Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1611–1622

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Fixed Point Property of Real Unital Abelian Banach Algebras and Their Closed Subalgebras Generated by an Element with Infinite Spectrum

Worapong Fupinwong

Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : g4865050@hotmail.com

Abstract A Banach space X is said to have the fixed point property if for each nonexpansive mapping $T : E \to E$ on a bounded closed convex subset E of X has a fixed point. Let X be an infinite dimensional unital Abelian real Banach algebra with $\Omega(X) \neq \emptyset$ satisfying: (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$. We prove that, for each element x_0 in X with infinite spectrum, the Banach algebra $\langle x_0 \rangle = \overline{\left\{\sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}$ generated by x_0 does not have the fixed point property.

MSC: 46J30; 47H09; 47H10

Keywords: the fixed point property; nonexpansive mapping; Abelian real Banach algebra; the Stone-Weierstrass approximation theorem

Submission date: 20.03.2020 / Acceptance date: 23.05.2020

1. Introduction

A Banach space X is said to have the fixed point property if for each nonexpansive mapping $T : E \to E$ on a bounded closed convex subset E of X has a fixed point, to have the weak fixed point property if for each nonexpansive mapping $T : E \to E$ on a weakly compact convex subset E of X has a fixed point.

In 1981, D. E. Alspach [1] proved that there exists an isometry $T : E \to E$ on a weakly compact convex subset E of the Lebesgue space $L_1[0, 1]$ without a fixed point. Consequently, $L_1[0, 1]$ does not have the weak fixed point property.

In 1983, J. Elton, P. K. Lin, E. Odell, and S. Szarek [2] showed that $C_{\mathbb{R}}(\alpha)$ has the weak fixed point property, if α is a compact ordinal with $\alpha < \omega^{\omega}$.

In 1997, A. T. Lau, P. F. Mah, and Ali Ülger [3] proved the following theorem.

Published by The Mathematical Association of Thailand.

Theorem 1.1. *Let X be a locally compact Hausdorff space. If $C_0(X)$ has the weak fixed point property, then X is dispersed.*

Moreover, by using Theorem 1.1, they proved the following corollaries.

Corollary 1.2. *Let G be a locally compact group. Then the C^*-algebra $C_0(G)$ has the weak fixed point property if and only if G is discrete.*

Corollary 1.3. *A von Neumann algebra $\mathcal{M}$ has the weak fixed point property if and only if $\mathcal{M}$ is finite dimensional.*

In 2005, Benavides and Pineda [4] proved the following results.

Theorem 1.4. *Let X be a ω-almost weakly orthogonal closed subspace of $C(K)$ where K is a metrizable compact space. Then X has the weak fixed point property.*

Theorem 1.5. *Let K be a metrizable compact space. Then, the following conditions are all equivalent:*
(1) $C(K)$ is ω-almost weakly orthogonal,
(2) $C(K)$ is ω-weakly orthogonal,
(3) $K^{(\omega)} = \emptyset$.

Corollary 1.6. *Let K be a compact set with $K^{(\omega)} = \emptyset$. Then $C(K)$ has the weak fixed point property.*

In 2008, A. T. Lau and M. Leinert [5] proved the following theorem.

Theorem 1.7. *$A(G)$ has the fixed point property if and only if G is finite.*

As a consequence, they obtained the following corollary.

Corollary 1.8. *$B(G)$ has the fixed point property if and only if G is finite.*

If X is a complex Banach algebra, condition (A) is defined by:
(A) *For each $x \in X$, there exists an element $y \in X$ such that $\tau(y) = \overline{\tau(x)}$, for each $\tau \in \Omega(X)$.*
Note that each C*-algebra satisfies condition (A).

In 2010, W. Fupinwong and S. Dhompongsa [6] proved that each infinite dimensional unital Abelian real Banach algebra X with $\Omega(X) \neq \emptyset$ satisfying (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$, does not have the fixed point property. Moreover, they proved the following theorem.

Theorem 1.9. *Let X be an infinite dimensional unital Abelian complex Banach algebra satisfying condition (A) and each of the following statements:*
(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$.
(ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$.
Then X does not have the fixed point property.

In 2010, by using Theorem 1.9, D. Alimohammadi and S. Moradi [7] obtain sufficient conditions to show that some unital unifromly closed subalgebras of $C(\Omega)$, where Ω is a compact space, do not have the fixed point property.

