Dedicated to Prof. Suthep Suantai on the occasion of his 60th anniversary

Pointwise Well-Posedness and Scalarization for Set Optimization Problems with a Partial Order Relation

Pakkapon Preechasilp[1,*] **and Rabian Wangkeeree**[2]

[1] *Program in Mathematics, Faculty of Education, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand*
email : pakkapon.p@psru.ac.th (P. Preechasilp)

[2] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
email : rabianw@nu.ac.th (R. Wangkeeree)

Abstract The present paper aims to obtain some relation among three kinds of pointwise well-posedness for set optimization problems with $\preceq_C^{m_1}$ order. By using a nonlinear scalarization function, we establish some relations between the three kinds of pointwise well-posedness for set optimization problems and the well-posedness of scalar optimization problems, respectively. Some global well-posedness notions are studied. Finally, we discuss the results to robust vector optimization problems.

MSC: 49K40; 90C31
Keywords: set optimization problem; nonlinear scalarization function; well-posedness

1. Introduction

In the last decades, set optimization have received an increasing study due to unify scalar as well as vector optimization. Many problems arise in different fields can be modeled as a set optimization problem, for example, game theory, welfare economics, risk measure, robust optimization, fuzzy optimization, and so on [1, 2]. The set criterion approach was first introduced by Kuroiwa [3, 4] in 1998. Recently, there were some concepts about how to compare elements of the power set have been developed (see Jahn and Ha [5], Karaman et al. [6]).

Well-posedness, initiated by Tikhonov [7] in 1966, requires the uniqueness of the existing optimal solution and the convergence of every minimizing sequence of approximate solutions to the unique minimum point. Roughly speaking, an objective function such that points with values close to the optimal value are actually close to a unique optimal solution. Later, many other notions of well-posedness for scalar optimization were defined

---

and studied, (see [8–10] and the references therein). Since then, there were many extensions of the conceptions of well-posedness to vector optimization appeared (see [11–16]). For set optimization problems, Zhang et al. [17] introduced a notion of pointwise well-posedness for set optimization problems with respect to lower set less order relation. They obtained the equivalent relations between the three kinds of well-posedness. Gutiérrez et al. [18] improved some results in Zhang et al. [17] in the case where the objective values are not cone-bounded sets. Long et al. [19] obtained the relations between the three kinds of pointwise well-posedness for set optimization problems and the well-posedness of three kinds of scalar optimization problems by using scalarizing function with respect to upper set less order relation. Crespi et al. [20] gave a sufficient condition of global well-posedness for set optimization problems with respect to lower set less order relation. Dhingra and Lalitha [21] gave sufficient conditions for well-setness for set optimization problem with respect to lower set less relation. Crespi et al. [22] obtained the characterizations for pointwise well-posedness in terms if upper continuity of minimal solution set map and the characterizations for global well-posedness in terms of compactness of minimal solution set map and compactness of the solution set with respect to lower(upper) set less order relation. There have been many studies of well-posedness and related results (see, e.g., [23–25] and the references therein).

The aim of this paper is to investigate the well-posedness notions in Zhang et al. [17] and Long et al. [19] for set optimization problems with respect to a partial order relation which introduced by Karaman et al. [6]. We obtain some relations between the three kinds of pointwise well-posedness. Moreover, we establish some relations between pointwise well-posedness of set optimization problems and well-posedness of scalar optimization problems by using a nonlinear scalarizing function. Some relation between pointwise well-posedness and global well-posedness are also studied. As an application, we discuss well-posedness of robust vector optimization problems.

The organization of the paper is as follows. Section 2 presents some necessary notations and lemmas. We introduce the concept of m_1-C-convexity for set-valued mappings and show some convexity properties for nonlinear scalarizing function. Section 3, we consider three kinds of pointwise well-posedness for set optimization problems with $\preceq_C^{m_1}$ order and give some relation among them. By virtue of a nonlinear scalarization function, we obtain, in Section 4, the relations between the three kinds of pointwise well-posedness for set optimization problems and the well-posedness of three kinds of scalar optimization problems, respectively. In Section 5, we study some notions of global well-posedness. Some discussions between the concepts of robust vector optimization and set optimization are presented in Section 6.

2. Preliminaries

Throughout this paper, unless otherwise stated, X, Y and Z are normed spaces, and M is a subset of Z. Let $C \subset Y$ be a convex, pointed cone with $0_Y \in C$. Let $\mathcal{P}_0(Y)$ and $\mathcal{B}^*(Y)$ denote the family of nonempty subsets of Y and the family of nonempty bounded subsets of Y, respectively. We now recall some order relations on the family of subsets of Y. For a set $A \subseteq Y$, we denote interior of A by $\text{int}(A)$. $B(x, \varepsilon)$ denotes the open ball centered at $x \in Y$ with radius ε.

For $A, B \in \mathcal{P}_0(Y)$ and $\lambda \in \mathbb{R}$, $\lambda A := \{\lambda a : a \in A\}$. Here, the symbols $A + B$, $A - B$, and $A \dot{-} B$ mean algebraic sum, algebraic difference, and geometric(Minkowski or Pontryagin)

difference of A and B, respectively. That is, $A + B := \{a + b : a \in A \text{ and } b \in B\}$, $A - B := \{a - b : a \in A \text{ and } b \in B\}$, $A \dot{-} B := \bigcap_{b \in B} (A - b) = \{y \in Y : y + B \subseteq A\}$.

By using the idea of Minkowski difference, Karaman et al. [6] introduced the following partial order relations on the family of nonempty bounded sets.

Definition 2.1. [6] Let $A, B \in \mathcal{P}_0(Y)$ be arbitrarily chosen sets. m_1-*order relation* $A \preceq_C^{m_1} B$ is defined by $A \preceq_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap C \neq \emptyset$.

Definition 2.2. [6] Let $A, B \in \mathcal{P}_0(Y)$ be arbitrarily chosen sets. *Strictly m_1-order relation* $A \prec_C^{m_1} B$ is defined by $A \prec_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap \text{int}(C) \neq \emptyset$.

Remark 2.3. Note that, if we take A and B as singletons and C is convex, pointed cone with $0_Y \in C$, then $\preceq_C^{m_1}$ is coincide to the vector order relation $\leq_C$ on Y, i.e.,

$$\{a\} \preceq_C^{m_1} \{b\} \Leftrightarrow a \leq_C b \Leftrightarrow b = a + c \text{ for some } c \in C.$$

The following results are basic properties for $\preceq_C^{m_1}$ which can be found in [6].

Lemma 2.4. Let $C \in \mathcal{P}_0(Y)$. The following statement hold:
  (i) $\preceq_C^{m_1}$ is compatible with addition, i.e., $A_1 \preceq_C^{m_1} A_2$ implies $A_1 + B \preceq_C^{m_1} A_2 + B$.
  (ii) C is cone if and only if $\preceq_C^{m_1}$ is compatible with scalar multiplication, i.e., $A_1 \preceq_C^{m_1} A_2$ implies $\alpha A_1 \preceq_C^{m_1} \alpha A_2$.
  (iii) $0_Y \in C$ if and only if $\preceq_C^{m_1}$ is reflexive, i.e., $A \preceq_C^{m_1} A$.
  (iv) If C is a cone, then C is convex if and only if $\preceq_C^{m_1}$ is transitive, i.e., $A_1 \preceq_C^{m_1} A_2$ and $A_2 \preceq_C^{m_1} A_3$ implies $A_1 \preceq_C^{m_1} A_3$.
  (v) If C is a cone, then C is pointed if and only if $\preceq_C^{m_1}$ is antisymmetric on $\mathcal{B}^*(Y)$, i.e., $A_1 \preceq_C^{m_1} A_2$ and $A_2 \preceq_C^{m_1} A_1$ implies $A_1 = A_2$.
  (vi) If C is convex cone with nonempty interior and $A \prec_C^{m_1} B$ and $B \preceq_C^{m_1} D$ (or $A \preceq_C^{m_1} B$ and $B \prec_C^{m_1} D$), then $A \prec_C^{m_1} D$.

