Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1597–1609

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

The Convergence Theorem for a Square α-Nonexpansive Mapping in a Hyperbolic Space

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Abstract In this paper, we prove Δ-convergence theorems of the generalized Picard normal S_5-iterative process to approximate a fixed point for square α-nonexpansive mappings. Moreover, we obtain some properties of such mappings on a nonempty subset of a hyperbolic space.

MSC: 47H09; 47H10

Keywords: fixed point set; square α-nonexpansive mapping; generalized Picard normal S_5-iterative; hyperbolic spaces

Submission date: 26.05.2020 / Acceptance date: 01.09.2020

1. Introduction

Let X be a metric space and let M be a nonempty closed convex subset of X. A mapping $T : M \to M$ is said to be nonexpansive, if $d(Tx, Ty) \leq d(x, y)$, for each $x, y \in M$. In 2011, Aoyama and Kohsaka [1] introduced the class of α-nonexpansive mappings in Banach spaces as follow: Let X be a Banach space and M be a nonempty closed and convex subset of X. A mapping $T : M \to M$ is said to be α-nonexpansive if for all $x, y \in M$ and $\alpha < 1$, $\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|x - Ty\|^2 + (1 - 2\alpha) \|x - y\|^2$. This class contains the class of nonexpansive mappings and is related to the class of firmly nonexpansive mappings in Banach spaces. Then $F(T)$ is nonempty if and only if there exists $x \in M$ such that $\{T^n x\}$ is bounded, where X is a uniformly convex Banach

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Published by The Mathematical Association of Thailand.

space, and M is a nonempty, closed and convex subset of X, and $T : M \to M$ is an α-nonexpansive mapping for some real number α such that $\alpha < 1$.

In 2013, Naraghirad *et al.* [2] considered appropriate Ishihawa iterate algorithms ensure weak and strong convergence to a fixed point of such a mapping. Their theorems are also extended to CAT(0) spaces as follow : Let $\{x_n\}$ be a sequence with $\{x_1\}$ in M defined by

$$\begin{cases} y_n = \beta_n T x_n \oplus (1-\beta_n) x_n, \\ x_{n+1} = \gamma_n T y_n \oplus (1-\gamma_n) x_n. \end{cases}$$

In 2016, Song *et al.* [3] introduced the concept of monotone α-nonexpansive mappings in an ordered Banach space E with the partial order $\leq$, which contains monotone α-nonexpansive mappings as special case. With the help of the Mann iteration. In 2017, Shukla *et al.* [4] introduced some existence and convergence results for monotone α-nonexpansive mappings in partially ordered hyperbolic metric spaces as follow : Let $\{u_n\}$ be defined by

$$\begin{cases} u_1 \in K, \\ v_n = \gamma_n T(u_n) \oplus (1-\gamma_n) u_n, \\ u_{n+1} = \beta_n T(v_n) \oplus (1-\beta_n) T(u_n). \end{cases}$$

In 2018, Mebawondu and Izuchukwu [5] introduced some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces as follow : Suppose that the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in C, \\ z_n = W(x_n, Tx_n, \beta_n), \\ y_n = W(z_n, Tz_n, \gamma_n), \\ x_{n+1} = W(Ty, 0, 0). \end{cases}$$

Recently, there are some works that relate to hyperbolic spaces such as CAT(0) spaces that appeared (see [6–17]).

In this paper, we prove convergence and Δ-convergence theorems of the generalized Picard normal S_5-iterative process to approximate a fixed point for α-nonexpansive mappings. Moreover, we prove some properties of such mappings on a nonempty subset of a hyperbolic space.

2. Preliminaries

Throughout this paper, we work in the setting of hyperbolic spaces which were introduced by Kohlenbach [18].

Definition 2.1. A hyperbolic space is a metric space (X, d) with a mapping $W : X^2 \times [0,1] \to X$ satisfying the following conditions.
(i) $d(u, W(x, y, \alpha)) \leq (1-\alpha) d(u, x) + \alpha d(u, y)$;
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
(iii) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha) d(x, y) + \alpha d(z, w)$.
for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$.