In 2011, S. Dhompongsa, W. Fupinwong, and W. Lawton [8] proved that a C^*-algebra has the fixed point property if and only if it is finite dimensional.

In 2012, W. Fupinwong [9] show that the unitality in Theorem 1.9 proved in [6] can be omitted.

In 2016, by using Urysohn's lemma and Schauder-Tychonoff fixed point theorem, D. Alimohammadi [10] proved the following result.

Theorem 1.10. *Let Ω be a locally compact Hausdorff space. Then the following statements are equivalent:*
(i) Ω is infinite set.
(ii) $C_0(\Omega)$ is infinite dimensional.
(iii) $C_0(\Omega)$ does not have the fixed point property.

In 2017, J. Daengsaen and W. Fupinwong [11] proved the following theorem.

Theorem 1.11. *Let X be an infinite dimensional real Abelian Banach algebra with $\Omega(X) \neq \emptyset$ and satisfying each of the following:*
(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$ for each $\tau \in \Omega(X)$ then $\|x\| \leq \|y\|$.
(ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$.
Then X does not have the fixed point property.

In 2018, P. Thongin and W. Fupinwong [12] proved that if X is an infinite dimensional complex unital Abelian Banach algebra satisfying condition (A) and satisfying (i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$, then there exists an element x_0 in X such that

$$\langle x_0 \rangle = \overline{\left\{ \sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}$$

does not have the fixed point property.

In this paper, if X is an infinite dimensional unital Abelian real Banach algebra with $\Omega(X) \neq \emptyset$ and satisfying; (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$, we prove that, for each x_0 in X with infinite spectrum,

$$\langle x_0 \rangle = \overline{\left\{ \sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}$$

does not have the fixed point property. Our result is a generalization of Theorem 1.11.

2. Preliminaries

Let X be a Banach space. We say that a mapping $T : E \to E$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for each $x, y \in E$, where E is a nonempty subset of X. A Banach space X is said to have the fixed point property if for each nonexpansive mapping $T : E \to E$ on a nonempty bounded closed convex subset E of X has a fixed point.

We define the spectrum of an element x of a real unital Banach algebra X to be the set

$$\sigma_X(x) = \{\lambda \in \mathbb{R} : \lambda 1 - x \notin Inv(X)\},$$

where $Inv(X)$ is the set of all invertible elements in X.

The spectral radius of x is defined to be

$$r_X(x) = \sup_{\lambda \in \sigma(x)} |\lambda|.$$

We say that a mapping $\tau : X \to \mathbb{R}$ is a character on a real algebra X if τ is a non-zero homomorphism. We denote by $\Omega(X)$ the set of all characters on X. If X is a real unital Abelian Banach algebra with $\Omega(X) \neq \emptyset$, it is known that $\Omega(X)$ is compact.

If Y is a real unital subalgebra of a real unital Banach algebra X with $\Omega(X) \neq \emptyset$, then $\Omega(Y) \neq \emptyset$. In fact, for each $\tau \in \Omega(X)$, the restriction $\tau|Y$ of τ on Y is in $\Omega(Y)$. Note that $\tau|Y$ is nonzero since Y is unital.

We denote by $C_{\mathbb{R}}(S)$ the real unital Banach algebra of continuous functions from a topological space S to $\mathbb{R}$ where the operations are defined pointwise and the norm is the sup-norm.

The following Theorem is known as the Stone-Weierstrass approximation theorem for $C_{\mathbb{R}}(S)$.

Theorem 2.1. *Let A be a subalgebra of $C_{\mathbb{R}}(S)$ satisfying the following conditions:*
(i) A separates the points of S.
(ii) A annihilates no point of S.
Then A is dense in $C_{\mathbb{R}}(S)$.

Let X be a real Abelian Banach algebra with $\Omega(X) \neq \emptyset$. The Gelfand representation $\varphi : X \to C_{\mathbb{R}}(\Omega(X))$ is defined by $x \mapsto \widehat{x}$, where $\widehat{x}$ is defined by

$$\widehat{x}(\tau) = \tau(x),$$

for each $\tau \in \Omega(X)$. If X is unital and Abelian, then $\sigma(x) = \{\tau(x) : \tau \in \Omega(X)\}$, for each $x \in X$. It is known that $r_X(x) = \|\widehat{x}\|_{\infty,X}$ if X is Abelian, where

$$\|\widehat{x}\|_{\infty,X} = \sup_{\tau\in\Omega(X)} |\widehat{x}(\tau)|.$$

3. Lemmas

Some lemmas are proved in this section. In the next section, we will use them to prove our main theorem.