Definition 2.5. [6] Let $\mathcal{S} \subseteq \mathcal{B}^*(Y)$ and $A \in \mathcal{S}$. Then, A is said to be
  (i) m_1-minimal set of $\mathcal{S}$ if there is not any $B \in \mathcal{S}$ with $B \preceq_C^{m_1} A$ and $A \neq B$;
  (ii) weakly m_1-minimal set of $\mathcal{S}$ if there is not any $B \in \mathcal{S}$ with $B \prec_C^{m_1} A$.

Let $F : X \to 2^Y$ be a set-valued mapping with nonempty bounded values and $K \subseteq X$ with $K \neq \emptyset$. The set optimization problem is defined as follows:

$$(SOP) \qquad \min F(x) \quad \text{subject to} \quad x \in K.$$

Definition 2.6. An element $\bar{x} \in K$ is said to be
  (i) m_1-minimal solution of (SOP) if $F(\bar{x})$ is m_1-minimal set of $\mathcal{F}(K)$;
  (ii) weakly m_1-minimal solution of (SOP) if $F(\bar{x})$ is weakly m_1-minimal set of $\mathcal{F}(K)$.
We denote $\mathcal{F}(K) = \bigcup_{x \in K} F(x)$.

Let $\text{Eff}_{m_1}(F, K)$ and $\text{WEff}_{m_1}(F, K)$ denote the set of m_1-minimal solution of (SOP) and weakly m_1-minimal solution of (SOP), respectively.

Remark 2.7. It is easily seen that $\bar{x} \in \text{Eff}_{m_1}(F, K)$ if and only if $F(x) \not\preceq_C^{m_1} F(\bar{x})$ for all $x \in K \backslash \{\bar{x}\}$ and $F(x) \neq F(\bar{x})$.

Suppose that $G : M \to 2^Y$ is a set-valued map.

Definition 2.8. [26, 27] The set-valued map G is said to be

   (i) *upper semicontinuous* (u.s.c) at $\bar{\mu} \in M$ if for any open set $V \subseteq Y$ with $G(\bar{\mu}) \subseteq V$, there exists a neighborhood $N(\bar{\mu})$ of $\bar{\mu}$ such that $G(\mu \in N(\bar{\mu}) \cap M) \subseteq V$;

   (ii) *lower semicontinuous* (l.s.c) at $\bar{\mu} \in M$ if for any open set $V \subseteq Y$ with $G(\bar{\mu}) \cap V \neq \emptyset$, there exists a neighborhood $N(\bar{\mu})$ of $\bar{\mu}$ such that $G(\mu) \cap V \neq \emptyset$ for all $\mu \in N(\bar{\mu}) \cap M$;

   (iii) *closed* at $\bar{\mu}$ if graph $G := \{(\mu, y) : y \in G(\mu)\}$ is closed set, i.e., for any sequence $(\mu_n, y_n) \in$ graph $G := \{(\mu, y) : y \in G(\mu)\}$ with $(\mu_n, y_n) \to (\bar{\mu}, \bar{y})$, then $(\bar{\mu}, \bar{y}) \in$ graph G.

   (iv) *compact* at $\bar{\mu}$ if for any sequence $(\mu_n, y_n) \in$ graph $G := \{(\mu, y) : y \in G(\mu)\}$ with $\mu_n \to \bar{\mu}$, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $\bar{y} \in G(\bar{\mu})$ such that $y_{n_k} \to \bar{y}$.

The following proposition is an important tool.

Proposition 2.9. [26, 28]

   (i) G is l.s.c at $\bar{\mu}$ if and only if any sequence $\{\bar{\mu}_n\} \subseteq M$ with $\bar{\mu}_n \to \bar{\mu}$ and any $\bar{y} \in G(\bar{\mu})$, there exists $\bar{y}_n \in G(\bar{\mu}_n)$ such that $\bar{y}_n \to \bar{y}$.

   (ii) If G has compact values at $\bar{\mu}$, then G is u.s.c at $\bar{\mu}$ if and only if for any sequence $\{\bar{\mu}_n\} \subseteq M$ with $\bar{\mu}_n \to \bar{\mu}$ and any $\bar{y}_n \in G(\bar{\mu}_n)$, there exist $\bar{y} \in G(\bar{\mu})$ and a subsequence $\{\bar{y}_{n_k}\}$ of $\{\bar{y}_n\}$ such that $\bar{y}_{n_k} \to \bar{y}$.

Let $e \in int\,C$. A scalarizing function $I_e^{m_1}(\cdot, \cdot) : \mathcal{P}_0(Y) \times \mathcal{P}_0(Y) \to \overline{R}$ is defined to reduce a set optimization problem with respect to $\preceq_C^{m_1}$ as follows.

$$I_e^{m_1}(A, B) := \inf\{t \in \mathbb{R} : A \preceq_C^{m_1} te + B\} \quad \text{for all } A, B \in \mathcal{P}_0(Y).$$

Lemma 2.10. [6] Let $A, B \in \mathcal{P}_0(Y)$ and $r \in \mathbb{R}$. The following statement hold.

   (i) If A is bounded, then $I_e^{m_1}(A, A) = 0$.

   (ii) If B is bounded, then $I_e^{m_1}(A, B) > -\infty$.

   (iii) $I_e^{m_1}(A, B) > -\infty$ if and only if $B \dot{-} A = \emptyset$.

   (iv) $I_e^{m_1}(A, B) < r$ if and only if $A \prec_C^{m_1} re + B$.

   (v) If $B \dot{-} A$ is compact, then $I_e^{m_1}(A, B) = r$ if and only if $A \preceq_{cl(C)}^{m_1} re + B$ and $A \npreceq_C^{m_1} (r - \varepsilon)e + B$ for all $\varepsilon > 0$.

Lemma 2.11. Let $A, B \in \mathcal{P}_0(Y)$. For each $r \geq 0$, $I_e^{m_1}((A + re), B) = I_e^{m_1}(A, B) + r$.

Proof. We first consider the set $\{t \in \mathbb{R} : A \preceq_C^{m_1} te + B\}$. For any $r > 0$, it follows from Lemma 2.4 (i) that

$$s \in \{t \in \mathbb{R} : A \preceq_C^{m_1} te + B\} \iff s + r \in \{t \in \mathbb{R} : A + re \preceq_C^{m_1} te + B\},$$

and consequently

$$\{t \in \mathbb{R} : A \preceq_C^{m_1} te + B\} + r = \{t \in \mathbb{R} : A + re \preceq_C^{m_1} te + B\}.$$

This implies that $I_e^{m_1}(A, B) + r = I_e^{m_1}((A + re), B)$. ∎

Lemma 2.12. For any $B \in \mathcal{P}_0(Y)$, if B is convex set, then $I_e^{m_1}(\cdot, B)$ is convex function.

Proof. Let $A_1, A_2 \in \mathcal{P}_0(Y)$. Obviously, in the case where $I_e^{m_1}(A_1, B) = +\infty$ or $I_e^{m_1}(A_2, B) = +\infty$. Assume that $I_e^{m_1}(A_1, B) < +\infty$ and $I_e^{m_1}(A_2, B) < +\infty$. Let $\alpha \in [0, 1]$. For any $t > 0$, we have

$$A_1 \preceq_C^{m_1} (I_e^{m_1}(A_1, B) + t)\,e + B \quad \text{and} \quad A_2 \preceq_C^{m_1} (I_e^{m_1}(A_2, B) + t)\,e + B.$$

Thanks to Lemma 2.4 (ii),

$$\alpha A_1 \preceq_C^{m_1} \alpha \left(I_e^{m_1}(A_1, B) + t \right) e + \alpha B$$

and

$$(1 - \alpha)A_2 \preceq_C^{m_1} (1 - \alpha) \left(I_e^{m_1}(A_2, B) + t \right) e + (1 - \alpha)B.$$

Also Lemma 2.4 (i) and convexity of B imply that

$$
\begin{aligned}
\alpha A_1 + (1 - \alpha)A_2 \\
\preceq_C^{m_1} \alpha \left(I_e^{m_1}(A_1, B) + t \right) e + \alpha B + (1 - \alpha) \left(I_e^{m_1}(A_2, B) + t \right) e + (1 - \alpha)B \\
= \alpha \left[\alpha I_e^{m_1}(A_1, B) + (1 - \alpha)I_e^{m_1}(A_2, B) + t \right] e + B.
\end{aligned}
$$

This means that $I_e^{m_1}(\alpha A_1 + (1 - \alpha)A_2, B) \le \alpha I_e^{m_1}(A_1, B) + (1 - \alpha)I_e^{m_1}(A_2, B)$. The proof is complete. ∎

Proposition 2.13. [6] Let $A_1, A_2, B \in \mathcal{P}_0(Y)$. Then,

    (i) If $A_1 \preceq_C^{m_1} A_2$, then $I_e^{m_1}(A_1, B) \le I_e^{m_1}(A_2, B)$.

