty$, *and* $\Sigma_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$;

(ii) $\lim_{n\to\infty}t_n=\infty$ *and* $\lim_{n\to\infty}\frac{|t_n-t_{n-1}|}{\alpha_n t_n}=0$.

Then the sequence $\{x_n\}$ generated by (1.6) converge strongly to $q\in\Omega$, which is the unique solution of the following variational inequality:

$$\langle (B-\gamma f)q, q-w\rangle \leq 0, \forall w\in\Omega.$$

Proof. From examples 1.1 and 1.2, we see that a nonexpansive semigroups is an asymptotically nonexpansive semigroups then from Theorem 3.2 can be prove this theorem. ∎

Theorem 3.4. *Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0,1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0,1), \{t_n\} \subseteq (0,\infty)$, define $\{x_n\}$ in the following manner:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds, \quad (3.21)$$

and satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, *and* $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.

Then the sequence $\{x_n\}$ generated by (3.21) converge strongly to $q = P_\Omega(q)$, which is the unique solution of the following variational inequality:

$$\langle (I-f)q, q-w\rangle \leq 0, \forall w \in \Omega.$$

Proof. Putting $\gamma = 1$ and $B = I$, iterative scheme (1.6) reduces to (3.21). The desired conclusion follows immediately from Theorem 3.2. This complete the proof. ∎

Acknowledgements

The author would like to thank Uttaradit Rajabhat University for financial support. Moreover, we would like to thank Prof. Dr. Somyot Plubiteng for providing valuable suggestions.

References

[1] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171–174.

[2] H.K. Xu, Strong asymptotic behavior of almost-orbits of nonlinear semigroups, Nonlinear. Anal. 46 (2001) 135–151.

[3] S. Ishikawa, Fixed point theorems for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974) 147–150.

[4] W.A. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.

[5] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[6] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Amer. Math. Soc. 43 (1991) 153–159.

[7] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276–286.

[8] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear. Anal. 64 (2006) 1140–1152.

[9] G. Stampacchia, Formes bilinéaires coercitives sur les ensembles convexes, Comptes Rendus de l'Académie des Sciences, Vol. 258 (1964) 4413–4416.

[10] A. Bensoussan, J.L. Lions, Application des Inequations Variationelles en Control et en Stochastiques, Dunod, Paris, 1978.

[11] R. Glowinski, P. Letallec, Augmented Kargrangin and Operator-splitting Methods in Control Theory, Springer-Verlag, New York, 1989.

[12] P.T. Harker, J.S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithm and applications, Math. Program. 48 (1990) 161–220.

[13] G. Marino, H. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318 (2006) 43–52.

[14] C. Byrne, Y. Censor, A. Gibali and S. Reich, Weak and strong convergence of algorithms for the split common null point problem, J. Nonlinear Convex Anal. 13 (2012) 759–775.

[15] D.J. Wen, Y.A. Chen, Iterative methods for split variational inclusion and fixed point problem of nonexpansive semigroup in Hilbert spaces, Journal of Inequalities and Applications (2015) 2015:24 DOI 10.1186/s13660-014-0528-9.

[16] H.K. Xu, Iterative algorithm for nonlinear operators, J. Lond. Math. Soc. 66 (2) (2002) 1–17.

[17] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer New York, 2011.

[18] Y. Shehu, An iterative method for nonexpansive semigroups, variational inclusions and generalized equilibrium problems, Math. Comput. Model. 55 (2012) 1301–1314.

[19] K.R. Kazmi, S.H. Rizvi, An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping, Optim. Lett. (2013). doi:10.1007/s11590-013-0629-2.

[20] A. Moudafi, Split monotone variational inclusions, J. Optim. Theory Appl. 150 (2011) 275–283.