

Dedicated to Prof. Suthep Suantai on the occasion of his 60th anniversary

Generalized Ulam-Hyers Stability, Well-Posedness and Limit Shadowing of a Fixed Point Problem for Chatterjea Contractive Mappings in M -Metric Spaces

Pathaithep Kumrod¹, Wutiphol Sintunavarat^{1,*}, Yeol Je Cho² and Yi-bin Xiao³

¹Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12120, Thailand
email : wutiphol@mathstat.sci.tu.ac.th (W. Sintunavarat)

²Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea

³School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, P. R. China

Abstract The purpose of this paper is to investigate new types of the Ulam-Hyers stability, the well-posedness and the limit shadowing type of the fixed point problem for Chatterjea contractive mappings in the framework of M -metric spaces. Some illustrative examples for showing main results in this paper are also given. The main results in this paper are motivated from the results of Pansuwan et al. [A. Pansuwan, W. Sintunavarat, J.Y. Choi, Y.J. Cho, Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problems in M -metric spaces, *J. Nonlinear Sci, Appl.* 9 (2016) 4489-4499].

MSC: 47H09; 47H10; 54H25

Keywords: M -metric space; Ulam-Hyers stability type (C); well-posedness type (C); limit shadowing type (C); fixed point problem

Submission date: 11.02.2020 / Acceptance date: 19.06.2020

1. INTRODUCTION

In 1922, Banach [1] proved one of famous results in mathematics which is called the Banach contraction principle. This principle is shown below.

Theorem 1.1 ([1]). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping satisfying the Banach contractive condition, that is,*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

*Corresponding author.

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.1 has been generalized and applied by many authors in many ways since it is simplicity and usefulness. Many authors investigated fixed point results for mappings satisfying other contractive conditions in metric spaces. Two other famous fixed results in metric spaces are presented by Kannan [2] and Chatterjea [3]. These results are as follows:

Theorem 1.2 ([2]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping satisfying the Kannan contractive condition, that is,

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad (1.2)$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Theorem 1.3 ([3]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping satisfying the Chatterjea contractive condition, that is,

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)] \quad (1.3)$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

One of the directions to study the fixed point theory is to prove fixed point results in new spaces. Several authors introduced some extensions of metric spaces and proved the fixed points results for mappings satisfying the Banach contractive condition, the Kannan contractive condition and the Chatterjea contractive condition in such spaces. We now give the background of some extensions of metric spaces, which are the motivation in this paper. In 1994, Matthews [4] introduced the concept of a partial metric space, which is an extension of a metric space. Many authors have studied some geometric structures of partial metric spaces and their topological properties. Motivated by the idea of Matthews's partial metric space, Asadi et al. [5] presented a new extension of a partial metric space, which is called an M -metric space. They studied the topological and geometric structures of M -metric spaces and established fixed point theorems for mappings satisfying the Banach contractive condition and the Kannan contractive condition in M -metric spaces as follows:

Theorem 1.4 ([5]). Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in [0, 1)$ such that

$$m(Tx, Ty) \leq km(x, y) \quad (1.4)$$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 1.5 ([5]). Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in [0, \frac{1}{2})$ such that

$$m(Tx, Ty) \leq k[m(x, Tx) + m(y, Ty)] \quad (1.5)$$

for all $x, y \in X$. Then T has a unique fixed point.

They also posed the following open problem:

Open problem in [5]: Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in [0, \frac{1}{2})$ such that

$$m(Tx, Ty) \leq k[m(x, Ty) + m(y, Tx)] \quad (1.6)$$

for all $x, y \in X$. Then does T has a unique fixed point?

There are many ways to solve this problem, for example, Monfared et al. [6] solved this open problem by considering a number k in more small interval as follows:

Theorem 1.6 ([6]). *Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. If there exists $k \in [0, \frac{\sqrt{3}-1}{2})$ such that*

$$m(Tx, Ty) \leq k[m(x, Ty) + m(y, Tx)] \quad (1.7)$$

for all $x, y \in X$. Then T has a unique fixed point.