    (ii) If $A_1 \preceq_C^{m_1} A_2$, then $I_e^{m_1}(B, A_1) \ge I_e^{m_1}(B, A_2)$.

    (iii) If B is compact and $A_1 \prec_C^{m_1} A_2$, then $I_e^{m_1}(A_1, B) < I_e^{m_1}(A_2, B)$.

    (iv) If A_1, A_2, B are compact and $A_1 \prec_C^{m_1} A_2$, then $I_e^{m_1}(A_1, B) > I_e^{m_1}(A_2, B)$.

Proposition 2.14. [6] Let $A, B \in \mathcal{P}_0(Y)$ and $I_e^{m_1}(A, B)$ be finite. Then, the following statements hold:

    (i) If $B \dot{-} A$ is compact and C is closed, $A \preceq_C^{m_1} B$ if and only if $I_e^{m_1}(A, B) \le 0$.

    (ii) $A \prec_C^{m_1} B$ if and only if $I_e^{m_1}(A, B) < 0$.

Definition 2.15. Let K be a convex subset of X. A set-valued map $G : K \to 2^Y$ is said to be

    (i) m_1-C- *convex* on K if, for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$G(tx_1 + (1 - t)x_2) \preceq_C^{m_1} tG(x_1) + (1 - t)G(x_2).$$

    (ii) *strictly m_1-C- convex* on K if, for any $x_1, x_2 \in K$ with $x_1 \ne x_2$ and $t \in (0, 1)$,

$$G(tx_1 + (1 - t)x_2) \prec_C^{m_1} tG(x_1) + (1 - t)G(x_2).$$

Remark 2.16. In the case where F is a single-valued mapping, (strict) m_1-C-convexity of F and the classic (strict) C-convexity of vector-valued map $f : K \to Y$ are coincide.

Lemma 2.17. Assume that K is convex and F is strictly m_1-C-convex on K with nonempty compact convex values, then $\mathrm{WEff}_{m_1}(F, K) = \mathrm{Eff}_{m_1}(F, K)$.

Proof. It is clear that $\mathrm{Eff}_{m_1}(F, K) \subseteq \mathrm{WEff}_{m_1}(F, K)$. It suffices to show that $\mathrm{WEff}_{m_1}(F, K) \subseteq \mathrm{Eff}_{m_1}(F, K)$. Let $\bar{x} \in \mathrm{WEff}_{m_1}(F, K)$. We show that $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$, if not there exists an $\tilde{x} \in K \backslash \{\bar{x}\}$ such that

$$F(\tilde{x}) \preceq_C^{m_1} F(\bar{x}) \quad \text{and } F(\tilde{x}) \ne F(\bar{x}). \tag{2.1}$$

By strict m_1-C-convexity of F, one has for any $t \in (0, 1)$

$$F(t\bar{x} + (1 - t)\tilde{x}) \prec_C^{m_1} tF(\bar{x}) + (1 - t)F(\tilde{x}). \tag{2.2}$$

It follow from (2.1), (2.2), Lemma 2.4 (i), (vi) and convexity of $F(\bar{x})$ that

$$
\begin{aligned}
F(t\bar{x} + (1 - t)\tilde{x}) \quad &\prec_C^{m_1} \quad tF(\bar{x}) + (1 - t)F(\tilde{x}) \\
&\preceq_C^{m_1} \quad tF(\bar{x}) + (1 - t)F(\bar{x}) = F(\bar{x})
\end{aligned}
$$

This means that $F(t\bar{x}+(1-t)\tilde{x}) \in \text{WEff}_{m_1}(F, K)$, which contradicts with $\bar{x} \in \text{WEff}_{m_1}(F, K)$. Hence, $\bar{x} \in \text{Eff}_{m_1}(F, K)$. ■

Example 2.18. Let $C = \mathbb{R}_+^2$ and $F : [0, 2] \to 2^{\mathbb{R}^2}$ be defined as follows.

$$F(x) = [0, x^2] \times [0, x].$$

It is clear that F is m_1-C-convex on convex set $[0, 2]$ with convex values. Lemma 2.17 implies $\text{WEff}_{m_1}(F, [0, 2]) = \text{Eff}_{m_1}(F, [0, 2])$. Indeed, we calculate that $\text{WEff}_{m_1}(F, [0, 2]) = \{0\} = \text{Eff}_{m_1}(F, [0, 2])$. However, if we define $G : \{1, 2, 3\} \to 2^{\mathbb{R}^2}$ as follows.

$$G(x) = \begin{cases} [0, 1] \times [1, 2] & \text{if } x = 1 \\ [1, 2] \times [0, 1] & \text{if } x = 2 \\ [2, 3] \times [0, 1] & \text{if } x = 3. \end{cases}$$

It is easily seen that $G(x) \not\prec_C^{m_1} G(y)$ for all $x \in \{1, 2, 3\}$. Thus $\text{WEff}_{m_1}(G, \{1, 2, 3\}) = \{1, 2, 3\}$. We have $3 \notin \text{Eff}_{m_1}(G, \{1, 2, 3\})$ since $F(2) \notin F(3)$ and $F(2) \preceq_C^{m_1} F(3)$. Hence $\text{Eff}_{m_1}(G, K) = \{1, 2\}$. We remark that G is not strictly m_1-C-convex and K is not convex set.

Proposition 2.19. If F is m_1-C-convex on a convex set $K \subseteq X$ and B is convex, then $I_e^{m_1}(F(\cdot), B)$ is convex for all $B \in \mathcal{P}_0(Y)$.

Proof. Let $x_1, x_2 \in K$ and $0 \le t \le 1$. Since F is m_1-C-convex, one has

$$F(tx_1 + (1-t)x_2) \preceq_C^{m_1} tF(x_1) + (1-t)F(x_2).$$

It follows from monotonicity and convexity of $I_e^{m_1}(\cdot, B)$ that

$$\begin{aligned} I_e^{m_1}(F(tx_1 + (1-t)x_2), B) &\le I_e^{m_1}((tF(x_1) + (1-t)F(x_2)), B) \\ &\le tI_e^{m_1}(F(x_1), B) + (1-t)I_e^{m_1}(F(x_2), B). \end{aligned}$$

■

3. Pointwise Well-Posedness

In this section, we consider three notions of pointwise well-posedness for a set optimization with $\preceq_C^{m_1}$ order. We first recall the notion of pointwise well-posedness for set optimization problems which was adapted from the definition in [18, 29].

Definition 3.1. Let $e \in int\, C$ and $\bar{x} \in \text{Eff}_{m_1}(F, K)$.