Lemma 3.1. *Let X be a real unital Abelian Banach algebra satisfying $\Omega(X) \neq \emptyset$ and*

$$\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0.$$

If Y is a unital subalgebra of X, then

$$\inf\{r_Y(x) : x \in Y, \|x\| = 1\} > 0.$$

Proof. If $x \in Y$, since $\sigma_Y(x) \supset \sigma_X(x)$, then

$$r_Y(x) = sup_{\lambda\in\sigma_Y(x)}|\lambda| \geq sup_{\lambda\in\sigma_X(x)}|\lambda| = r_X(x).$$

So

$$\inf\{r_Y(x) : x \in Y, \|x\| = 1\} \geq \inf\{r_X(x) : x \in X, \|x\| = 1\} > 0.$$

Therefore, $\inf\{r_Y(x) : x \in Y, \|x\| = 1\}$ is greater than zero. ∎

Lemma 3.2. *Let X be an infinite dimensional real unital Abelian Banach algebra with $\Omega(X) \neq \emptyset$, and let x_0 be an element in X with infinite spectrum. Then $\{x_0^n : n \in \mathbb{N}\}$ is linearly independent. Consequently,*

$$\langle x_0 \rangle = \overline{\left\{ \sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}$$

infinite dimensional.

Proof. Assume that

$$\sum_{i=1}^{k} \alpha_i x_0^i = 0,$$

where $\alpha_i \in \mathbb{R}$. Let $\{\lambda_1, \lambda_2, \lambda_3, ...\} \subseteq \sigma_X(x_0)$, with $\lambda_i \neq \lambda_j$ for each $i \neq j$. Since $\sigma_X(x_0) = \{\tau(x_0) : \tau \in \Omega(X)\}$, write $\lambda_j = \tau_j(x_0)$, where $\tau_j \in \Omega(X)$. Then

$$\sum_{i=1}^{k} \alpha_i \left(\tau_j(x_0)\right)^i = \tau_j \left(\sum_{i=1}^{k} \alpha_i x_0^i \right) = 0,$$

therefore,

$$\sum_{i=1}^{k} \alpha_i \lambda_j^i = 0,$$

for each $j \in \mathbb{N}$. Hence $\alpha_i = 0$, for each $i \in \{1, 2, 3, ..., k\}$. ■

Lemma 3.3. *Let X be an infinite dimensional real unital Abelian Banach algebra with $\Omega(X) \neq \emptyset$, and let x_0 be an element in X with infinite spectrum. Define*

$$Z = \overline{\left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}.$$

If X satisfies

$$\inf\{\|\widehat{x}\|_{\infty,X} : x \in X, \|x\| = 1\} > 0,$$

then Z is a real unital Abelian Banach algebra satisfying the following conditions:
(i) The Gelfand representation φ from Z into $C_{\mathbb{R}}(\Omega(Z))$ is a bounded isomorphism.
(ii) The inverse φ^{-1} is also a bounded isomorphism.

Proof. (i) Since $ker(\varphi) = \{0\}$, φ is injective. We have $\varphi(Z)$ is a subalgebra of $C_{\mathbb{R}}(\Omega(Z))$ separating the points of $\Omega(Z)$, and annihilating no point of $\Omega(Z)$. Moreover, $\varphi(Z)$ is complete, so $\varphi(Z)$ is closed. Indeed, if $\{\widehat{z_n}\}$ is a Cauchy sequence in $\varphi(Z)$, assume to the contrary that $\{z_n\}$ is not Cauchy, then there exists $\varepsilon_0 > 0$ and subsequences $\{z'_n\}$ and $\{z''_n\}$ of $\{z_n\}$ such that

