Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1581–1596
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Fixed Point Optimization Method for Image Restoration

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Abstract In this paper, we introduce the modified MM-algorithm (MMMA) and forward-backward modified MM-algorithm (FBMMMA). Furthermore, we prove that any sequence generated by the proposed algorithms converge to a set of solution. As applications, we apply the FBMMMA algorithm to solving image restoration problems. We found that our algorithm has a higher efficiency than the others in the literature.

MSC: 47H09; 47H10
Keywords: fixed point optimization method; proximal methods; forward-backward algorithm; image restoration problem

Submission date: 24.05.2020 / Acceptance date: 23.07.2020

1. Introduction

The fixed point theory for contractive mappings including the Banach contraction principle plays an important role in proving that the algorithms strongly converge to a unique solution to the problem. The Picard iteration is the most popular fixed point algorithm for solving problems in real life. Currently, there is widespread development of

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Published by The Mathematical Association of Thailand.

algorithms in applying and solving various kinds of real world problems, such as Mann iteration [1], Ishikawa iteration [2], S-iteration [3], Noor iteration [4], SP-iteration [5] and MM-iteration [6]. There is a lot of research that extends these methods (e.g. [7–13]). In this paper, we focus on image restoration by using fixed point optimization method.

Image restoration plays an important part in various areas of applied sciences such as medical and astronomical imaging. The mathematical model for the image restoration problem is formulated by the linear model:

$$b = Yu + \varepsilon, \tag{1.1}$$

where u is an original image, Y is a blurring matrix and ε is a noise term.

In order to solve the problem (1.1), Tibshirani [14] introduced the least absolute shrinkage and selection operator (LASSO) for solving the following minimization problem:

$$\min_u \left\{ \frac{1}{2}\|Yu - b\|_2^2 + \eta\|u\|_1 \right\}, \tag{1.2}$$

where $\eta > 0$, $\|u\|_1 = \sum_{i=1}^k u_i$, and $\|u\|_2 = \sqrt{\sum_{i=1}^k |u_i|^2}$.

In optimization, it often needs to solve a convex minimization problem which includes (1.2) as a special case is the following convex minimization problem:

$$\min_{u\in\mathbb{R}^n} \{\Phi(u) + \Psi(u)\}, \tag{1.3}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is proper convex and lower semi-continuous, and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, whose gradient is Lipschitz continuous ($L > 0$).

From convex minimization problem (1.3), the optimality conditions are

$$0 \in \nabla\Psi(u) + \partial\Phi(u), \tag{1.4}$$

where $\partial\Phi$ is the subdifferential of Φ defined by

$$\partial\Phi(u) := \{z \in \mathbb{R}^n : \Psi(u) \geq \Psi(v) + \langle z, u - v\rangle, \quad \forall v \in \mathbb{R}^n\}$$

and $\nabla\Psi$ is the gradient of Ψ.

Lions and Mercier [15] introduced forward-backward splitting (FBS) algorithm for problem (1.3) as follows:

$$u_{k+1} = \mathrm{prox}_{\gamma_k\Phi}(I_d - \gamma_k\nabla\Psi)(u_k),\ \gamma \in (0, 2/L),\ k \in \mathbb{N}, \tag{1.5}$$

where $u_1 \in \mathbb{R}^n$, γ_k is the step-size, I_d is an identity operator and prox_Φ is the proximity operator of Φ defined by

$$\mathrm{prox}_\Phi(u) := \arg\min_x \left\{ \Phi(x) + \frac{1}{2}\|u - x\|_2^2 \right\}. \tag{1.6}$$

Moudafi and Oliny [16] presented the inertial forward-backward splitting (IFBS) as follows:

$$\begin{cases} p_k = u_k + \theta_k(u_k - u_{k-1}), \\ u_{k+1} = \mathrm{prox}_{\gamma_k\Phi}(p_k - \gamma_k\nabla\Psi(u_k)),\ \gamma_k \in (0, 2/L),\ k \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $u_0, u_1 \in \mathbb{R}^n$, θ_k is the inertial parameter which controls the momentum $u_k - u_{k-1}$.