    (i) A sequence $\{x_n\} \subseteq K$ is said to be an *e-minimizing sequence* for Problem (SOP) at $\bar{x}$ if there exists $\{\varepsilon_n\} \subseteq \mathbb{R}_+ \backslash \{0\}$ with $\varepsilon_n \to 0$ such that

$$F(x_n) \preceq_C^{m_1} F(\bar{x}) + \varepsilon_n e.$$

    (ii) A sequence $\{x_n\} \subseteq K$ is said to be *a minimizing sequence* for Problem (SOP) at $\bar{x}$ if there exists $\{c_n\} \subseteq C \backslash \{0\}$ with $c_n \to 0$ such that

$$F(x_n) \preceq_C^{m_1} F(\bar{x}) + c_n.$$

Definition 3.2. Problem (SOP) is said to be

    (i) *well-posed* at $\bar{x} \in \text{Eff}_{m_1}(F, K)$ if for any minimizing sequence for problem (SOP) at $\bar{x}$ converges to $\bar{x}$;

    (ii) *e-well-posed* at $\bar{x} \in \text{Eff}_{m_1}(F, K)$ if for any e-minimizing sequence for problem (SOP) at $\bar{x}$ converges to $\bar{x}$;

(iii) *generalized well-posed* at $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$ if for any minimizing sequence for problem (SOP) at $\bar{x}$ there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ that converges to an element of $\mathrm{Eff}_{m_1}(F, K)$;

(iv) *generalized e-well-posed* at $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$ if for any e-minimizing sequence for problem (SOP) at $\bar{x}$ there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ that converges to an element of $\mathrm{Eff}_{m_1}(F, K)$.

Remark 3.3. Problem (SOP) is well-posed if and only if it is generalized well-posed and $\mathrm{Eff}_{m_1}(F, K)$ is a singleton.

The following lemma give the equivalent between two kinds of minimizing sequences.

Lemma 3.4. Let $e \in int\, C$, $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$ and $\{x_n\} \subseteq K$. The following statements are equivalent:

(i) $\{x_n\}$ is an e-minimizing sequence for problem (SOP) at $\bar{x}$.

(ii) $\{x_n\}$ is an minimizing sequence for problem (SOP) at $\bar{x}$.

Proof. Obviously, (i) implies (ii). Conversely, we assume that $\{x_n\}$ is an minimizing sequence for problem (SOP) at $\bar{x}$. Then there exists a sequence $\{c_n\} \subseteq C\backslash\{0\}$ with $c_n \to 0$ such that

$$F(x_n) \preceq_C^{m_1} F(\bar{x}) + c_n \quad \text{for all } n.$$

By Durea [9, Lamma 2.2] there exists a sequence $\{\alpha_n\} \subseteq \mathbb{R}\backslash\{0\}$ with $\alpha_n \to 0$ such that $\alpha e \in c_n + C$, this means that $c_n \preceq_C^{m_1} \alpha_n e$ and so $F(\bar{x}) + c_n \preceq_C^{m_1} F(\bar{x}) + \alpha_n e$. It follows from Lemma 2.4 (i) that

$$F(x_n) \preceq_C^{m_1} F(\bar{x}) + \alpha_n e \quad \text{for all } n.$$

Hence, the proof is complete. ∎

Remark 3.5. (generalized) well-posedness and (generalized) e-well-posedness are coincide.

We now recall three kinds of pointwise well-posedness for set optimization problems in Long et al. [19].

Definition 3.6. Problem (SOP) is said to be

(i) *L-well-posed* at $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$ if for every minimizing sequence at $\bar{x}$ has a subsequence that converges to an element of $\mathrm{Eff}_{m_1}(F, K)$;

(ii) *DH-well-posed* at $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$ if

$$\inf_{\alpha>0} \mathrm{diam} L(\bar{x}, c, \alpha) = 0, \quad \text{for each } c \in C,$$

where $L(\bar{x}, c, \alpha) := \{x \in K : F(x) \preceq_C^{m_1} F(\bar{x}) + \alpha c\}$;

(iii) *B-well-posed* at $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$ if the set-valued mapping $Q_{\bar{x}} : C \to 2^K$, defined as

$$Q_{\bar{x}}(c) := \{x \in K : F(x) \preceq_C^{m_1} F(\bar{x}) + c\}, \quad \text{for each } c \in C,$$

is upper semicontinuous at $c = 0$.

Definition 3.6 (i), (ii) and (iii) generalize Definition of well-posedness in Loridan [14], Dentcheva and Helbig [12] and Bednarczuck [11] from the vector-valued case to set-valued case with respect to order $\preceq_C^{m_1}$, respectively.

Remark 3.7. It is clear that if $\bar{x} \in \text{Eff}_{m_1}(F, K)$, then $\bar{x} \in Q_{\bar{x}}(0) \subseteq \text{Eff}_{m_1}(F, K)$.

Example 3.8. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}_2^+$, $K = [0, 1]$, and $F : K \to 2^{\mathbb{R}^2}$ be defined as

$$F(x) = B\left((1 - x, x), 1\right).$$

It is clear that $\text{Eff}_{m_1}(F, K) = [0, 1]$ and $Q_x(0) = x$. Hence, $\text{Eff}_{m_1}(F, K) = [0, 1] \supset Q_{\bar{x}}(0)$ for all $\bar{x} \in \text{Eff}_{m_1}(F, K)$.

We first have the following result which gives a characterization for pointwise DH-well-posed.

Proposition 3.9. Let $e \in int\, C$ and $\bar{x} \in \text{Eff}_{m_1}(F, K)$. Problem (SOP) is DH-well-posed at $\bar{x}$ if and only if $\lim_{\alpha \to 0} \text{diam} L(\bar{x}, e, \alpha) = 0$.

Proof. The necessity is obvious. Conversely, we suppose that $\lim_{\alpha \to 0} \text{diam} L(\bar{x}, e, \alpha) = 0$. Let $c \in C$. Then there exists $\lambda > 0$ such that $\lambda e - c \in C$, i.e. $\lambda e \in c + C$. This means that $c \preceq_C^{m_1} \lambda e$. By Lamma 2.4 (i), one has

$$F(x) \preceq_C^{m_1} F(\bar{x}) + \alpha c \preceq_C^{m_1} F(\bar{x}) + \alpha \lambda e,$$

which implies that $L(\bar{x}, c, \alpha) \subseteq L(\bar{x}, e, \lambda \alpha)$. By our hypothesis, $\lim_{\lambda \alpha \to 0} L(\bar{x}, e, \lambda \alpha)$ implies $\inf_{\alpha > 0} \text{diam} L(\bar{x}, c, \alpha) = 0$. Hence we have desired. ∎

Remark 3.10. From the prove in Proposition 3.9, it is easy to verify that the problem (SOP) is DH-well-posed at $\bar{x}$ if and only if it is e-well-posed at $\bar{x}$

We now give some characterizations among pointwise B-well-posed, pointwise L-well-posed, and DH-well-posed for set optimization problems.

Proposition 3.11. Let $\bar{x} \in \text{Eff}_{m_1}(F, K)$.
   (i) If problem (SOP) is DH-well-posed at $\bar{x}$, then it is L-well-posed at $\bar{x}$.
   (ii) If Problem (SOP) is L-well-posed at $\bar{x}$ and $\text{Eff}_{m_1}(F, K)$ is singleton, then it is DH-well-posed at $\bar{x}$.

Proof. (i) Suppose that problem (SOP) is DH-well-posed at $\bar{x}$. Let $\{x_n\}$ be any minimizing sequence at $\bar{x}$. Let $e \in int\, C$. It follows from Lemma 3.4 that $\{x_n\}$ is e-minimizing sequence at $\bar{x}$. Then there exists $\{\varepsilon_n\} \subseteq \mathbb{R}\backslash\{0\}$ with $\varepsilon_n \to 0$ such that

$$F(x_n) \preceq_C^{m_1} F(\bar{x}) + \varepsilon_n e.$$

This means that $x_n \in L(\bar{x}, e, \varepsilon_n)$ for all n, which implies that

$$\|x_n - \bar{x}\| \le \text{diam} L(\bar{x}, e, \varepsilon_n) \to 0 \text{ as } n \to +\infty.$$

Thus, Problem (SOP) is L-well-posed at $\bar{x}$.