$$\|z'_n - z''_n\| \geq \varepsilon_0,$$

for each $n \in \mathbb{N}$. Define $y_n = (z'_n - z''_n)/\varepsilon_0$. Thus $\|y_n\| \geq 1$, for each $n \in \mathbb{N}$. Since $\{\widehat{z_n}\}$ is Cauchy, so $\lim \widehat{y_n} = \widehat{0}$. From Lemma 3.1, we have

$$\inf\{r_Z(x) : x \in Z, \|x\| = 1\} > 0.$$

Hence

$$\begin{aligned} 0 < \inf\{r_Z(x) : x \in Z, \|x\| = 1\} &= \inf\{\|\widehat{x}\|_{\infty,Z} : x \in Z, \|x\| = 1\} \\ &\leq \inf_{n\in\mathbb{N}} \left\| \widehat{\left(\frac{y_n}{\|y_n\|}\right)} \right\|_{\infty,Z} \\ &\leq \inf_{n\in\mathbb{N}} \|\widehat{y_n}\|_{\infty,Z} = 0, \end{aligned}$$

which is a contradiction. Hence we conclude that $\{z_n\}$ is a Cauchy sequence. Then $\{z_n\}$ is a convergent sequence in Z, say $\lim z_n = z_0 \in Z$. From, for each $n \in \mathbb{N}$,

$$\|\widehat{z_n} - \widehat{z_0}\|_{\infty,Z} = \|\varphi(z_n - z_0)\|_{\infty,Z} \leq \|z_n - z_0\|,$$

it follows that

$$\lim \|\widehat{z_n} - \widehat{z_0}\|_{\infty,Z} = 0.$$

So $\varphi(Z)$ is complete. It follows from the Stone-Weierstrass theorem that φ is surjective.
(ii) From the open mapping theorem, φ^{-1} is a bounded isomorphism. ∎

Lemma 3.4. *Let X be an infinite dimensional real unital Abelian Banach algebra with $\Omega(X) \neq \emptyset$. If there exists x_0 in X with infinite spectrum, then there exists $y \in \langle x_0 \rangle$ satisfying the following conditions:*
(i) $1 \in \sigma_X(y) \subset [0, 1]$.
(ii) There exists a strictly decreasing sequence in $\sigma_X(y)$.

Proof. Note that $Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}$ is unital and Abelian, so the spectrum of x_0 is $\widehat{x_0}(\Omega(Z))$. Hence

$$\sigma_X\left(\frac{x_0^2}{r_X(x_0^2)}\right) = \left\{\tau\left(\frac{x_0^2}{r_X(x_0^2)}\right) : \tau \in \Omega(Z)\right\} \subset [0, 1].$$

Let $\{a_n\}$ be an infinite sequence in $\sigma_X\left(\frac{x_0^2}{r_X(x_0^2)}\right)$. We may assume that $\{a_n\}$ is strictly increasing and $a_1 > 0$.

Define a continuous function $g : [0, 1] \to [0, 1]$ by

$$g(t) = \begin{cases} \frac{t}{a_1}, & \text{if } t \in [0, a_1], \\ \frac{1-t}{1-a_1}, & \text{if } t \in [a_1, 1]. \end{cases}$$

So g is joining the points $(0, 0)$ and $(a_1, 1)$, and $g(1) = 0$. Let

$$\widehat{y} = g \circ \left(\widehat{\frac{x_0^2}{r_X(x_0^2)}}\right).$$

It follows from Lemma 3.3 that $y \in Z$. Since $g(0) = 0$, so $y \in \langle x_0 \rangle$. We have $\{g(a_n)\}$ is a strictly decreasing sequence in $\sigma_X(y)$. Moreover, $1 = g(a_1) \in \sigma_X(y) \subset [0, 1]$. ∎

Lemma 3.5. *Let X be an infinite dimensional real unital Abelian Banach algebra satisfying $\Omega(X) \neq \emptyset$ and*

$$\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0,$$

and let x_0 be an element in X with infinite spectrum and $\tau(x_0) \in \mathbb{R}$, for each $\tau \in \Omega(X)$. Then there exists a sequence $\{z_n\}$ in $\langle x_0 \rangle$ such that $\{\tau(z_n) : \tau \in \Omega(Z)\} \subset [0, 1]$, for each $n \in \mathbb{N}$, and $\{(\widehat{z_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(Z)$, where