Beck and Teboulle [17] introduced fast iterative shrinkage thresholding algorithm (FISTA) as follows:

$$\begin{cases} p_k = \text{prox}_{\frac{1}{L}\Phi}(u_k - \frac{1}{L}\nabla\Psi(u_k)), \\ t_{k+1} = \dfrac{1+\sqrt{1+4t_k^2}}{2}, \quad \theta_k = \dfrac{t_k - 1}{t_{k+1}}, \\ u_{k+1} = p_k + \theta_k(p_k - p_{k-1}), \quad k \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $u_1 = p_0 \in \mathbb{R}^n$, $t_1 = 1$. They proved the convergence rate of the FISTA and applied to the image restoration problem.

Verma and Shukla [18] introduced a new accelerated proximal gradient algorithm (NAGA) as follow:

$$\begin{cases} p_k = u_k + \theta_k(u_k - u_{k-1}), \\ u_{k+1} = T_k((1-\delta_k)p_k + \delta_k T p_k), \quad k \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $u_0, u_1 \in \mathbb{R}^n$, T_k is the forward-backward operator of Φ and Ψ with respect to $\gamma_k \in (0, 2/L)$. They proved the convergence of the NAGA and applied to solving the convex minimization problem with sparsity-inducing regularizes for multitask learning framework.

Phon-on et al. [19] proposed modified inertial S-iteration process as follows:

$$\begin{cases} p_k = u_k + \theta_k(u_k - u_{k-1}), \\ q_k = (1-\rho_k)p_k + \rho_k T_1 p_k, \\ u_{k+1} = (1-\delta_k)T_1 p_k + \delta_k T_2 q_k, \quad k \in \mathbb{N}, \end{cases} \tag{1.10}$$

where $T_1, T_2 : C \to C$ are nonexpansive mappings, $\{\theta_k\}$, $\{\rho_k\}$, and $\{\delta_k\}$ satisfy

(D1) $\sum_{k=1}^{\infty} \theta_k < \infty$, $\{\theta_k\} \subset [0, \theta]$, $0 \le \theta < 1$, $\{\rho_k\}$, $\{\delta_k\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 0.5)$;

(D2) $\{T_i(p_k) - p_k\}$ is bounded for $i = 1, 2$;

(D3) $\{T_i(p_k) - x\}$ is bounded for any x is in common fixed points of T_1 and T_2.

They proved the weak and strong convergence for finding common fixed points of T_1 and T_2.

In fact, letting $T_1 = T_2 = T$, we obtain

$$\begin{cases} p_k = u_k + \theta_k(u_k - u_{k-1}), \\ q_k = (1-\rho_k)p_k + \rho_k T p_k, \\ u_{k+1} = (1-\delta_k)T p_k + \delta_k T q_k, \quad k \in \mathbb{N}, \end{cases} \tag{1.11}$$

where sequences $\{\rho_k\}$ and $\{\delta_k\}$ are in the interval $(0, 1)$, θ_k is the inertial parameter which controls the momentum $u_k - u_{k-1}$, and $T : C \to C$ is a nonexpansive mapping. Then, T have fixed point which (1.11) is called inertial S-iteration (IS).

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and C be a nonempty closed convex subset of $\mathcal{H}$.