Some definitions on hyperbolic space are considered as follow:

Definition 2.2. [19] Let X be hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$. A nonempty subset $M \subseteq X$ is said to be convex, if $W(x,y,\alpha) \in M$ for all $x, y \in M$ and $\alpha \in [0,1]$. A hyperbolic space is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that for all $u, x, y \in X$

$$d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r,$$

provided $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq \epsilon r$. A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is known as a modulus of uniform convexity of X. η is said to be monotone, if it decreases with r (for a fixed ϵ), i.e., $\forall \epsilon > 0$, $\forall r_1 \geq r_2 > 0$ $[\eta(r_2,\epsilon) \leq \eta(r_1,\epsilon)]$. We denote the unit sphere and the closed unit ball centered at the origin of M by S_M and B_M, respectively. We also denote the closed ball with radius $r > 0$ centered at the origin of M by rB_M.

Definition 2.3. [20] Let $\{x_n\}$ be a bounded sequence in a hyperbolic space (X,d). For $x \in X$, we define a continuous functional $r(\cdot,x_n) : X \to [0,\infty)$ by

$$r(x,x_n) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,x_n) : x \in X\}.$$

The asymptotic center $A_M(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $M \subseteq X$ is the set

$$A_M(\{x_n\}) = \{x \in X : r(x,x_n) \leq r(y,x_n), \quad \forall y \in M\}.$$

This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot,x_n)$ in M. If the asymptotic center is taken with respect to X, then it is simply denoted by $A_M(\{x_n\})$. It is known that uniformly convex hyperbolic spaces enjoy the property that ounded sequences have unique asymptotic centers with respect to closed convex subsets.

Definition 2.4. Recall that a sequence $\{x_n\}$ in X is said to be Δ-convergent which converges to a point $x \in X$ if x is the unique asymptotic centers of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$. Moreover, if $x_n \to x$, then $\Delta - \lim_{n\to\infty} x_n = x$ (see [18],[21]).

Lemma 2.5. [20] *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unqiue asymtotic center with respect to any nonempty closed convex subset M of X.*

Lemma 2.6. [20] *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n,p) \leq c, \limsup_{n\to\infty} d(y_n,p) \leq c$ *and* $\limsup_{n\to\infty} d(W(x_n,y_n,\alpha_n)$
$,p) = c$*, for some $c \geq 0$. Then* $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 2.7. ([21–23]) *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in M has a unique asymptotic center in M.*

Lemma 2.8. [5] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A_M(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A_M(\{x_{n_k}\}) = \{x_1\}$. and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

Definition 2.9. Let M be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to M if for all $x \in M$ and $n \geq 1$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Next, we defined Picard Normal S_5-iteration process (PNS_5) in hyperbolic spaces as follow : Let M be a nonempty closed convex subset of a hyperbolic space X and $T : M \to M$ be a mapping which asymptotically Suzuki-generalized nonexpansive, for any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Tu_n, 0, 0) \\ u_n = W(v_n, Tv_n, \beta_n) \\ v_n = W(y_n, Ty_n, \gamma_n) \\ y_n = W(z_n, Tz_n, \delta_n) \\ z_n = W(x_n, Tx_n, \zeta_n), \ n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\zeta_n\}$ in $(0, 1)$.

3. Main Results

In this section, we will prove some properties for class of α-nonexpansive mappings in hyperbolic spaces.

Definition 3.1. Let (X, d) be a metric space and M be nonempty subset of X. Then $T : M \to M$ is said to be a square α-nonexpansive mapping (or α-nonexpansive mapping), if $\alpha < 1$ such that

$$d^2(Tx, Ty) \leq \alpha d^2(Tx, y) + \alpha d^2(x, Ty) + (1 - 2\alpha)d^2(x, y),$$

for all $x, y \in M$.

Now, we give example for a square α-nonexpansive mapping as follows :

Example 3.2. Let M be a nonempty closed and convex subset of a complete hyperbolic space X, and let $S, T : M \to M$ be firmly nonexpansive mappings such that $S(M)$ and $T(M)$ are contained by rB_M for some positive real number r. Let α and δ be real numbers such that $0 < \alpha \leq 1$ and $\delta \geq (1 + 2/\sqrt{\alpha})r$. Then the mapping $U : M \to M$ is defined by

$$Ux = \begin{cases} Sx & (x \in \delta B_M); \\ Tx & (\text{otherwise}), \end{cases} \tag{3.1}$$

then U is a square α-nonexpansive (See [1]).