The other two answers of the open problem in [5] was given by Kumrod and Sintunavarat [7] as follows:

Theorem 1.7 ([7]). *Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in [0, \frac{1}{2})$ such that*

$$m(Tx, Ty) \leq k[m(x, Ty) + m(y, Tx)] \quad (1.8)$$

for all $x, y \in X$. If there is $x_0 \in X$ such that

$$m(T^n x_0, T^n x_0) \leq m(T^{n-1} x_0, T^{n-1} x_0) \quad (1.9)$$

for all $n \in \mathbb{N}$, then T has a unique fixed point.

Theorem 1.8 ([7]). *Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in [0, \frac{1}{2})$ such that*

$$m(Tx, Ty) \leq k[m(x, Ty) + m(y, Tx)] \quad (1.10)$$

for all $x, y \in X$. If there is $x_0 \in X$ such that

$$m(T^{n-1} x_0, T^{n-1} x_0) \leq m(T^n x_0, T^n x_0) \quad (1.11)$$

for all $n \in \mathbb{N}$, then T has a unique fixed point.

Not only the existence and uniqueness of a fixed point but also the Ulam-Hyers stability, the well-posedness, and the limit shadowing property of the fixed point problem are popular research topics in the fixed point theory. For instance, the study on the Ulam-Hyers stability for the fixed point problem can see in [8–11]. The research on the well-posedness and the limit shadowing property of the fixed point problem can see in [12–14] and references therein.

Very recently, Pansuwan et al. [15] first introduced various types of the Ulam-Hyers stability, the well-posedness and the limit shadowing property in M -metric spaces. They also investigated the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem for mappings satisfying the Banach contractive condition and the Kannan contractive condition in M -metric spaces.

To the best of our knowledge, there is no discussion so far concerning the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem for mappings satisfying the Chatterjea contractive condition in M -metric spaces. Inspired by the above view, the aim of this paper is to introduce the new types of the Ulam-Hyers stability, the well-posedness and the limit shadowing property in M -metric spaces for investigation of the fixed point problem for mappings satisfying the Chatterjea contractive condition in M -metric spaces. Furthermore, some illustrative examples for showing the usage of our main results are given.

2. PRELIMINARIES

In this section, we give some basic definitions and their properties which are needed in this paper.

Definition 2.1 ([4]). Let X be a nonempty set. A mapping $p : X \times X \rightarrow [0, \infty)$ is called a partial metric if p satisfies the following conditions for all $x, y, z \in X$:

- (p₁) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;
- (p₂) $p(x, x) \leq p(x, y)$;
- (p₃) $p(x, y) = p(y, x)$;
- (p₄) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Also, the pair (X, p) is called a partial metric space.

Each partial metric p on a nonempty set X generates a T_0 topology τ_p on X which has a base as the family $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$ of open p -balls, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

Definition 2.2 ([4]). Let (X, p) be a partial metric space.

- (1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n).$$

- (2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m)$$

exists and is finite.

- (3) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = p(x, x).$$

It is obvious that a metric is evidently a partial metric, but the converse does not hold in general. The following examples show that, in general, a partial metric space need not necessarily be a metric space.

Example 2.3 ([4]). Let $X = [0, \infty)$ and $p : X \times X \rightarrow [0, \infty)$ be defined by

$$p(x, y) = \max\{x, y\}$$

for all $x, y \in X$. Then p is a p -metric, but it is not metric.

Example 2.4 ([4]). Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and $p : X \times X \rightarrow [0, \infty)$ be defined by

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$$

for all $[a, b], [c, d] \in X$. Then p is a p -metric, but it is not a metric.

Next, we recall some notations, definitions, lemmas and examples of M -metric spaces which are needed in main results. For a nonempty set X and a given mapping $m : X \times X \rightarrow [0, \infty)$, the following notation are used in the sequel:

- (1) $m_{x,y} := \min\{m(x, x), m(y, y)\}$ for all $x, y \in X$;
- (2) $M_{x,y} := \max\{m(x, x), m(y, y)\}$ for all $x, y \in X$.