(ii) Suppose that the problem (SOP) is L-well-posed at $\bar{x}$ and $\text{Eff}_{m_1}(F, K)$ is a singleton. We will show that (SOP) is DH-well-posed at $\bar{x}$. If not, there exists $c \in C$ and some $\delta > 0$ such that

$$\text{diam} L(\bar{x}, c, \alpha) > \delta \text{ for every } \alpha > 0.$$

Let $\alpha_n = \frac{1}{n+1}$. Then there exists $x_n \in L(\bar{x}, c, \alpha_n)$ such that

$$\|x_n - \bar{x}\| > \frac{\delta}{2} \text{ for every } n. \tag{3.1}$$

By definition of $L(\bar{x}, e, \alpha_n)$ and Lemma 3.4, one has $\{x_n\}$ is minimizing sequence at $\bar{x}$. Since the problem (SOP) is L-well-posed at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to +\infty$, which contradicts with (3.1). Here, we complete the proof. ∎

The previous proposition generalizes results in [16, Proposition 3.3] and [9, Proposition 3.1 (i)] from vector-valued case to set-valued criteria.

Proposition 3.12. Let $\bar{x} \in \mathrm{Eff}_{m_1}(F, K)$.

    (i) If problem (SOP) is L-well-posed at $\bar{x}$ and $\mathrm{Eff}_{m_1}(F, K) = Q_{\bar{x}}(0)$, then it is B-well-posed at $\bar{x}$.

    (ii) If problem (SOP) is B-well-posed at $\bar{x}$, $\mathrm{Eff}_{m_1}(F, K) = Q_{\bar{x}}(0)$ and $\mathrm{Eff}_{m_1}(F, K)$ is compact set, then it is L-well-posed at $\bar{x}$.

Proof. (i) We show that a set-valued mapping $Q_{\bar{x}}(\cdot)$ defined in Definition 3.6 (iii) is upper semicontinuous at 0. If not, there exists a open set V such that $Q_{\bar{x}}(\cdot) \subset V$ and for each $\{c_n\} \subseteq C$ with $c_n \to 0$, there exists $x_n \in Q_{\bar{x}}(c_n)$ such that

$$x_n \notin V \quad \text{for all } n. \tag{3.2}$$

Since $x_n \in Q_{\bar{x}}(0)$, one has

$$F(x_n) \preceq_C^{m_1} F(\bar{x}) + c_n.$$

That is, $\{x_n\}$ is a minimizing sequence at $\bar{x}$. Thanks to L-well-posedness of the problem (SOP) at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \to \tilde{x} \in \mathrm{Eff}_{m_1}(F, K) \quad \text{as} \quad k \to +\infty.$$

It follows from our hypothesis that

$$x_{n_k} \to \tilde{x} \in \mathrm{Eff}_{m_1}(F, K) = Q_{\bar{x}}(0) \subset V,$$

which leads to a contradiction with (3.2). Therefore, problem (SOP) is B-well-posed at $\bar{x}$.

    (ii) It follows from Proposition 2.9 (ii). ∎

4. SCALARIZATION RESULTS

In this section, we establish some relations between pointwise well-posedness of set optimization problems and well-posedness of scalar optimization problems.

Consider the scalar optimization problem (OP) as follows:

$$(OP) \quad \min f(x) \quad \text{subject to} \quad x \in S,$$

where $f : S \subseteq X \to \overline{\mathbb{R}}$. Denote by $\inf_S f$ the infimum of f over S and by $\arg\min(f, S)$ the solution set of problem (OP). Also, let $\alpha\text{-}\arg\min(f, S) = \{x \in S : f(x) \leq \inf_S F + \alpha\}$, where α is a positive scalar.

Definition 4.1. [8, 9] The scalar optimization problem (OP) is said to be

    (i) *Tikhonov well-posed* if $\arg\min(f, S)$ is a singleton and every minimizing sequence, i.e. $x_n \subseteq S, f(x_n) \to \inf_S f$ converges to $\arg\min(f, S)$;

    (ii) *generalized well-posed* if $\arg\min(f, S) \neq \emptyset$ and for every minimizing sequence in S there exists a subsequence that converges to an element of $\arg\min(f, S)$.

Remark 4.2. It is easily seen that the scalar optimization problem (OP) is Tikhonov well-posed if and only if it is generalized well-posed and $\arg\min(f, S)$ is a singleton set.

The following results characterizes the well-posedness for scalar optimization problems.

Theorem 4.3. [9, 10] If $\arg\min(f, S) \neq \emptyset$, then problem (OP) is Tikhonov well-posed if and only if $\inf_{\alpha>0} \operatorname{diam}(\alpha - \arg\min(f, S)) = 0$.

Theorem 4.4. [10] If problem (OP) is Tikhonov well-posed, then $\arg\min(f, S)$ is a singleton and $(OP) \inf_{\alpha>0} \operatorname{diam}(\alpha - \arg\min(f, S)) = 0$.

Theorem 4.5. [8, 9] The problem (OP) is generalized well-posed if and only if the set-valued maps $D : \mathbb{R}_+ \to 2^S$ defined by $D(\alpha) = \alpha - \arg\min(f, S)$ is upper semicontinuous at 0 and $\arg\min(f, S)$ is compact.

Consider a scalar optimization problem with respect to the scalarizing function $I_e^{m_1}$ as follows:

$$(OP_I) \quad \min I_e^{m_1}(F(x), F(\bar{x})) \quad \text{subject to} \quad x \in K,$$

Theorem 4.6. [6, Corollary 8 and Corollary 9] Let $F : X \to 2^Y$ be a compact valued on X and C be closed. Then,

    (i) $\bar{x} \in X$ is a solution of (SOP) implies

$$\bar{x} \in \arg\min\left(I_e^{m_1}(F(\cdot), F(\bar{x})), X\right).$$

    (ii) $\{\bar{x}\} = \arg\min\left(I_e^{m_1}(F(\cdot), F(\bar{x})), X\right)$ implies $\bar{x} \in X$ is a solution of (SOP).

We now establish a relationships between pointwise well-posedness of the set optimization problem and well-posedness of a scalar optimization problem.

Theorem 4.7. Let $\bar{x} \in \operatorname{Eff}_{m_1}(F, K)$ and $e \in int\, C$. Assume that $F(\bar{x})$ is compact. Problem (SOP) is DH-well-posed at $\bar{x}$ if and only if Problem (OP_I) is Tikhonov well-posed.

Proof. We first show that, for every $\alpha > 0$,

$$\alpha - \arg\min(K, I_e^{m_1}(F(x), F(\bar{x}))) = L(\bar{x}, e, \alpha). \tag{4.1}$$

Let $x \in \alpha - \arg\min(K, I_e^{m_1}(F(x), F(\bar{x})))$. Then, for every $x \in K$

$$0 \leq I_e^{m_1}(F(x), F(\bar{x})) \leq \inf_{y \in K} I_e^{m_1}(F(y), F(\bar{x})) + \alpha = \alpha.$$

By Lemma 2.10 (iv) and (v),

$$F(x) \preceq_C^{m_1} \alpha e + F(\bar{x}).$$

Then $x \in L(\bar{x}, e, \alpha)$, and so $\alpha - \arg\min(K, I_e^{m_1}(F(x), F(\bar{x}))) \subseteq L(\bar{x}, e, \alpha)$.

Conversely, let $x \in L(\bar{x}, e, \alpha)$. Then $F(x) \preceq_C^{m_1} \alpha e + F(\bar{x})$. By Lemma 2.13 (i), Lemma 2.11 and Lemma 2.10 (i), one has

$$I_e^{m_1}(F(x), F(\bar{x})) \leq I_e^{m_1}(F(\bar{x}), F(\bar{x}) + \alpha e) \leq I_e^{m_1}(F(\bar{x}), F(\bar{x})) + \alpha = \alpha,$$

and so

$$I_e^{m_1}(F(x), F(\bar{x})) \leq \alpha \leq \inf_K I_e^{m_1}(F(x), F(\bar{x})) + \alpha.$$

This means that $x \in \alpha - \arg\min(K, I_e^{m_1}(F(x), F(\bar{x})))$. Hence, (4.1) holds.