From lemma of Naraghirad [2], we obtain the lemma as follow :

Lemma 3.3. *Let M be a nonempty subset of a hyperbolic space X. Let $T : M \to M$ be a square α-nonexpansive mapping for some $\alpha < 1$. Let $x, y \in M$, then the following assertions hold* *(i) If* $0 \le \alpha < 1$*, then*

$$d^2(x,Ty) \le \frac{1+\alpha}{1-\alpha} d^2(x,Tx) + \frac{2}{1-\alpha}(\alpha d(x,y) + d(Tx,Ty))d(x,Tx) + d^2(x,y)$$

(ii) If $\alpha < 0$*, then*

$$d^2(x,Ty) \le d^2(x,Tx) + \frac{2}{1-\alpha}[(-\alpha)d(x,y) + d(Tx,Ty)]d(x,Tx) + d^2(x,y)$$

Proof. let $x, y \in M$.
(i) Suppose that $0 \le \alpha < 1$. Consider

$$\begin{aligned}
d^2(x,Ty) &\le (d(x,Tx) + d(Tx,Ty))^2 \\
&= d^2(x,Tx) + d^2(Tx,Ty) + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-2\alpha)d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha(d(Tx,x) + d(x,y))^2 + \alpha d^2(x,Ty) + (1-2\alpha)d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,x) + \alpha d^2(x,y) + 2\alpha d(Tx,x)d(x,y) + \alpha d^2(x,Ty) \\
&\quad + (1-2\alpha)d^2(x,y) + 2d(x,Tx)d(Tx,Ty) \\
&= (1+\alpha)d^2(x,Tx) + 2\alpha d(Tx,x)d(x,y) + \alpha d^2(x,Ty) \\
&\quad + (1-\alpha)d^2(x,y) + 2d(x,Tx)d(Tx,Ty).
\end{aligned}$$

We obtain that

$$d^2(x,Ty) \le \frac{(1+\alpha)}{1-\alpha} d^2(x,Tx) + \frac{2}{1-\alpha}(\alpha d(x,y) + d(Tx,Ty))d(Tx,x) + d^2(x,y).$$

(ii) Suppose that $\alpha < 0$. Consider

$$\begin{aligned}
d^2(x,Ty) &\le (d(x,Tx) + d(Tx,Ty))^2 \\
&= d^2(x,Tx) + d^2(Tx,Ty) + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-2\alpha)d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&= d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) - \alpha d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) \\
&\quad - \alpha[d^2(x,Tx) + d^2(Tx,y) + 2d(x,Tx)d(Tx,y)] + 2d(x,Tx)d(Tx,Ty) \\
&= (1-\alpha)d^2(x,Tx) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) \\
&\quad - 2\alpha d(x,Tx)d(Tx,y) + 2d(x,Tx)d(Tx,Ty) \\
&= (1-\alpha)d^2(x,Tx) + \alpha d^2(x,Tx) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) \\
&\quad + 2[(\alpha)d(Tx,y) + d(Tx,Ty)]d(x,Tx),
\end{aligned}$$

this implies that
$d^2(x,Ty) \le d^2(x,Tx) + \frac{2}{1-\alpha}[(-\alpha)d(Tx,y) + d(Tx,Ty)]d(x,Tx) + d^2(x,y)$. ■

Lemma 3.4. *Let M be a nonempty closed and convex subset of a hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping for some real number $\alpha < 1$. In case $0 \le \alpha < 1$, we have $F(T) \neq \emptyset$ if and only if $\{T^n x\}_{n=1}^{\infty}$ is bounded for some $x \in M$. If M is compact, then $F(T) \neq \emptyset$.*

Proof. Assume that $0 \le \alpha < 1$. The necessity is obvious. We verify the sufficiency. Suppose that $\{T^n x\}_{n=1}^{\infty}$ is bounded for some x in M. Set $x_n := T^n x$ for $n = 1, 2, \ldots$. By the boundedness of $\{x_n\}_{n=1}^{\infty}$, there exists z in X such that $A_M(\{x_n\}) = \{z\}$. It follows from Lemma 2.6 that $z \in M$. Furthermore, we have

$$d^2(x_n, Tz) \le \alpha d^2(x_n, z) + \alpha d^2(x_{n-1}, Tz) + (1-2\alpha) d^2(x_n, z), \qquad \forall n = 1, 2, \ldots.$$