Definition 2.5 ([5]). Let X be a nonempty set. A mapping $m : X \times X \rightarrow [0, \infty)$ is called an m -metric if m satisfies the following conditions for all $x, y, z \in X$:

- (m_1) $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$;
- (m_2) $m_{x,y} \leq m(x, y)$;
- (m_3) $m(x, y) = m(y, x)$;
- (m_4) $(m(x, y) - m_{x,y}) \leq (m(x, z) - m_{x,z}) + (m(z, y) - m_{z,y})$.

Also, the pair (X, m) is called an M -metric space.

Lemma 2.6 ([5]). *Every partial metric space is an M -metric space, but the converse does not hold in general.*

Example 2.7 ([5]). Let $X = [0, \infty)$ and a function $m : X \times X \rightarrow [0, \infty)$ be defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then m is an m -metric, but it is not a partial metric.

Example 2.8 ([5]). Let $X = \{1, 2, 3\}$ and a function $m : X \times X \rightarrow [0, \infty)$ defined by

$$m(x, y) = \begin{cases} 1, & x = y = 1, \\ 9, & x = y = 2, \\ 5, & x = y = 3, \\ 10, & x, y \in \{1, 2\}, x \neq y, \\ 7, & x, y \in \{1, 3\}, x \neq y, \\ 8, & x, y \in \{2, 3\}, x \neq y. \end{cases}$$

Then m is an m -metric, but it is not a partial metric.

Definition 2.9 ([5]). Let (X, m) be an M -metric space.

- (1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$\lim_{n \rightarrow \infty} (m(x_n, x) - m_{x_n,x}) = 0.$$

- (2) A sequence $\{x_n\}$ in an M -metric space (X, m) is called an m -Cauchy sequence if

$$\lim_{n,m \rightarrow \infty} (m(x_n, x_m) - m_{x_n,x_m}), \quad \lim_{n,m \rightarrow \infty} (M_{x_n,x_m} - m_{x_n,x_m})$$

exist and are finite;

- (3) An M -metric space (X, m) is said to be M -complete if every m -Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n \rightarrow \infty} (m(x_n, x) - m_{x_n,x}) = 0, \quad \lim_{n \rightarrow \infty} (M_{x_n,x} - m_{x_n,x}) = 0.$$

3. MAIN RESULTS

First, we introduce the following definitions which are some generalizations and modifications of ideas in [15].

Definition 3.1. Let (X, m) be an M -metric space and $T : X \rightarrow X$ be a given mapping. The fixed point problem of T is said to be Ulam-Hyers stable type (C) if there exists

$c_1, c_2 > 0$ such that, for any $\epsilon > 0$ and $w^* \in X$ which is an ϵ -solution of the fixed point problem

$$x = Tx, \tag{3.1}$$

that is, w^* satisfies the inequality

$$m(w^*, Tw^*) \leq \epsilon, \tag{3.2}$$

there exists a solution $x^* \in X$ of the equation (3.1) such that

$$m(x^*, w^*) - c_2m(x^*, x^*) \leq c_1\epsilon. \tag{3.3}$$

Remark 3.2. It is clear that every Ulam-Hyers stability type (K) of the fixed point problem in [15] is necessarily the Ulam-Hyers stability type (C) , but the converse is not always true.

Definition 3.3. Let (X, m) be an M -metric space and $T : X \rightarrow X$ be a given mapping. The fixed point problem of T is said to be well-posed type (C) if T has a unique fixed point x^* and there exists $c > 0$ such that for any sequence $\{x_n\}$ in X ,

$$\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0 \implies \limsup_{n \rightarrow \infty} m(x_n, x^*) \leq cm(x^*, x^*). \tag{3.4}$$

Definition 3.4. Let (X, m) be an M -metric space and $T : X \rightarrow X$ be a given mapping. The fixed point problem of T has the limit shadowing property type (C) in X if there exists $c > 0$ such that for any sequence $\{x_n\}$ in X with $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$, there exists $z \in X$ such that

$$\limsup_{n \rightarrow \infty} m(T^n z, x_n) \leq cm(z, z). \tag{3.5}$$

Now, we give the main results in this paper.