Suppose that problem (SOP) is DH-well-posed at $\bar{x}$. It follows from Proposition 3.9 that $\lim\limits_{\alpha \to 0} \operatorname{diam} L(\bar{x}, e, \alpha) = 0$. This fact together with (4.1) yields

$$\lim_{\alpha \to 0} \operatorname{diam} (\alpha - \arg \min(K, I_e^{m_1}(F(x), F(\bar{x})))) = 0.$$

Since $\bar{x} \in \operatorname{Eff}_{m_1}(F, K)$, one has $\arg \min(K, I_e^{m_1}(F(x), F(\bar{x})) \neq \emptyset$. By Theorem 4.3, problem (OP_I) is Tikhonov well-posed.

On the other hand, assume that the problem (OP_I) is Tikhonov well-posed. Then, by Theorem 4.4, $\lim\limits_{\alpha \to 0} \operatorname{diam} (\alpha - \arg \min(K, I_e^{m_1}(F(x), F(\bar{x})))) = 0$. Thanks to (4.1) and Proposition 3.9, we get problem (SOP) is DH-well-posed at $\bar{x}$. ∎

Lemma 4.8. Let $e \in int\, C$. Problem (SOP) is B-well-posed at $\bar{x}$ if and only if the set-valued map $Q_{\bar{x}}^+(\alpha) : \mathbb{R}_+ \to 2^K$ defined as

$$Q_{\bar{x}}^+(\alpha) := \{x \in K : F(x) \preceq_C^{m_1} F(\bar{x}) + \alpha e\}, \quad \text{for each } \alpha \in \mathbb{R}_+ \tag{4.2}$$

is upper semicontinuous at $\alpha = 0$.

Proof. Suppose that the problem (SOP) is B-well-posed at $\bar{x} \in \operatorname{Eff}_{m_1}(F, K)$. Let V be an open set such that $Q_{\bar{x}}^+(0) \subset V$. So, $Q_{\bar{x}}(\mathbf{0}) \subset V$ and there exists $\delta > 0$ such that

$$Q_{\bar{x}}(c) \subset V, \quad \forall c \in B(0, \delta) \cap C.$$

There exists $\beta > 0$ such that $\rho e \in B(\mathbf{0}, \delta)$ for all $\rho \in [0, \beta)$. Putting $\alpha \in [0, \beta)$ and $x \in Q_{\bar{x}}^+(\alpha)$, we have

$$F(x) \preceq_C^{m_1} F(\bar{x}) + \alpha e.$$

This implies that $x \in Q_{\bar{x}}(\alpha e)$. Since $\alpha e \in B(\mathbf{0}, \delta)$, one has $x \in V$ and so

$$Q_{\bar{x}}^+(\alpha) \subset V \quad \text{for all } \alpha \in [0, \beta),$$

since α is arbitrarily chosen. Hence, $Q_{\bar{x}}^+(\cdot)$ is u.s.c at $\alpha = 0$.

Conversely, suppose that $Q_{\bar{x}}^+$ is upper semicontinuous at 0. Let V be an open set such that $Q_{\bar{x}}(0) \subset V$. Then $Q_{\bar{x}}^+(0) \subset V$. It follows that there exists a positive number β such that $Q_{\bar{x}}^+(\alpha) \subset V$ for every $\alpha \in [0, \beta)$. Let $\gamma \in [0, \beta)$. Then there exists a positive number ρ such that $B(\mathbf{0}, \rho) \subset \gamma e - C$. Let $c \in B(\mathbf{0}, \rho) \cap C$ and $x \in Q_{\bar{x}}(c)$. Thus, $F(x) \preceq_C^{m_1} F(\bar{x}) + c$ implies

$$F(x) \preceq_C^{m_1} F(\bar{x}) + c \preceq_C^{m_1} F(\bar{x}) + \gamma e.$$

This means that $x \in Q_{\bar{x}}(\gamma) \subset V$. Hence, we have desired. ∎

Theorem 4.9. Let $\bar{x} \in \operatorname{Eff}_{m_1}(F, K)$ and $e \in int\, C$. Assume that $F(\bar{x})$ is compact. Problem (SOP) is B-well-posed at $\bar{x}$ iff Problem (OP_I) satisfies

$$W(\alpha) := \alpha - \arg \min(K, I_e^{m_1}(F(x), F(\bar{x}))) \text{ is upper semicontinuous at } \alpha = 0. \tag{4.3}$$

Proof. Suppose that Problem (SOP) is B-well-posed at $\bar{x}$. It follows from Lemma 4.8 that the set-valued mapping $Q_{\bar{x}}^+ : \mathbb{R}_+ \to 2^K$ is define in (4.2) is upper semicontinuous at $\alpha = 0$. From the prove in Theorem 4.7 implies

$$Q_{\bar{x}}^+(\alpha) = L(\bar{x}, e, \alpha) = \alpha - \arg \min(K, I_e^{m_1}(F(x), F(\bar{x}))) = W(\alpha).$$

Conversely, we assume that $W(\alpha)$ is upper semicontinuous at $\alpha = 0$. Thus $Q_{\bar{x}}^+(\alpha)$ is also. Lemma 4.8 implies that Problem (SOP) is B-well-posed at $\bar{x}$. ∎

Theorem 4.10. Let $\bar{x} \in \text{Eff}_{m_1}(F, K)$ and $e \in int\, C$. Assume that F is compact valued on K. If problem (SOP) is L-well-posed at $\bar{x}$, then Problem (OP_I) is Tikhonov well-posed in generalized sense.

Proof. Let $\bar{x} \in \text{Eff}_{m_1}(F, K)$ and $e \in int\, C$. Assume that (SOP) is L-well-posed at $\bar{x}$. Let $\{x_n\}$ be a minimizing sequence to problem (OP_I). Then,

$$I_e^{m_1}(F(x_n), F(\bar{x})) \to \inf_{x \in K} I_e^{m_1}(F(x), F(\bar{x})) = 0.$$

This implies that there exists $\{\varepsilon_n\} \subseteq (0, +\infty)$ with $\varepsilon_n \to 0$ such that

$$0 \leq \inf\{t \in \mathbb{R} : A \preceq_C^{m_1} te + B\} < \varepsilon_n.$$

So, there exists $t_n \in [0, \varepsilon_n)$ such that

$$F(x_n) \preceq_C^{m_1} t_n e + F(\bar{x}).$$

Thus, $\{x_n\}$ is e-minimizing sequence to problem (SOP). Thanks to Lemma 3.4 $\{x_n\}$ is minimizing sequence to problem (SOP). Since (SOP) is L-well-posed at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converse to an element $\tilde{x} \in \text{Eff}_{m_1}(F, K)$. Then, by Theorem 4.6, $\tilde{x} \in \arg\min (I_e^{m_1}(F(\cdot), F(\bar{x})), X)$. Therefore, problem (OP_I) is Tikhonov well-posedness in the generalized sense. ∎

5. Global Well-Posedness

In this following, we give sufficiency results for generalized e-well-posedness and extended-e-well-posedness notions.

Theorem 5.1. Let K be a compact set. If F is l.s.c, compact on K and $F(u) = F(v)$ for all $u, v \in \text{Eff}_{m_1}(F, K)$, then (SOP) is generalized e-well-posed.

Proof. Let $\{x_n\}$ be a generalized e-minimizing sequence. Then there exists a real sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and a sequence $\{z_n\} \subseteq \text{Eff}_{m_1}(F, K)$ such that

$$F(x_n) \preceq_C^{m_1} F(z_n) + \varepsilon_n e.$$

Since K is compact, there exist a subsequence $\{x_{n_k}\}$ and $\tilde{x} \in K$ such that $x_{n_k} \to \tilde{x}$. For any $\tilde{z} \in \text{Eff}_{m_2}(F, K)$, $F(\tilde{z}) = F(z_n)$ for all n. We then have

$$F(x_n) \preceq_C^{m_1} F(\tilde{z}) + \varepsilon_n e.$$

From definition of a partial order $\preceq_C^{m_1}$, there exists $\tilde{c} \in C$ such that

$$\tilde{c} + F(x_n) \subseteq F(\tilde{z}) + \varepsilon_n e.$$

Let $\hat{\xi} \in F(\tilde{x})$. Since F is l.s.c, there exists $\hat{\xi}_n \in F(x_n)$ such that $\hat{\xi}_n \to \hat{\xi}$.