A nonlinear operator $T : C \to C$ is called

- L-Lipschitz operator, if there exists $L > 0$ such that
$$\|Tu - Tv\| \leq L\|u - v\|, \quad \text{for all } u, v \in C; \tag{2.1}$$
- nonexpansive, if
$$\|Tu - Tv\| \leq \|u - v\|, \quad \text{for all } u, v \in C. \tag{2.2}$$

Next, we denote by $Fix(T)$ the set of all fixed points of T, $Fix(T) := \{x \in C : Tx = x\}$, $\omega_w(u_k)$ denote the set of all weak-cluster points of a bounded sequence $\{u_k\}$ in C, $\{T_k\}$ and Λ be families of nonexpansive operators of C into itself such that $\Upsilon := \bigcap_{k=1}^{\infty} Fix(T_k) \supset Fix(\Lambda) \neq \emptyset$, where $Fix(\Lambda)$ is the set of all common fixed points of Λ.

Nakajo et al. [20] introduced the NST-condition (I) with Λ, A sequence $\{T_k\}$ is said to satisfy the NST if for every bounded sequence $\{u_k\}$ in C,
$$\lim_{k\to\infty} \|u_k - T_k u_k\| = 0 \quad \text{implies} \quad \lim_{k\to\infty} \|u_k - Tu_k\| = 0, \quad \forall\, T \in \Lambda. \tag{2.3}$$
Nakajo et al. [21] introduced the NST*-condition (I) with Λ, A sequence $\{T_k\}$ is said to satisfy the NST if for every bounded sequence $\{u_k\}$ in C,
$$\lim_{k\to\infty} \|u_k - T_k u_k\| = \lim_{k\to\infty} \|u_k - u_{k+1}\| = 0 \quad \text{implies} \quad \omega_w(u_k) \subset \Upsilon. \tag{2.4}$$

Lemma 2.1. [22] *For a real Hilbert space $\mathcal{H}$, let $\Phi : \mathcal{H} \to \mathbb{R}\cup\{\infty\}$ be a proper convex and lower semi-continuous function, and $\Psi : \mathcal{H} \to \mathbb{R}$ be convex differentiable with gradient $\nabla\Psi$ being L-Lipschitz constant for some $L > 0$. If $\{T_k\}$ is the forward-backward operator of Φ and Ψ with respect to $\gamma_k \in (0, 2/L)$ such that γ_k converges to γ, then $\{T_k\}$ satisfies NST-condition (I) with T, where T is the forward-backward operator of Φ and Ψ with respect to $\gamma \in (0, 2/L)$.*

Lemma 2.2. [23] *Let $\{\delta_k\}$ and $\{\theta_k\}$ be sequences of nonnegative real numbers such that*
$$\delta_{k+1} \leq (1 + \theta_k)\delta_k + \theta_k\delta_{k-1}, \quad k \in \mathbb{N}.$$
Then the following holds
$$\delta_{k+1} \leq \mathcal{M}\prod_{j=1}^{k}(1 + 2\theta_j), \quad \text{where } \mathcal{M} = \max\{\delta_1, \delta_2\}.$$
Moreover, if $\sum_{k=1}^{\infty}\theta_k < \infty$, then $\{\delta_k\}$ is bounded.

Lemma 2.3 ([24])**.** *Let $\{\delta_k\}$, $\{\theta_k\}$, and $\{\rho_k\}$ be sequences of nonnegative real numbers such that $\delta_{k+1} \leq (1 + \rho_k)\delta_k + \theta_k$, $k \in \mathbb{N}$. If $\sum_{k=1}^{\infty}\rho_k < \infty$ and $\sum_{k=1}^{\infty}\theta_k < \infty$, then $\lim_{k\to\infty}\delta_k$ exists.*

Lemma 2.4 ([25])**.** *Let $\mathcal{H}$ be a Hilbert space and $\{u_k\}$ be a sequence in $\mathcal{H}$ such that there exists a nonempty set $\Upsilon \subset \mathcal{H}$ satisfying*

(i) *For every $u \in \Upsilon$, $\lim_{k\to\infty}\|u_k - u\|$ exists;*

(ii) *Each weak-cluster point of the sequence $\{u_k\}$ is in Υ.*

Then, $\{u_k\}$ converges weakly to a point in Υ.