Theorem 3.5. Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in [0, \frac{1}{2})$ such that

$$m(Tx, Ty) \leq k[m(x, Ty) + m(y, Tx)] \tag{3.6}$$

for all $x, y \in X$. If there exists $x_0 \in X$ such that either

$$m(T^n x_0, T^n x_0) \leq m(T^{n-1} x_0, T^{n-1} x_0) \quad \text{for all } n \in \mathbb{N} \tag{3.7}$$

or

$$m(T^{n-1} x_0, T^{n-1} x_0) \leq m(T^n x_0, T^n x_0) \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Then the following assertions hold:

- (1) the fixed point problem of T is Ulam-Hyers stable type (C) ;
- (2) the fixed point problem of T is well-posed type (C) ;
- (3) the fixed point problem of T has the limit shadowing property type (C) in X .

Proof. In [7], the authors shown that T has a unique fixed point. Let x^* be a unique fixed point of T .

First, we will show that the fixed point problem of T is Ulam-Hyers stable type (C) . Let $\epsilon > 0$ and $w^* \in X$ be a solution of the inequality (3.2), that is,

$$m(w^*, Tw^*) \leq \epsilon.$$

By using (m_4) , we have

$$\begin{aligned}
 m(x^*, w^*) &\leq [m(x^*, Tw^*) - m_{x^*, Tw^*}] + [m(Tw^*, w^*) - m_{Tw^*, w^*}] + m_{x^*, w^*} \\
 &= [m(Tx^*, Tw^*) - m_{x^*, Tw^*}] + [m(Tw^*, w^*) - m_{Tw^*, w^*}] + m_{x^*, w^*} \\
 &\leq m(Tx^*, Tw^*) - km_{x^*, Tw^*} + m(Tw^*, w^*) + m(x^*, x^*) \\
 &\leq k[m(x^*, Tw^*) + m(Tx^*, w^*)] - km_{x^*, Tw^*} + m(Tw^*, w^*) + m(x^*, x^*) \\
 &= k[m(x^*, Tw^*) - m_{x^*, Tw^*}] + km(x^*, w^*) + m(Tw^*, w^*) + m(x^*, x^*) \\
 &\leq k[m(x^*, w^*) - m_{x^*, w^*} + m(w^*, Tw^*) - m_{w^*, Tw^*}] + km(x^*, w^*) \\
 &\quad + m(Tw^*, w^*) + m(x^*, x^*) \\
 &\leq 2km(x^*, w^*) + (1 + k)\epsilon + m(x^*, x^*),
 \end{aligned}$$

that is,

$$m(x^*, w^*) - \frac{1}{1 - 2k}m(x^*, x^*) \leq \frac{1 + k}{1 - 2k}\epsilon.$$

Then the fixed point problem of T is Ulam-Hyers stable type (C) , that is, the condition (3.3) holds with $c_1 := \frac{1+k}{1-2k}$ and $c_2 := \frac{1}{1-2k}$.

Next, we claim that the fixed point problem of T is well-posed type (C) . Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$. By (m_4) , we have

$$\begin{aligned}
 &m(x^*, x_n) \\
 &\leq [m(x^*, Tx_n) - m_{x^*, Tx_n}] + [m(Tx_n, x_n) - m_{Tx_n, x_n}] + m_{x^*, x_n} \\
 &= [m(Tx^*, Tx_n) - m_{x^*, Tx_n}] + [m(Tx_n, x_n) - m_{Tx_n, x_n}] + m_{x^*, x_n} \\
 &\leq m(Tx^*, Tx_n) - km_{x^*, Tx_n} + m(Tx_n, x_n) + m(x^*, x^*) \\
 &\leq k[m(x^*, Tx_n) + m(Tx^*, x_n)] - km_{x^*, Tx_n} + m(Tx_n, x_n) + m(x^*, x^*) \\
 &\leq k[m(x^*, Tx_n) - m_{x^*, Tx_n}] + km(x^*, x_n) + m(Tx_n, x_n) + m(x^*, x^*) \\
 &\leq k[m(x^*, x_n) - m_{x^*, x_n} + m(x_n, Tx_n) - m_{x_n, Tx_n}] + km(x^*, x_n) \\
 &\quad + m(Tx_n, x_n) + m(x^*, x^*) \\
 &\leq 2km(x^*, x_n) + (1 + k)m(Tx_n, x_n) + m(x^*, x^*),
 \end{aligned}$$

that is,

$$m(x^*, x_n) \leq \frac{1 + k}{1 - 2k}m(Tx_n, x_n) + \frac{1}{1 - 2k}m(x^*, x^*)$$

for all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$\limsup_{n \rightarrow \infty} m(x_n, x^*) \leq \frac{1}{1 - 2k}m(x^*, x^*). \tag{3.9}$$