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}.$$

Proof. From Lemma 3.1, it follows that

$$\inf\{r_Z(x) : z \in Z, \|z\| = 1\} > 0.$$

From Lemma 3.2, Z is infinite dimensional, then, from Lemma 2.10 (iii) in [6], there exists $z_0 = \sum_{i=0}^{k} \alpha_i x_0^i \in Z$ such that $\{\tau(z_0) : \tau \in \Omega(Z)\}$ is infinite. Let $z_1 = \sum_{i=1}^{k} \alpha_i x_0^i$. It can be seen that $z_1 \in \langle x_0 \rangle$ and $\sigma(z_1)$ is infinite. From Lemma 3.4, we may assume without generality that z_1 satisfies

$$1 \in \sigma_Z(z_1) \subset [0, 1]$$

and there exists a strictly decreasing sequence of real numbers in $\sigma_Z(z_1)$, say $\{a_n\}$. Moreover, we may assume that $a_1 < 1$.

Define a continuous function $g_1 : [0, 1] \to [0, 1]$ by

$$g_1(t) = \begin{cases} \frac{t}{a_1}, & \text{if } t \in [0, a_1], \\ 1 + \frac{(g_1(a_2)-1)(t-a_1)}{2(1-a_1)}, & \text{if } t \in [a_1, 1]. \end{cases}$$

So g_1 is joining the points $(0, 0)$ and $(a_1, 1)$, and $g_1(1) \in (g_1(a_2), 1)$.

Define $\widehat{z_2} = g_1 \circ \widehat{z_1} : \Omega(Z) \to \mathbb{R}$, and define a continuous function $g_2 : [0, 1] \to [0, 1]$ by

$$g_2(t) = \begin{cases} \frac{t}{g_1(a_2)}, & \text{if } t \in [0, g_1(a_2)], \\ 1 + \frac{(g_2(g_1(a_3))-1)(t-g_1(a_2))}{2(1-g_1(a_2))}, & \text{if } t \in [g_1(a_2), 1]. \end{cases}$$

So g_2 is joining the points $(0, 0)$ and $(g_1(a_2), 1)$, and $g_2(1) \in (g_2(g_1(a_3)), 1)$.

Define $\widehat{z_3} = g_2 \circ \widehat{z_2} : \Omega(Z) \to \mathbb{R}$. Continuing in this process, we have a sequence $\{z_n\}$ in Z with $1 \in \{\tau(z_n) : \tau \in \Omega(Z)\} \subset [0, 1]$, for each $n \in \mathbb{N}$, and $\{(\widehat{z_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(Z)$. Note that, from Lemma 3.3, $\{z_n\}$ is in Z .

Since $g_n(0) = 0$, for each $n \in \mathbb{N}$, it follows that $z_n \in \langle x_0 \rangle$, for each $n \in \mathbb{N}$. ∎

Lemma 3.6. *Let X be an infinite dimensional real unital Abelian Banach algebra satisfying $\Omega(X) \neq \emptyset$ and*

$$\inf\{r(x)_X : x \in X, \|x\| = 1\} > 0,$$

and let x_0 be an element in X with infinite spectrum. Assume that there exists a bounded sequence $\{y_n\}$ in $\langle x_0 \rangle$ which contains no convergent subsequences and such that $\{\tau(y_n) : \tau \in \Omega(Z)\}$ is finite, for each $n \in \mathbb{N}$, where

$$Z = \overline{\left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}.$$

Then there exists an element $z_0 \in \langle x_0 \rangle$ such that $\{\tau(z_0) : \tau \in \Omega(Z)\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.

Proof. It follows form Lemma 3.2 and Lemma 3.3 that Z is an infinite dimensional real unital Abelian Banach algebra with $\Omega(Z) \neq \emptyset$ and homeomorphic to $C_{\mathbb{R}}(\Omega(Z))$. Assume that there exists a bounded sequence $\{y_n\}$ in $\langle x_0 \rangle$ which contains no convergent subsequences and such that $\{\tau(y_n) : \tau \in \Omega(Z)\}$ is finite, for each $n \in \mathbb{N}$. From the proof of Lemma 2.10 (ii) in [6], we have

$$\Omega(Z) = (\cup_{n \in \mathbb{N}} G_n) \cup F,$$

where F is a closed set in $\Omega(Z)$, G_n is closed and open for each $n \in \mathbb{N}$, and $\{F, G_1, G_2, ...\}$ is a partition of $\Omega(Z)$. Define $\tau_Z : Z \to \mathbb{R}$ by

$$\tau_Z(\sum_{i=0}^{k} \alpha_i x_0^i) = \alpha_0,$$

for each $\sum_{i=0}^{k} \alpha_i x_0^i \in Z$. So $\tau_Z \in \Omega(Z)$. There are two cases to be considered. If τ_Z is in F, define $\psi : \Omega(Z) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} 1, & \text{if } \tau \in G_1, \\ \frac{1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 0, & \text{if } \tau \in F. \end{cases}$$