This implies that

$$\limsup_{n\to\infty} d^2(x_n, Tz) \le \alpha \limsup_{n\to\infty} d^2(x_n, z) + \alpha \limsup_{n\to\infty} d^2(x_{n-1}, Tz) + (1-2\alpha) \limsup_{n\to\infty} d^2(x_n, z).$$

We obtain

$$\limsup_{n\to\infty} d^2(x_n, Tz) \le \limsup_{n\to\infty} d^2(x_n, z).$$

Consequently, $Tz \in A_M(\{x_n\}) = \{z\}$, we obtain that $F(T) \neq \emptyset$.
Next, we assume that $\alpha < 0$ and M is compact. In particular, T is continuous and the sequence of $x_n := T^n x$ for any $x \in M$ is bounded. We adapt in [Lemmas 3.1 and 3.2][24], we have μ is a Banach limit, i.e., μ is a bounded unital positive linear functional of l_∞ such that $\mu \circ s = \mu$, where s is the left shift operator on l_∞. We write μ_n, a_n for the value of $\mu(a)$ with $a = (a_n)$ in l_∞ as usual. In particular, $\mu_n\, a_{n+1} = \mu(s(a)) = \mu(a) = \mu_n a_n$. We get

$$\mu_n d^2(x_n, Ty) \le \mu_n d^2(x_n, y), \qquad \forall y \in M, \tag{3.2}$$

and

$$g(y) := \mu_n d^2(x_n, y)$$

defines a continuous function from M into $\mathbb{R}$.

By compactness, there exists y in M such that $g(y) = \inf g(M)$. Suppose that there is another z in M such that $g(z) = g(y)$. Let m be the midpoint by definition 2.1, we see that g is convex. Thus, $g(m) = g(y)$ too. Observing the comparison triangles in $\mathbb{E}^2$, we have

$$d^2(x_n, y) + d^2(x_n, z) \ge 2d^2(x_n, m) + \frac{1}{2} d^2(y, z), \qquad \forall n = 1, 2, \ldots.$$

Consequently,

$$\mu_n d^2(x_n, y) + \mu_n d^2(x_n, z) \ge 2\mu_n d^2(x_n, m) + \frac{1}{2} \mu_n d^2(y, z).$$

So,

$$g(y) + g(z) \ge 2g(m) + \frac{1}{2} d^2(y, z).$$

Since $g(y) = g(z) = g(m)$, we have $y = z$. Finally, it follows from (3.2) that $g(Ty) \le g(y) = \inf g(M)$. By uniqueness, we have $Ty = y \in F(T)$. ∎

Lemma 3.5. *Let M be a nonempty closed and convex subset of a hyperbolic space X. Let $T : M \to M$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Proof. Let $\{x_n\} \subset F(T)$ such that $\{x_n\}$ converges to y for some $y \in M$. We will show that $y \in F(T)$. We consider $d^2(x_n, Ty) \leq \alpha d^2(x_n, y) + \alpha d^2(Ty, x_n) + (1-2\alpha)d^2(x_n, y)$. So, we get $(1-\alpha)d^2(x_n, Ty) \leq (1-\alpha)d^2(x_n, y)$ implies that, $d(x_n, Ty) \leq d(x_n, y)$. Since $\lim_{n\to\infty} d(x_n, y) = 0$, then by Sandwish theorem, we obtain that $\lim_{n\to\infty} d(x_n, Ty) = 0$. By uniqueness of limit, we get that $Ty = y$. Hence $y \in F(T)$, and then $F(T)$ is closed.
Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$. By definition of T, we obtain that

$$d^2(x, Tz) \leq \alpha d^2(Tx, z) + \alpha d^2(Tz, x) + (1-2\alpha)d^2(x, z).$$