Consequently, the fixed point problem of T is well-posed type (C) , that is, the condition (3.4) holds with $c = \frac{1}{1-2k}$.

To prove T has the limit shadowing property type (C) , let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$. Since x^* is a fixed point of T , from (3.9), it follows that

$$\limsup_{n \rightarrow \infty} m(x_n, T^n x^*) = \limsup_{n \rightarrow \infty} m(x_n, x^*) \leq \frac{1}{1 - 2k}m(x^*, x^*).$$

Therefore, the fixed point problem of T has the limit shadowing property type (C) , that is, the condition (3.5) holds with $c = \frac{1}{1-2k}$. This completes the proof. ■

Now, we give examples to illustrate Theorem 3.5.

Example 3.6. Let $X = [0, \infty)$ and a function $m : X \times X \rightarrow [0, \infty)$ be defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then (X, m) is a complete M -metric space. Define a mapping $T : X \rightarrow X$ by

$$Tx = \begin{cases} 0, & 0 \leq x < 3, \\ \frac{x}{1+x}, & x \geq 3. \end{cases}$$

From [7], we obtain T is a Chatterjea contractive mapping. It is easy to see that $x^* = 0$ is a unique solution of the fixed point problem of T .

First, we show that the condition (3.3) holds with $c_1 \geq 1$ and $c_2 > 0$, that is, the fixed point problem of T is Ulam-Hyers stable type (C) . Assume that $\epsilon > 0$ and $w^* \in X$ is an ϵ -solution of the fixed point problem of T . Then we consider the following two cases:

Case 1. If $w^* \in [0, 3)$, then we have

$$\begin{aligned} (m(w^*, Tw^*) \leq \epsilon &\implies \frac{w^*}{2} \leq \epsilon) \\ \implies (m(x^*, w^*) - c_2m(x^*, x^*) = m(0, w^*) &= \frac{w^*}{2} \leq \epsilon \leq c_1\epsilon). \end{aligned}$$

Case 2. If $w^* \in [3, \infty)$, then we get

$$\begin{aligned} (m(w^*, Tw^*) \leq \epsilon &\implies \frac{w^*}{2} + \frac{w^*}{2(1+w^*)} \leq \epsilon) \\ \implies (m(x^*, w^*) - c_2m(x^*, x^*) = m(0, w^*) &= \frac{w^*}{2} \leq \epsilon \leq c_1\epsilon). \end{aligned}$$

From Case 1 and Case 2, we conclude that the fixed point problem of T is Ulam-Hyers stable type (C) .

Next, we prove that the fixed point problem of T is well-posed type (C) . Assume that $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0.$$

For each $n \in \mathbb{N}$, we have

$$m(x_n, x^*) = m(x_n, 0) = \frac{x_n}{2} \leq \frac{1}{2}(x_n + Tx_n) = m(x_n, Tx_n).$$

Since $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$, it follows that

$$\limsup_{n \rightarrow \infty} m(x_n, x^*) = 0 = cm(x^*, x^*)$$

for all $c > 0$ and so the fixed point problem of T is well-posed type (C) .

Finally, we show that the fixed point problem of T has the limit shadowing property type (C) in X . Let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$. Then there exists $z \in [0, 3)$ such that

$$m(T^n z, x_n) = m(0, x_n) = \frac{x_n}{2} \leq \frac{1}{2}(x_n + Tx_n) = m(x_n, Tx_n)$$

for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$, we have

$$\lim_{n \rightarrow \infty} m(T^n z, x_n) = 0 \leq cm(z, z)$$

for all $c > 0$. This implies that the fixed point problem of T has the limit shadowing property type (C) in X .