If τ_Z is in G_{n_0}, for some $n_0 \in \mathbb{N}$, without loss of generality, we may assume that $n_0 = 1$, define $\psi : \Omega(Z) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} 0, & \text{if } \tau \in G_1, \\ \frac{n-1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 1, & \text{if } \tau \in F. \end{cases}$$

For each case, the inverse image of each closed set in $\psi(\Omega(Z))$ is closed, so $\psi \in C_{\mathbb{R}}(\Omega(Z))$. Let $\varphi : Z \to C_{\mathbb{R}}(\Omega(Z))$ be the Gelfand representation. Therefore, $\varphi^{-1}(\psi)$ is an element in Z. Write $z_0 = \varphi^{-1}(\psi)$. Then $\{\tau(z_0) : \tau \in \Omega(Z)\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. Moreover, $z_0 \in \langle x_0 \rangle$ since $\tau_Z(z_0) = \psi(\tau_Z) = 0$. ■

Lemma 3.7. *Let X be an infinite dimensional real unital Abelian Banach algebra satisfying $\Omega(X) \neq \emptyset$ and the following conditions:*
(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$,
(ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$.
Let x_0 be an element in X with infinite spectrum, and let $x \in \langle x_0 \rangle_{\mathbb{R}}$ with $(\widehat{x})^{-1}\{1\} \neq \emptyset$, and $0 \leq \tau(x) \leq 1$, for each $\tau \in \Omega(Z)$, where

$$Z = \overline{\left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}}.$$

Define

$$E = \{z \in \langle x_0 \rangle : 0 \leq \tau(z) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(z) = 1 \text{ if } \tau \in A\},$$

where $A = (\widehat{x})^{-1}\{1\}$, and define $T : E \to E$ by

$$z \mapsto xz.$$

Then E is a nonempty bounded closed convex subset of $\langle x_0 \rangle$ and $T : E \to E$ is a nonexpansive mapping.

Proof. Obviously, E is closed and convex. E is nonempty since $x \in E$. From Lemma 3.1, we have

$$\inf\{r_Z(x) : x \in Z, \|x\| = 1\} > 0.$$

If $z \in E$, then

$$\inf\{r_Z(x) : x \in Z, \|x\| = 1\} = \inf\{\|\widehat{x}\|_{\infty,Z} : x \in Z, \|x\| = 1\} \leq \left\|\widehat{\left(\frac{z}{\|z\|}\right)}\right\|_{\infty,Z} \leq \frac{1}{\|z\|}.$$

Then, for each $z \in E$,

$$\|z\| \leq \frac{1}{\inf\{r_Z(x) : x \in Z, \|x\| = 1\}}.$$

Therefore, E is bounded.

Let $\omega \in \Omega(X)$, and let $z, z' \in E$. Note that the restriction $\omega_{|Z}$ of ω on Z is in $\Omega(Z)$. Then

$$\begin{aligned}|\omega(Tz - Tz')| &= |\omega_{|Z}(Tz - Tz')|,\\ &= |\omega_{|Z}(xz - xz')|,\\ &= |\omega_{|Z}x||\omega_{|Z}(z - z')|,\\ &\leq |\omega_{|Z}(z - z')|,\\ &= |\omega(z - z')|.\end{aligned}$$

From (i), we have

$$\|Tz - Tz'\| \leq \|z - z'\|.$$

So T is nonexpansive. ∎

4. Main Results

We now present the main theorem.