Thus, $(\tilde{z}, \tilde{c} + \hat{\xi}_n - \varepsilon_n e) \in \text{graph}(F)$. Then, there exist subsequence $\{\varepsilon_{n_k}\}$ and $\{\hat{\xi}_{n_k}\}$ such that

$$\tilde{c} + \hat{\xi}_{n_k} - \varepsilon_{n_k} e \to \tilde{c} + \hat{\xi} \in F(\tilde{z}).$$

Since $\hat{\xi}$ is an arbitrary element in $F(\tilde{x})$, one has $\tilde{c} + F(\tilde{x}) \subseteq F(\tilde{z})$. That is, $F(\tilde{x}) \preceq_C^{m_1} F(\tilde{z})$. Since $\tilde{z} \in \text{Eff}_{m_1}(F, K)$, we have $\tilde{x} \in \text{Eff}_{m_1}(F, K)$. The proof is complete. ∎

The following example shows that the compactness of F cannot be dropped.

Example 5.2. Let $K = [0,1]$ and $C = \mathbb{R}^2_+$. A set-valued mapping $F : K \to 2^{\mathbb{R}^2_+}$ is defined by

$$F(x) = \begin{cases} [0,1] \times [0,1], & \text{if } 0 < x \le 1, \\ [2,3] \times [2,3], & \text{if } x = 0. \end{cases}$$

Let $e = (1,1)$. We see that $\text{Eff}_{m_1}(F,K) = (0,1]$, K is compact and $F(u) = F(v)$ for all $u, v \in \text{Eff}_{m_1}(F,K)$. But F is not compact at 0. Indeed, $\left(\frac{1}{n}, \left(0, \frac{1}{n} \right) \right) \in \text{graph}(G)$ with $\frac{1}{n} \to 0$, but $\left(0, \frac{1}{n} \right) \to (0,0) \notin F(0)$. Let $x_n = \varepsilon_n = \frac{1}{n}$. Then clearly that x_n is a generalized e-minimizing sequence and $x_n \to 0 \notin \text{Eff}_{m_1}(F,K)$.

We recall the extended e-well-posedness notion defined for m_1-weak-minimal solutions which developed from the concept in Zhang et al. [17].

Definition 5.3. A sequence $\{x_n\}$ is an *extended e-minimizing sequence* if there exists a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}$ with $\varepsilon_n \to 0$ such that

$$F(x) + \varepsilon_n e \not\prec_C^{m_1} F(x_n), \quad \forall \ x \in K.$$

(SOP) is said to be *extended e-well-posed* if for each extended e-minimizing sequence there exist a subsequence $\{x_{n_k}\}$ and $\bar{x} \in \text{WEff}_{m_1}(F,K)$ such that $x_{n_k} \to \bar{x}$.

Theorem 5.4. If K is a compact set, F is a compact on K, then (SOP) is extended e-well-posed.

Proof. Let $\{x_n\}$ be an extended e-minimizing sequence. Then there exists a real sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$F(x_n) + \varepsilon_n e \not\prec_C^{m_1} F(y) \quad \text{for all} \quad y \in K. \tag{5.1}$$

Since K is compact, there exist a subsequence $\{x_{n_k}\}$ and $\tilde{x} \in K$ such that $x_{n_k} \to \tilde{x}$. Let $x \in K$. It follows from (5.1) for any $\tilde{c} \in \text{int}\,(C)$,

$$F(x_{n_k}) + \varepsilon_{n_k} e + \tilde{c} \not\subseteq F(x).$$

Then, there exists a sequence $\xi_{n_k} \in F(x_{n_k})$ such that

$$\xi_{n_k} + \varepsilon_{n_k} e + \tilde{c} \notin F(x) \Rightarrow \xi_{n_k} + \varepsilon_{n_k} e \notin F(x) - \tilde{c}.$$

Since F is a compact map, there exist subsequence $\{\xi_{n_{k_l}}\}$ of $\{\xi_{n_k}\}$ and $\bar{\xi} \in F(\bar{x})$ such that $\xi_{n_{k_l}} \to \bar{\xi}$. Hence, $\xi_{n_{k_l}} + \varepsilon_{n_k} e \to \bar{\xi} \notin -\text{int}\,(C) + F(x)$. One has $F(\bar{x}) \not\subseteq F(x) - \text{int}(C)$ implies $F(x) \not\prec_C^{m_1} F(\bar{x})$. Therefore, $\bar{x} \in \text{WEff}_{m_1}(F,K)$. This show that (SOP) is extended e-well-posed. ∎

Remark 5.5. We observe that the problem (SOP) in Example 5.2 is not extended e-well-posed. Indeed, if $e = (1,1)$ and $x_n = \varepsilon_n = \frac{1}{n}$, then x_n is an extended e-minimizing sequence. We see that $x_n \to 0 \notin \text{WEff}_{m_1}(F,K)$.

We recall the global well-posedness notion defined for m_1-weak-minimal solutions which developed form the concept in Crespi et al. [20].

Definition 5.6. (SOP) is said to be *globally well-posed* if for every extended e-minimizing sequence $\{x_n\}$ there exists a subsequence $\{x_{n_k}\}$ such that

$$d(x_{n_k}, \text{WEff}_{m_1}) \to 0,$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$.

Proposition 5.7. Problem (SOP) is extended e-well-posed, then (SOP) is globally well-posed. Conversely, if problem (SOP) is globally well-posed and $\mathrm{WEff}_{m_1}(F, K)$ is compact, then (SOP) is extended e-well-posed.

Proof. Since (SOP) is extended e-well-posed, every extended e-minimizing sequence there exist a subsequence $\{x_{n_k}\}$ and $\bar{x} \in \mathrm{WEff}_{m_1}(F, K)$ such that $x_{n_k} \to \bar{x}$. One has

$$d(x_{n_k}, \mathrm{WEff}_{m_1}(F, K)) \le \|x_{n_k} - \bar{x}\| \to 0.$$

Hence, (SOP) is globally well-posed.

Conversely, we assume that (SOP) is globally well-posed. From the definition of $d(x_{n_k}, \mathrm{WEff}_{m_1}(F, K))$, we can find a sequence $\bar{x}_{n_k} \in \mathrm{WEff}_{m_1}(F, K)$ such that $\|x_{n_k} - \bar{x}_{n_k}\| \to 0$. It follow from the compactness of $\mathrm{WEff}_{m_1}(F, K)$, there exist a subsequence $\{\bar{x}_{n_{k_l}}\}$ of $\{\bar{x}_{n_k}\}$ and $\bar{x} \in \mathrm{WEff}_{m_1}(F, K)$ such that $\bar{x}_{n_{k_l}} \to \bar{x}$. Then we have

$$\|x_{n_{k_l}} - \bar{x}\| \le \|x_{n_{k_l}} - \bar{x}_{n_{k_l}}\| + \|\bar{x}_{n_{k_l}} - \bar{x}\| \to 0.$$

Therefore, (SOP) is extended e-well-posed. ∎

6. $\preceq_C^{m_1}$-Robustness

In this section we discuss solution concepts for set optimization problems with the $\preceq_C^{m_1}$ to study robust vector optimization problems. Detailed overviews of the existing robustness concepts for vector optimization problem can be found in [2, 30]. Let $\mathcal{U} \subseteq \mathbb{R}^m$ be a nonempty compact set. We consider the following uncertain vector-optimization problem

$$\min_{x \in K} f(x, \xi) \qquad\qquad (UVOP)$$

where $f : K \times \mathcal{U} \to Y$ and $\xi \in \mathcal{U}$ is the uncertain parameter of the problem. In this setting, the objective function f depends on scenarios ξ which are unknown to uncertain. This means that, finding the valued $\bar{x} \in K$ such that

$$(f(\bar{x}, \xi) - C) \cap f(K, \xi) = \{f(\bar{x}, \xi)\}.$$

For studying the robust counterpart problem of the uncertain vector optimization problem (UVOP), we consider the following set optimization problem:

$$\min_{x \in K} f(x, \mathcal{U})$$

where $f(x, \mathcal{U}) = \{f(x, \xi) : \xi \in \mathcal{U}\}$ is compact subset in Y.