Example 3.7. Let $X = [0, \infty)$ and $m : X \times X \rightarrow [0, \infty)$ be a function defined by

$$m(x, y) = \frac{x + y}{2}$$

for all $x, y \in X$. Then (X, m) is a complete M -metric space. Define a mapping $T : X \rightarrow X$ by

$$Tx = \begin{cases} x^2, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{4}, & x \geq \frac{1}{2}. \end{cases}$$

It follows from [7] that T is a Chatterjea contractive mapping. It is clear that $x^* = 0$ is a unique solution of the fixed point problem of T .

First, we show that the condition (3.3) holds with $c_1 \geq 1$ and $c_2 > 0$, that is, the fixed point problem of T is Ulam-Hyers stable type (C) . Assume that $\epsilon > 0$ and $w^* \in X$ is an ϵ -solution of the fixed point problem of T . We consider the following two cases:

Case 1. If $w^* \in [0, \frac{1}{2})$, then we have

$$\begin{aligned} (m(w^*, Tw^*) \leq \epsilon &\implies \frac{1}{2}(w^* + (w^*)^2) \leq \epsilon) \\ \implies (m(x^*, w^*) - c_2m(x^*, x^*) = \frac{w^*}{2} &\leq \frac{1}{2}(w^* + (w^*)^2) \leq c_1\epsilon). \end{aligned}$$

Case 2. If $w^* \in [3, \infty)$, then we have

$$\begin{aligned} (m(w^*, Tw^*) \leq \epsilon &\implies \frac{1}{2}(w^* + \frac{1}{4}) \leq \epsilon) \\ \implies (m(x^*, w^*) - c_2m(x^*, x^*) = \frac{w^*}{2} &< \frac{1}{2}(w^* + \frac{1}{4}) \leq c_1\epsilon). \end{aligned}$$

By the above two cases, we conclude that the fixed point problem of T is Ulam-Hyers stable type (C) .

Next, we will show that the fixed point problem of T is well-posed type (C) . We assume that $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$. For each $n \in \mathbb{N}$, we have

$$m(x_n, x^*) = m(x_n, 0) = \frac{x_n}{2} \leq \frac{1}{2}(x_n + Tx_n) = m(x_n, Tx_n).$$

Since $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$, it follows that

$$\limsup_{n \rightarrow \infty} m(x_n, x^*) = 0 = cm(x^*, x^*)$$

for all $c > 0$ and so the fixed point problem of T is well-posed type (C) .

Finally, we show that the fixed point problem of T has the limit shadowing property type (C) in X . Let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$. Then there exists $z = 0 \in X$ such that

$$m(T^n z, x_n) = m(0, x_n) = \frac{x_n}{2} < \frac{1}{2}(x_n + Tx_n) = m(x_n, Tx_n)$$

for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} m(x_n, Tx_n) = 0$, we have

$$\limsup_{n \rightarrow \infty} m(T^n z, x_n) = 0 = cm(z, z)$$

for all $c > 0$, which show that the fixed point problem of T has the limit shadowing property type (C) in X .

In the same way, we immediately obtain the following result:

Theorem 3.8. *Let (X, m) be a complete M -metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exists $k \in \left[0, \frac{\sqrt{3}-1}{2}\right)$ such that*

$$m(Tx, Ty) \leq k[m(x, Ty) + m(y, Tx)] \quad (3.10)$$

for all $x, y \in X$. Then the following assertions hold:

- (1) the fixed point problem of T is Ulam-Hyers stable type (C);
- (2) the fixed point problem of T is well-posed type (C);
- (3) the fixed point problem of T has the limit shadowing property type (C) in X .

Remark 3.9. It follows from Examples 3.6 and 3.7 that T satisfies the Chatterjea contractive condition with $k = \frac{1}{4}, \frac{1}{3} \in \left[0, \frac{\sqrt{3}-1}{2}\right)$, respectively and so the examples also support Theorem 3.8.

FUNDING

This work was supported by Thammasat University Research Unit in Fixed Points and Optimization.

COMPETING INTERESTS

The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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