So, we get $(1-\alpha)d^2(x, Tz) \leq (1-\alpha)d^2(x, z)$,

$$d^2(x, Tz) \leq d^2(x, z) \Longrightarrow d(x, Tz) \leq d(x, z). \tag{3.3}$$

In the other hand, we get

$$d^2(y, Tz) \leq d^2(y, z) \Longrightarrow d(y, Tz) \leq d(y, z) \tag{3.4}$$

Let $z = W(x, y, \eta)$ where $\eta \in [0, 1]$. From (3.3) and (3.4), we obtain

$$\begin{aligned} d(x, y) &\leq d(x, Tz) + d(Tz, y) \\ &\leq d(x, z) + d(z, y) \qquad (3.5) \\ &= d^2(x, W(x, y, \eta)) + d(W(x, y, \eta), y) \\ &\leq (1-\eta)d(x, x) + \eta d(x, y) + (1-\eta)d(x, y) + \eta d(y, y) \\ &= d(x, y). \end{aligned}$$

So $d(x, Tz) = d(x, z)$ and $d(y, Tz) = d(y, z)$, because if $d(x, Tz) < d(x, z)$ or $d(y, Tz) < d(y, z)$, which is a contradiction to $d(x, y) < d(x, y)$. Hence $Tz = z$ Therefor $W(x, y, \eta) \in F(T)$, and then $F(T)$ is convex. ∎

Theorem 3.6. *Let M be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping and $\{x_n\}$ be a bounded sequence in M such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$. Then $x \in F(T)$.*

Proof. Let $\{x_n\}$ be a bounded seqence in X, By Lemma 2.5 we get $\{x_n\}$ has a unique asymptotic center in M. Since, $\Delta\text{-}\lim_{n\to\infty} x_n = x$, we have that $A(\{x_n\}) = \{x\}$. Using Lemma 3.3 and the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

(i) $d^2(x_n, Tx) \leq \frac{1+\alpha}{1-\alpha} d^2(x_n, Tx_n) + \frac{2}{1-\alpha}(\alpha d(x_n, x) + d(Tx_n, Tx))d(x_n, Tx_n) + d^2(x_n, x)$, where $0 \leq \alpha < 1$,

(ii) $d^2(x_n, Tx) \leq d^2(x_n, Tx_n) + \frac{2}{1-\alpha}[(-\alpha)d(x_n, x) + d(Tx_n, Tx)]d(x_n, Tx_n) + d^2(x_n, x)$, where $\alpha < 0$.

Taking limit superior as $n \to \infty$ with both sides, we obtain that
Case (i) : $0 \leq \alpha < 1$,

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n, Tx) &\leq \frac{1+\alpha}{1-\alpha}\limsup_{n\to\infty} d^2(x_n, Tx_n)\\ &\quad + \frac{2}{1-\alpha}\limsup_{n\to\infty}(\alpha d(x,x) + d(Tx_n, Tx))d(x_n, Tx_n)\\ &\quad + \limsup_{n\to\infty} d^2(x_n, x)\\ &= \limsup_{n\to\infty} d^2(x_n, x).\end{aligned}$$

Case (ii) : $\alpha < 0$,

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n, Tx) &\leq \limsup_{n\to\infty} d^2(x_n, Tx_n)\\ &\quad + \frac{2}{1-\alpha}\limsup_{n\to\infty}[(-\alpha)d(x_n,x) + d(Tx_n, Tx)]d(x_n, Tx_n)\\ &\quad + \limsup_{n\to\infty} d^2(x_n, x)\\ &= \limsup_{n\to\infty} d^2(x_n, x).\end{aligned}$$

So, we get $\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x)$. By the uniqueness of asymptotic center, we obtain that $Tx = x$. Therefore $x \in F(T)$. ∎

Now we recall the quasi nonexpansive mappings as follow: A mapping $T : M \to M$ is said to be quasi-nonexpansive, if

$$d(Tx, p) \leq d(x, p),$$

for each $x \in M$ and $p \in F(T)$.