Theorem 4.1. *Let X be an infinite dimensional real unital Abelian Banach algebra with $\Omega(X) \neq \emptyset$, and let X satisfy the following conditions:*
(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$,
(ii) $\inf\{r_X(x) : x \in X, \|x\| = 1\} > 0$.
If x_0 is an element in X with infinite spectrum, then the closed subalgebra

$$\langle x_0 \rangle = \overline{\left\{\sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}$$

does not have the fixed point property.

Proof. From Lemma 3.2, $\langle x_0 \rangle$ and Z are infinite dimensional real Abelian Banach algebras, where

$$Z = \overline{\left\{\sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\right\}}.$$

Note that $\Omega(Z) \neq \emptyset$ since $\Omega(X) \neq \emptyset$.

From Lemma 3.5, there exists a sequence $\{z_n\}$ in $\langle x_0 \rangle$ such that, for each $n \in \mathbb{N}$,

$$\{\tau(z_n) : \tau \in \Omega(Z)\} \subset [0, 1],$$

and $\{(\widehat{z_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(Z)$.

Let $A_n = (\widehat{z_n})^{-1}\{1\}$, and define $T_n : E_n \to E_n$ by

$$x \mapsto z_n x,$$

where

$$E_n = \{x \in \langle x_0 \rangle : 0 \leq \tau(x) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(x) = 1 \text{ if } \tau \in A_n\}.$$

From Lemma 3.7, $T_n : E_n \to E_n$ is a nonexpansive mapping on the bounded closed convex set E_n, for each $n \in \mathbb{N}$.

Assume to the contrary that $\langle x_0 \rangle$ has fixed point property. So T_n has a fixed point, for each $n \in \mathbb{N}$. Let y_n be a fixed point of T_n, for each $n \in \mathbb{N}$. We have $y_n = z_n y_n$, hence $\widehat{y_n} = \widehat{z_n}\widehat{y_n}$, and then

$$\widehat{y_n}(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is not in } A_n, \\ 1, & \text{if } \tau \text{ is in } A_n, \end{cases}$$

for each $n \in \mathbb{N}$. It follows that $\|\widehat{y_m} - \widehat{y_n}\|_{\infty,Z} = 1$, if $m \neq n$, since $A_1, A_2, A_3, ...$ are pairwise disjoint. Then $\{\widehat{y_n}\}$ has no convergent subsequences. From Lemma 3.3, Z and $C_{\mathbb{R}}(\Omega(Z))$ are homeomorphic, so $\{y_n\}$ has no convergent subsequences. Note that $\{y_n\}$ is in $\langle x_0 \rangle$. It follows from Lemma 3.6 that there exists an element z_0 in $\langle x_0 \rangle_{\mathbb{R}}$ such that $\{\tau(z_0) : \tau \in \Omega(Z)\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.

Write $A_0 = (\widehat{z_0})^{-1}\{1\}$, define $T_0 : E_0 \to E_0$ by

$$x \mapsto z_0 x,$$

where

$$E_0 = \{x \in \langle x_0 \rangle : 0 \leq \tau(x) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(x) = 1 \text{ if } \tau \in A_0\}.$$

It follows from Lemma 3.7 that T_0 is a nonexpansive mapping on a nonempty bounded closed convex subset E_0 in $\langle x_0 \rangle_{\mathbb{R}}$. So T_0 has a fixed point in E_0, say y_0. There are two cases to be considered.

Case(1) $\{\tau(z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$:

We have $\widehat{y_0} = \widehat{z_0}\widehat{y_0}$ since y_0 is a fixed point of T_0. Then

$$\widehat{y_0}(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is not in } A_0, \\ 1, & \text{if } \tau \text{ is in } A_0. \end{cases}$$

Therefore,

$$(\widehat{y_0})^{-1}\{1\} = (\widehat{z_0})^{-1}\{1\} = A_0$$

and

$$\Omega(Z)\backslash A_0 = (\widehat{y_0})^{-1}\{0\} = \bigcup_{n=0}^{\infty} \left((\widehat{z_0})^{-1}\{\frac{n}{n+1}\} \right).$$

It follows from

$$\{\tau(z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$$

that $\left\{ (\widehat{z_0})^{-1}\{\frac{n}{n+1}\} : n \in \mathbb{N} \right\} \bigcup \{(\widehat{z_0})^{-1}\{0\}\}$ is a nonempty pairwise disjoint open covering of the compact set $\Omega(Z)\backslash A_0$, which is a contradiction.