Lemma 2.5 ([26])**.** *Let $\mathcal{H}$ be a real Hilbert space. Then the following results hold:*

(i) $\|u \pm v\|^2 = \|u\|^2 \pm 2\langle u, v\rangle + \|v\|^2, \ \forall\, u, v \in \mathcal{H}$;

(ii) $\|\delta u+(1-\delta)v\|^2 = \delta\|u\|^2+(1-\delta)\|v\|^2-\delta(1-\delta)\|u-v\|^2, \quad \forall\delta \in [0,1],\ \forall\ u,v \in \mathcal{H}.$

3. Main Results

In this section, we present a modified MM-algorithm which is called MMMA for finding a common fixed point of a countable family of nonexpansive operators in a real Hilbert space as follow assumptions:

$(A1)$ $\mathcal{H}$ is a real Hilbert space;

$(A2)$ $\{T_k : \mathcal{H} \to \mathcal{H}\}$ is a family of nonexpansive operators;

$(A3)$ $\{T_k\}$ satisfies the NST*-condition;

$(A4)$ $\Upsilon := \bigcap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$.

Algorithm 1 : A modified MM-algorithm (MMMA)

Initial: Given $u_0, u_1 \in \mathcal{H}$ arbitrarily and $k = 1$.
Step 1. Compute

$$\begin{cases} p_k = u_k + \theta_k(u_k - u_{k-1}), \\ q_k = (1-\rho_k)p_k + \rho_k T_k p_k, \\ s_k = (1-\delta_k-\beta_k)q_k + \delta_k T_k q_k + \beta_k T_k p_k, \\ u_{k+1} = T_k s_k. \end{cases}$$

Set $k =: k+1$ and go back to **Step 1.**

Theorem 3.1. *Let $\{u_k\}$ be a sequence generated by MMMA where $\{\rho_k\}$, $\{\delta_k\}$, and $\{\beta_k\}$ are sequences in $[0,1]$ satisfying $(\delta_k+\beta_k) \in [0,1]$, $\theta_k \in (0,1)$ and $\sum_{k=1}^{\infty}\theta_k < \infty$. Then $\|u_{k+1}-u^*\| \leq \mathcal{M}\prod_{j=1}^{k}(1+2\theta_j)$, where $\mathcal{M} = \max\{\|u_1-u^*\|, \|u_2-u^*\|\}$ and $u^* \in \Upsilon$*

Proof. Let $u^* \in \Upsilon$. Using MMMA, we have

$$\begin{aligned} \|p_k - u^*\| &= \|u_k + \theta_k(u_k-u_{k-1}) - u^*\| \\ &\leq \|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\|. \end{aligned} \tag{3.1}$$

Using MMMA, T_k is a family of nonexpansive operators, and (3.1), we have

$$\begin{aligned} \|q_k - u^*\| &= \|(1-\rho_k)p_k + \rho_k T_k p_k - u^*\| \\ &\leq (1-\rho_k)\|p_k - u^*\| + \rho_k\|T_k p_k - u^*\| \\ &\leq (1-\rho_k)\|p_k - u^*\| + \rho_k\|p_k - u^*\| \\ &= \|p_k - u^*\| \\ &\leq \|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\|. \end{aligned} \tag{3.2}$$

Using MMMA, T_k is a family of nonexpansive operators, (3.2), and (3.1), we have

$$\begin{aligned}\|s_k - u^*\| &= \|(1-\delta_k-\beta_k)q_k + \delta_k T_k q_k + \beta_k T_k p_k - u^*\| \\ &\le (1-\delta_k-\beta_k)\|q_k - u^*\| + \delta_k\|T_k q_k - u^*\| + \beta_k\|T_k p_k - u^*\| \\ &\le (1-\delta_k-\beta_k)\|q_k - u^*\| + \delta_k\|q_k - u^*\| + \beta_k\|p_k - u^*\| \\ &= (1-\beta_k)\|q_k - u^*\| + \beta_k\|p_k - u^*\| \\ &\le (1-\beta_k)(\|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\|) \\ &\quad + \beta_k(\|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\|) \\ &= \|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\|.\end{aligned} \tag{3.3}$$