Lemma 3.7. *Let M be a nonempty subset of a hyperbolic space X. Let $T : M \to M$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

Proof. Let $T : M \to M$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, we let $p \in F(T)$ and $x \in M$. We consider

$$\begin{aligned}d^2(Tx, Tp) &\leq \alpha d^2(Tx, p) + \alpha d^2(x, p) + (1-2\alpha)d^2(x, p)\\ &= \alpha d^2(Tx, p) + (1-\alpha)d^2(x, p),\end{aligned}$$

we obtain that

$$d^2(Tx, Tp) \leq d^2(x, p),$$

implies that

$$d(Tx, p) \leq d(x, p).$$

Hence T is quasi-nonexpansive. ∎

New, we recall Picard normal S_5-iteration process (PNS_5). Let M be a nonempty closed convex subset of a hyperbolic space X and $T : M \to M$ be a mapping which a square α-nonexpansive , for any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Tu_n, 0, 0) \\ u_n = W(v_n, Tv_n, \beta_n) \\ v_n = W(y_n, Ty_n, \gamma_n) \\ y_n = W(z_n, Tz_n, \delta_n) \\ z_n = W(x_n, Tx_n, \zeta_n), \ n \in \mathbb{N}, \end{cases} \tag{3.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$.

Theorem 3.8. *Let M be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (2.1) then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Step1: We prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. Let $p \in F(T)$. Since T is an α-nonexpansive mapping and Lemma 3.7, we get

$$\begin{aligned} d(u_n, p) &= d(W(v_n, Tv_n\beta_n), p) \\ &\le (1-\beta_n)d(v_n, p) + \beta_n d(Tv_n, p) \\ &= (1-\beta_n)d(v_n, p) + \beta_n d(Tv_n, p) \\ &\le (1-\beta_n)d(v_n, p) + \beta_n d(v_n, p) \\ &= d(v_n, p), \end{aligned} \tag{3.7}$$

$$\begin{aligned} d(v_n, p) &= d(W(y_n, Ty_n, \gamma_n), p) \\ &\le (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &= (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &\le (1-\gamma_n)d(y_n, p) + \gamma_n d(y_n, p) \\ &= d(y_n, p), \end{aligned} \tag{3.8}$$

$$\begin{aligned} d(y_n, p) &= d(W(z_n, Tz_n, \delta_n), p) \\ &\le (1-\delta_n)d(z_n, p) + \delta_n d(Tz_n, p) \\ &= (1-\delta_n)d(z_n, p) + \delta_n d(Tz_n, Tp) \\ &\le (1-\delta_n)d(z_n, p) + \delta_n d(z_n, p) \\ &= d(z_n, p), \end{aligned} \tag{3.9}$$

$$\begin{aligned} d(z_n, p) &= d(W(x_n, Tx_n, \zeta_n), p) \\ &\le (1-\zeta_n)d(x_n, p) + \zeta_n d(Tx_n, p) \\ &= (1-\zeta_n)d(x_n, p) + \zeta_n d(Tx_n, Tp) \\ &\le (1-\zeta_n)d(x_n, p) + \zeta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.10}$$

By (3.7),(3.8),(3.9), and (3.10), we have

$$\begin{aligned} d(x_{n+1},p) &= d(W(Tu_n,0,0),p) \\ &= d(Tu_n,p) \\ &\le d(u_n,p) \\ &\le d(v_n,p) \\ &\le d(y_n,p) \\ &\le d(z_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{3.11}$$

We obtain $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F$.
Step 2: We will show that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Suppose that $\lim_{n\to\infty} d(x_n,p) = c \ge 0$. If $c = 0$, then

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

Next, we consider $c > 0$. By (3.11), we obtain that

$$d(x_{n+1},p) \le d(u_n,p) \le d(v_n,p) \le d(y_n,p) \le d(z_n,p) \le d(x_n,p). \tag{3.12}$$

Taking limsup in (3.12), we get

$$\limsup_{n\to\infty} d(u_n,p) \le \limsup_{n\to\infty} d(v_n,p) \le \limsup_{n\to\infty} d(y_n,p) \le \limsup_{n\to\infty} d(z_n,p) \le c \tag{3.13}$$

Since $d(Tx_n,p) \le d(x_n,p)$, we have

$$\lim_{n\to\infty} \sup d(Tx_n,p) \le c. \tag{3.14}$$

Since $d(x_{n+1},p) \le (z_n,p)$, as $n \to \infty$, we get

$$c = \liminf_{n\to\infty} d(x_{n+1},p) \le \liminf_{n\to\infty} d(z_n,p) \le \limsup_{n\to\infty} d(z_n,p) \le c. \tag{3.15}$$

