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Meromorphic Functions that share two finite values with their derivative (II)

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Abstract : The purpose of this paper is to study a meromorphic functions which share two finite nonzero values with their derivatives, and the result is proved: Let f be a nonconstant meromorphic function, a, b be a nonzero distinct finite complex constant. If f and f' share a CM, and share b IM and $N_{(2}(r, \frac{1}{f'-b}) = S(r, f)$ then $f = f'$.

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1 Introduction

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We use the standard notations in Nevanlinna theory of meromorphic functions such as the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$, and the proximity function $m(r, f)$ (see, e.g., [1]).
The notation $S(r, f)$ is used to define any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure.

Two meromorphic functions f and g share the value $a \in \hat{\mathbb{C}}$ if $f^{-1}\{a\} = g^{-1}\{a\}$. And one says that f and g share the value a **CM** if the value a is shared by f and by g and moreover if $f(z_0) = a$ with multiplicity p implies that $g(z_0) = a$ with multiplicity p, here $p = p(z_0)$.
The usual sharing is also denoted by sharing **IM** (IM =ignoring multiplicity).
In sharing **CM** the abbreviation CM stands for counting multiplicities.
Obviously **sharing CM** $\Longrightarrow$ **sharing IM**.
But the converse must not be true !
In 1976, Rubel and Yang [2] proved that if f is an entire function and shares two finite values CM with f', then $f = f'$.

E. Mues and N. Steinmetz [3] have shown that "CM" can be replaced by "IM" . (another proof of this result for nonzero shared values is in [4]).
On the other hand, the meromorhic function [3]

$$f(z) = (\frac{1}{2} - \frac{\sqrt{5}}{2} itan(\frac{\sqrt{5}}{4} iz))^2 \quad (1.1)$$

shares 0 by DM and 1 by DM with f' ; while the meromorhic function [4]

$$f(z) = \frac{2A}{1 - Be^{-2z}}, A \neq 0, B \neq 0 \quad (1.2)$$

shares 0 (picard value) and A by DM with f' ; and $f \neq f'$ in both (1.1) and (1.2) .
Mues and Steinmetz [6], and Gundersen [5] improved this result and proved the following theorem.

Theorem 1.1. *Let f be a nonconstant meromorphic function, a and b be two distinct finite values. If f and f' share the values a and b CM, then $f = f'$.*

Qing Cai ZHANG [7], results that (1.2) is unique in some sense.

Theorem 1.2. *Let f be a nonconstant meromorphic function, b be a nonzero finite complex constant. If f and f' share 0 CM, and share b IM, then $f = f'$ or $f = \frac{2b}{1-ce^{-2z}}$, where c is a nonzero finite complex constant.*

2 Preliminaries

Theorem 2.1. *the first fundamental theorem of value distribution theory*
Let $0 < R_0 \leq \infty$. *and let* $a \in \mathbb{C}$.
Let f *be meromorphic in the disk* $\{z \in \mathbb{C} : |z| < R_0\}$.
Assume that at the point $z = 0$ *the function* $f - a$ *has the expansion* $f(z) - a = c_k(a)z^k + \cdots,\ c_k(a) \neq 0$.
Then we have for $0 < r < R_0, m(r, \frac{1}{f-a}) + N(r, \frac{1}{f-a}) = T(r, f) + \eta(r, a)$ *with the estimate.*

we can write the first fundamental theorem in the form
$m(r, \frac{1}{f-a}) + N(r, \frac{1}{f-a}) = T(r, f) + O(1) \quad \text{if } r \to R_0$.

Theorem 2.2. *If* f *is meromorphic in* $\mathbb{C}$, $f \not\equiv$ *constant, then*
$m(r, \frac{f'}{f}) = S(r, f)$.

Theorem 2.3. *Left main-inequality*
Let f *be meromorphic in* $\mathbb{C}$ *and not constant.*
Let $a_1, a_2, \ldots, a_q$ *be* $q \geq 1$ *pairwise different complex numbers. Then we have*

$$\sum_{\nu=1}^{q} m(r, \frac{1}{f-a_\nu}) \leq m(r, \frac{1}{f'}) + S(r, f).$$

Theorem 2.4. *second fundamental theorem of value distribution theory.*
Let f *be meromorphic in* $\mathbb{C}$ *and not constant.*
Let $a_1, a_2, \ldots, a_q$ *be* $q \geq 1$ *pairwise different complex numbers. Then we have*

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{\nu=1}^{q} \overline{N}(r, \frac{1}{f-a_\nu}) + S(r, f).$$

3 Main Results

Theorem 3.1. *Let f be a nonconstant meromorphic function let $a, b \neq 0$ be distinct finite complex constants. If f, f' share the value a CM and the value b IM and $N_{(2}(r, \frac{1}{f'-b}) = S(r,f)$, then $f \equiv f'$. where $N_{(2}(r, \frac{1}{f'-b})$ is the counting function which only includes multiple zero of $f'(z) - b$*

Proof.