Case(2) $\{\tau(z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$:

Define $T : E \to E$ by

$$1 + x \mapsto (1 - z_0)(1 + x),$$

for each $1 + x \in E$, where

$E = \{1 + x \in 1 + \langle x_0\rangle : 0 \leq \tau(1 + x) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(1 + x) = 1 \text{ if } \tau \in A\}$,

where $A = (\widehat{1 - z_0})^{-1}\{1\}$. Then

$$\{\tau(1 - z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}.$$

Define $S : Z \to Z$ by

$$\lambda + x \mapsto (-\lambda + 1) + x.$$

Note that S^2 is the identity mapping. It follows that $STS : S(E) \to S(E)$ is a nonexpansive mapping on a nonempty bounded closed convex subset $S(E)$ of $\langle x_0\rangle$.

Since $\langle x_0\rangle$ has the fixed point property, so STS has a fixed point. Then T has a fixed point, say $1 + y_0$. Therefore,

$$(\widehat{1 + y_0})(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is not in } A, \\ 1, & \text{if } \tau \text{ is in } A. \end{cases}$$

Then

$$(\widehat{1 + y_0})^{-1}\{1\} = (\widehat{1 - z_0})^{-1}\{1\} = A$$

and

$$\Omega(Z)\backslash A = (\widehat{1 + y_0})^{-1}\{0\} = \bigcup_{n=0}^{\infty} \left((\widehat{1 - z_0})^{-1}\{\frac{n}{n+1}\} \right).$$

From

$$\{\tau(1 - z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\},$$

we have $\left\{(\widehat{1 - z_0})^{-1}\{\frac{n}{n+1}\} : n \in \mathbb{N}\right\} \bigcup \left\{(\widehat{1 - z_0})^{-1}\{0\}\right\}$ is a nonempty pairwise disjoint open covering of the compact set $\Omega(Z)\backslash A$, which is a contradiction.

So we conclude that $\langle x_0\rangle$ does not have the fixed point property. ∎

Acknowledgements

This research was supported by Chiang Mai University.

References

[1] D. Alspach, A fixed point free nonexpansive map., Proc. Amer. Math. Soc. 82 (1981) 423–424.

[2] J. Elton, P.K. Lin, E. Odell, S. Szarek, Remarks on the fixed point problem for nonexpansive maps, Contemp. Math. 18 (1983) 87–120.

[3] A.T. Lau, P.F. Mah, Ali Ülger, Fixed point property and normal structure for Banach spaces associated to locally compact groups, Proc. Amer. Math. Soc. 125 (1997) 2021–2027.

[4] T. Domínguez Benavides, M.A. Japón Pineda, Fixed points of nonexpansive mappings in spaces of continuous functions, Proc. Amer. Math. Soc. 133 (2005) 3037–3046.

[5] A.T. Lau, M. Leinert, Fixed point property and the Fourier algebra of a locally compact group, Trans. Amer. Math. Soc. 360 (2008) (12) 6389–6402.

[6] W. Fupinwong, S. Dhompongsa, The fixed point property of unital abelian Banach algebra, Fixed Point Theory Appl. (2010) Article ID 34959.

[7] D. Alimohammadi, S. Moradi, On the fixed point property of unital uniformly closed subalgebras of $C(X)$, Fixed Point Theory Appl. (2010) Article ID 268450.

[8] S. Dhompongsa, W. Fupinwong, W. Lawton, Fixed point properties of C^*-algebra, J. Math. Anal. Appl. 374 (2011) 22–28.

[9] W. Fupinwong, Nonexpansive mappings on Abelian Banach algebras and their fixed points, Fixed point Theory Appl. (2012).

[10] D. Alimohammadi, Nonexpansive mappings on complex C^*-algebras and thier fixed points, Int. J. Nonlinear Anal. Appl. 7 (2016) 21–29.

[11] J. Daengsaen, W. Fupinwong, Fixed points of nonexpansive mappings on real Abelian Banach algebras, Proceedings of the 22^{nd} Annual Meeting in Mathematics (AMM 2017), Chiang Mai University, Thailand, 2–4 June 2017.

[12] P. Thongin, W. Fupinwong, The fixed point property of a Banach algebra generated by an element with infinite spectrum, J. Funct. Spaces (2018) Article ID 9045790.