From (3.14) and (3.15), we have

$$\lim_{n\to\infty} d(z_n,p) = c,$$

it implies that

$$\lim_{n\to\infty} d(W(x_n,Tx_n,\gamma_n),p) = c.$$

By Lemma 2.6, we obtain that

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{3.16}$$

Step 3: Let $\mathcal{W}_\Delta(x_n) := \bigcup A_M(\{\mu_n\})$, where the union is taken over all subsequence $\{\mu_n\}$ of $\{x_n\}$. Next, we prove that $\mathcal{W}_\Delta(x_n) \subset F(T)$ and contains only one point. Let $u \in \mathcal{W}_\Delta(x_n)$, there exists a subsequence $\{\mu_n\}$ of $\{x_n\}$ such that $A_M(\{\mu_n\}) = \{u\}$. By Lemma 2.5 we let subsequence $\{\nu_n\}$ of $\{\mu_n\}$ such that $\Delta - \lim_{n\to\infty} \nu_n = v$, for some $v \in M$. Since, $\lim_{n\to\infty} d(\nu_n,T\nu_n) = 0$, we have $v \in F(T)$. Hence, $\{d(u_n,v)\}$ converges and by lemma 2.8, we have that $v = u \in F(T)$. Hence, $\mathcal{W}_\Delta(x_n) \subset F(T)$. Let $A_M(\{x_n\}) = x$ and $\{\mu_n\}$ be arbitrary subsequence of $\{x_n\}$ such that $A_M(\{\mu_n\}) = \{u\}$. We have that $\{d(x_n,u)\}$ converges, since $u \in F(T)$. Thus, by Lemma 2.8, we have that $u = x \in F(T)$. and $\mathcal{W}_\Delta(x_n) = \{x\}$. Therefore, $\{x_n\}$ Δ-converges to a common fixed point of T. ■

Theorem 3.9. *Let M be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (2.1). Then $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Proof. First, we show that the fixed point set $F(T)$ is closed, let $\{x_n\}$ be a sequence in $F(T)$ which converges to some point $z \in M$.

$$d(x_n, Tz) = d(Tx_n, Tz) \leq d(x_n, z).$$

By taking the limit of both sides we obtain

$$\lim_{n\to\infty} d(x_n, Tz) \leq \lim_{n\to\infty} d(x_n, z) = 0.$$

In view of the uniqueness of the limit, we have $z = Tz$, so that $F(T)$ is closed. Suppose that

$$\lim_{n\to\infty} \inf d(x_n, F(T)) = 0.$$

From (3.11),

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)),$$

then $\lim_{n\to\infty} d(x_n, F(T))$ exists. Hence we know $\lim_{n\to\infty} d(x_n, F(T)) = 0$.
We have $\lim_{n\to\infty} d(x_n, z) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, z)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely, consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{3^k}$, for all $k \geq 1$ where $\{p_k\}$ is in $F(T)$. By (3.11), we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{3^k},$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{+1}}, p_k) \\ &< \frac{1}{3^{k+1}} + \frac{1}{3^k} \\ &< \frac{1}{3^{k-1}}. \end{aligned}$$

This show that $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{p_k\}$ is convergent sequence. Let $\lim_{n\to\infty} p_k = p$. In fact, since $d(x_{n_k}, p) \leq d(x_{n_k}, p_k) + d(p_k, p) \to 0$ as $k \to \infty$, we have $\lim_{k\to\infty} d(x_{n_k}, p) = 0$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, the sequence $\{x_n\}$ converges to p.
∎

Theorem 3.10. *Let M be a nonempty compact convex subset of a complete hyperbolic space X with monotone modulus of uniformly convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \leq \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$. In case $\alpha \leq 0$, we assume that $\limsup_{k\to\infty} \beta_{n_k} < 1$. Let $\{x_n\}$ be a sequence with x_1 in M defined by (2.1). Then $\{x_n\}$ converges in metric to a fixed point of T.*

Proof. We use Lemma 3.3 and Lemma 3.4, and replacing $\|\cdot, \cdot\|$ with $d(\cdot, \cdot)$ in the proof of [Theroem 3.4][2], we conclude the desired result. ∎

Competing interests
The authors declare that they have no competing interests.

Authorś contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The author would like to thank the anonymous referee who provided useful and detailed comments on a previous/earlier version of the manuscript. The author would like to thank the Science and Applied Science center, Kamphaengphet Rajabhat University.

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