We prove by contradiction. Suppose that $f \not\equiv f'$.
Let $\alpha = \frac{f'}{(f-a)(f-b)} - \frac{f''}{(f'-a)(f'-b)}[= \frac{1}{a-b}(\frac{H'}{H})$ where $H = (\frac{f-a}{f'-a})(\frac{f'-b}{f-b})]$
If $\alpha \equiv 0$ then $f \equiv f'$. get a contradiction.
If $\alpha \not\equiv 0$ we have $m(r,\alpha) = S(r,f)$ and $N(r,\alpha) = \overline{N}_{(2}(r, \frac{1}{f'-b}) = S(r,f)$.
Then $T(r,\alpha) = S(r,f)$.
For f has a pole at z_0 of order $p \geq 2$ then $\alpha(z_0) = 0$.
Hence $N_{(2}(r,f) \leq N(r, \frac{1}{\alpha}) \leq T(r,\alpha) + O(1) = S(r,f)$.
Let $G = \frac{(f-a)(f-b)}{f'-f}$.
Then

$$\begin{aligned} N(r,G) &\leq N(r, \frac{1}{f'-f}) - N(r, \frac{1}{f-a}) - N(r, \frac{1}{f-b}) + N_{(2}(r,f) \\ &= N(r, \frac{1}{f'-f}) - N(r, \frac{1}{f-a}) - N(r, \frac{1}{f-b}) + S(r,f). \end{aligned}$$

And $\frac{1}{G}$ has no poles then $N(r, \frac{1}{G}) = 0$.
Let $\Omega = \frac{G'}{G} + \frac{\alpha'}{\alpha} - 1$.
If z_0 is a simple pole of f then $\Omega(z_0) = 0$.
Let $N_{1)}(r,f)$ be the counting function of the simple pole of f .

If $\Omega \not\equiv 0$ then

$$\begin{aligned} N_{1)}(r,f) &\leq N(r, \frac{1}{\Omega}) \\ &\leq T(r,\Omega) + O(1) \\ &\leq \overline{N}(r,G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r,\alpha) + \overline{N}(r, \frac{1}{\alpha}) + S(r,f) \\ &\leq N(r, \frac{1}{f'-f}) - N(r, \frac{1}{f-a}) - N(r, \frac{1}{f-b}) + S(r,f). \end{aligned}$$

That is

$$N_{1)}(r,f) \leq N(r, \frac{1}{f'-f}) - N(r, \frac{1}{f-a}) - N(r, \frac{1}{f-b}) + S(r,f). \quad (3.1)$$

Since

$$N(r, \frac{1}{f'-f}) \leq T(r, f'-f) + O(1) \leq T(r,f) + \overline{N}(r,f) + S(r,f). \quad (3.2)$$

From(1)and(2) and $N_{(2}(r,f)=S(r,f)$ we have

$$N(r,\frac{1}{f-a})+N(r,\frac{1}{f-b})\leq T(r,f)+S(r,f). \tag{3.3}$$

By Left main inequality

$$m(r,\frac{1}{f-a})+m(r,\frac{1}{f-b})\leq m(r,\frac{1}{f'})+S(r,f). \tag{3.4}$$

From (3) and (4) we have

$$T(r,f)\leq m(r,\frac{1}{f'})+S(r,f). \tag{3.5}$$

Since f, f' share a CM and share b IM so
$N(r,\frac{1}{f'-a})=N(r,\frac{1}{f-a})$ and
$N(r,\frac{1}{f'-b})=N_{1)}(r,\frac{1}{f'-b})+N_{(2}(r,\frac{1}{f'-b})\leq N(r,\frac{1}{f-b})+S(r,f)$.
Hence
$N(r,\frac{1}{f'-a})+N(r,\frac{1}{f'-b})\leq N(r,\frac{1}{f-a})+N(r,\frac{1}{f-b})+S(rf)$.
By (3) we have

$$N(r,\frac{1}{f'-a})+N(r,\frac{1}{f'-b})\leq T(r,f)+S(r,f). \tag{3.6}$$

Again By Left main inequality

$$m(r,\frac{1}{f'})+m(r,\frac{1}{f'-a})+m(r,\frac{1}{f'-b})\leq m(r,\frac{1}{f''})+S(r,f). \tag{3.7}$$

From (5),(6),(7) we have
$2T(r,f')\leq m(r,\frac{1}{f''})+S(r,f)\leq T(r,f'')+S(r,f)$.
Since $T(r,f'')\leq T(r,f')+\overline{N}(r,f')+S(r,f)$.
so

$$T(r,f')\leq \overline{N}(r,f)+S(r,f). \tag{3.8}$$

Since $2\overline{N}(r,f)\leq N(r,f')$ and (8) so we have $\overline{N}(r,f)=S(r,f)$.
Use (8) again we have
$T(r,f')=S(r,f)$.
From (5) then $T(r,f)=S(r,f)$ get a contradiction.
If $\Omega\equiv 0$.

Then

$$\begin{aligned} \frac{G'}{G}+\frac{\alpha'}{\alpha} &= 1 \\ G\cdot\alpha &= Ce^z \\ \frac{(f-a)(f-b)}{f'-f} &= (\frac{C}{\alpha})e^z \\ f'-f &= \frac{\alpha}{C}e^{-z}(f-a)(f-b) \\ f'-b &= (f-b)[1+\frac{\alpha}{C}e^{-z}(f-a)] \end{aligned}$$

If there exist some z_1 such that $f'(z_1)=b$ multiplicity p where $p\geq 2$ then we have $[1+\frac{\alpha}{C}e^{-z}(f(z_1)-a)]=0$ multiplicity $p-1$.
But $\alpha(z_1)=\infty$ and $e^{-z_1}\neq 0,\infty$ and $f(z_1)-a=b-a\neq 0$ which impossible.
That is b is a simple b point of f' conclude that f,f' share b CM.
Hence $f\equiv f'$ contradiction.
Therefore $f\equiv f'$.

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