We define "the robust counterpart" of $(UVOP)$ as the set optimization problem with $\preceq_C^{m_1}$ relation. This means that $\bar{x} \in K$ is a solution of the robust counterpart of $(UVOP)$ if

there is no any $x \in K$ with $f(x, \mathcal{U}) \preceq_C^{m_1} f(\bar{x}, \mathcal{U})$ and $f(x, \mathcal{U}) \ne f(\bar{x}, \mathcal{U})$.

In same way, $\bar{x} \in K$ is a solution of the weak robust counterpart of $(UVOP)$ if

there is not any $x \in K$ with $f(x, \mathcal{U}) \prec_C^{m_1} f(\bar{x}, \mathcal{U})$.

Let $\mathrm{Eff}(f(x, \mathcal{U}), K)$ and $\mathrm{WEff}(f(x, \mathcal{U}), K)$ be the set of all (weak) solutions of the robust counterpart optimization problem, respectively.

The following two corollaries are immediately obtained form Theorem 5.1 and Theorem 5.4, respectively.

Corollary 6.1. Let K is a compact set. If $f(\cdot, \mathcal{U})$ is l.s.c and compact on K and $f(u, \mathcal{U}) = f(v, \mathcal{U})$ for all $u, v \in \mathrm{Eff}_{m_1}(f(x, \mathcal{U}), K)$, then $(UVOP)$ is generalized e-well-posed.

Corollary 6.2. If K is a compact set, $f(\cdot, \mathcal{U})$ is compact on K, then $(UVOP)$ is extended e-well-posed.

7. Conclusions

In this paper we discuss the notions of pointwise well-posedness and global well-posedness for set optimization problem with $\preceq_C^{m_1}$ order. We introduce the concept of m_1-C-convexity for set-valued mappings and show some convexity properties for nonlinear scalarizing function. We consider pointwise B-well-poseness, pointwise L-well-poseness and pointwise DH-well-poseness for set optimization problems with $\preceq_C^{m_1}$ order. We gave some relations among the three kinds of pointwise well-posedness. Moreover, we obtain some relations between pointwise well-posedness of set optimization problems and well-posedness of scalar optimization problems by using a nonlinear scalarizing function. Some relation between pointwise well-posedness and global well-posedness are also studied. Finally, we discuss the generalized e-well-posedness and extended e-well-posedness of robust vector optimization.

Acknowledgements

References

[1] A.H. Hamel, F. Heyde, A. Löhne, B. Rudloff, C. Schrage, Set Optimization and Applications - The State of the Art, Springer Proc. Math. Stat. 151, Springer, Heidelberg, 2015.

[2] A.A. Khan, C. Tammer, C. Zălinescu, Set-Valued Optimization: An Introduction with Applications, Springer, Berlin, 2015.

[3] D. Kuroiwa, The natural criteria in set optimization, RIMS Kokyuroku 1031 (1998) 85–90.

[4] D. Kuroiwa, Some duality theorems of set optimization, RIMS Kokyuroku 1079 (1999) 15–19.

[5] J. Jahn, T.X. Ha, New order relations in set optimization, J. Optim. Theory Appl. 148 (2011) 209–236.

[6] S. Karaman, M. Soyertem, I.A. Güvenç, D. Tozkan, M. Küçük, Y. Küçük, Partial order relations on family of sets and scalarization for set optimization, Positivity 22 (2018) 783–802.

[7] A.N. Tikhonov, On the stability of the functional optimization problem, USSR Comput. Math. Math. Phys. 6 (1966) 764–767.

[8] A.L. Dontchev, T. Zolezzi, Well-Posed Optimization Problems, Lecture Notes in Mathematics, Vol 1543, Springer, Berlin, 1993.

[9] M. Durea, Scalarization for pointwise well-posed vectorial problems, Math. Methods Oper. Res. 66 (2007) 409–418.

[10] M. Furi, A. Vignoli, About well-posed optimization problems for functionals in metric spaces, J. Optim. Theory Appl. 5 (1970) 225–229.

[11] E.M. Badnarczuck, Well posedness of vector optimization problem, In: J. Jahn, W. Krabs, (eds.) Recent Advances and Historical Development of Vector Optimization Problems, Lecture Notes in Economics and Mathematical Systems, Vol. 294, Springer, Berlin (1987), 51–61.

[12] D. Dentcheva, H. Helbig, On variational principles, level sets, well-posedness and ε-solutions in vector optimization, J. Optim. Theory Appl. 89 (1996) 325–349.

[13] X.X. Huang, X.Q. Yang, Levitin-Polyak well-posedness of constrained vector optimization problems, J. Global Optim. 37 (2007) 287–304.

[14] P. Loridan, Well-posedness in vector optimization, In: R. Lucchetti, J. Recalski, (eds.) Recent Developments in Cariational Well-posedness Problems. Mathematics and Its Applications, Vol. 331., Kluwer, Dordrecht (1995), 171–192.

[15] R.E. Lucchetti, E. Miglierina, Stability for convex vector optimization problems, Optimization 53 (2004) 517–528.

[16] E. Miglierina, E. Molho, M. Rocca, Well-posedness and scalarization in vector optimization, J. Optim. Theory Anal. 126 (2005) 391–409.

[17] W.Y. Zhang, S.J. Li, K.L. Teo, Well-posedness for set optimization problems, Nonlinear Anal. 71 (2009) 3769–3778.

[18] C. Gutiérrez, E. Miglierina, E. Malho, V. Novo, Pointwise well-posedness in set optimization with cone proper sets, Nonlinear Anal. 75 (2012) 1822–1833.

[19] X.J. Long, J.W. Peng, Z.Y.Peng, Scalarization and pointwise well-posedness for set optimization problems, J. Glob. Optim. 62 (2015) 763–773.

[20] G.P. Crespi, D. Kuroiwa, M. Rocca, Convexity and global well-posedness in set-optimization, Taiwanese J. Math. 18 (2014) 1897–1908.

[21] M. Dhingra, C.S. Lalitha, Well-setness and scalarization in set optimization, Optim. Lett. 10 (2016) 1657–1667.

[22] G.P. Crespi, M. Dhingra, C.S. Lalitha, Pointwise and global well-posedness in set optimization: a direct approach, Ann. Oper. Res. 269 (2018) 149–166.

[23] L.Q. Anh, T.Q. Duy, D.V. Hien, D. Kuroiwa, N. Petrot, Convergence of Solutions to Set Optimization Problems with the Set Less Order Relation, J. Optim. Theory Appl. (2020)

[24] P. Preechasilp, R. Wangkeeree, A note on semicontinuity of the solution mapping for parametric set optimization problems, Optim. Lett 13 (2019) 1085–1094.

[25] P.T. Vui, L.Q. Anh, R. Wangkeeree, Levitin-Polyak well-posedness for set optimization problems involving set order relations, Positivity 23 (2019) 599–616.

[26] J.P. Aubin, I. Ekeland, Applied Nonlinear Analysis, Wiley, New York, 1984.

[27] D.T. Luc, Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems, Vol. 39, Springer, Berlin, 1989.

[28] k F. Ferro, A minimax theorem for vector-valued functions, J. Optim. Theory Appl. 60 (1989) 19–31.

[29] X.J. Long, J.W. Peng, Generalized B-well-posedness for set optimization problems, J. Optim. Theory Appl. 157 (2013) 612–623.

[30] G.P. Crespi, D. Kuroiwa, M. Rocca, Quasiconvexity of set-valued maps assures well-posedness of robust vector optimization, Ann. Oper. Res. 251 (2017) 89–104.