Using MMMA, T_k is a family of nonexpansive operators, and (3.3), we have

$$\begin{aligned}\|u_{k+1} - u^*\| &= \|T_k s_k - u^*\| \\ &\le \|s_k - u^*\| \\ &\le \|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\| \\ &\le (1+\theta_k)\|u_k - u^*\| + \theta_k\|u_k - u_{k-1}\|.\end{aligned} \tag{3.4}$$

Applying Lemma 2.2, we obtain that $\|u_{k+1} - u^*\| \le \mathcal{M}\prod_{j=1}^{k}(1+2\theta_j)$, where $\mathcal{M} = \max\{\|u_1 - u^*\|, \|u_2 - u^*\|\}$, it follows that $\{u_k\}$ is bounded.
This implies $\sum_{k=1}^{\infty}\theta_k\|u_k - u_{k-1}\| < \infty$. ∎

Theorem 3.2. *Let $\{u_k\}$ be a sequence generated by MMMA where $\{\rho_k\}$, $\{\delta_k\}$, and $\{\beta_k\}$ are sequences in $[0,1]$ satisfying $(\delta_k + \beta_k) \in [0,1]$, $\theta_k \in (0,1)$ and $\sum_{k=1}^{\infty}\theta_k < \infty$. Then $\{u_k\}$ converges weakly to a point in Υ.*

Proof. Using (3.4) and Lemma 2.3, we get $\lim_{k\to\infty}\|u_k - u^*\|$ exists. Using Lemma 2.5(i), we get

$$\|p_k - u^*\|^2 \le \|u_k - u^*\|^2 + \theta_k^2\|u_k - u_{k-1}\|^2 + 2\theta_k\|u_k - u^*\|\|u_k - u_{k-1}\|. \tag{3.5}$$

Using Lemma 2.5(ii), we get

$$\begin{aligned}\|q_k - u^*\|^2 &= \|(1-\rho_k)p_k + \rho_k T_k p_k - u^*\|^2 \\ &= (1-\rho_k)\|p_k - u^*\|^2 + \rho_k\|T_k p_k - u^*\|^2 - \rho_k(1-\rho_k)\|p_k - T_k p_k\|^2 \\ &\le (1-\rho_k)\|p_k - u^*\|^2 + \rho_k\|p_k - u^*\|^2 - \rho_k(1-\rho_k)\|p_k - T_k p_k\|^2 \\ &= \|p_k - u^*\|^2 - \rho_k(1-\rho_k)\|p_k - T_k p_k\|^2.\end{aligned} \tag{3.6}$$

Using Lemma 2.5(ii), (3.5), and (3.6), we obtain

$$\begin{aligned}\|u_{k+1} - u^*\|^2 &= \|T_k s_k - u^*\|^2 \\ &\le \|s_k - u^*\|^2 \\ &\le \|(1-\delta_k-\beta_k)q_k + \delta_k T_k q_k + \beta_k T_k p_k - u^*\|^2\end{aligned}$$

$$
\begin{aligned}
&= (1-\delta_k-\beta_k)\|q_k-u^*\|^2+\delta_k\|T_kq_k-u^*\|^2+\beta_k\|T_kp_k-u^*\|^2\\
&\quad -\delta_k(1-\delta_k-\beta_k)\|q_k-T_kq_k\|^2\\
&\le (1-\delta_k-\beta_k)\|q_k-u^*\|^2+\delta_k\|q_k-u^*\|^2+\beta_k\|p_k-u^*\|^2\\
&\quad -\delta_k(1-\delta_k-\beta_k)\|q_k-T_kq_k\|^2\\
&= (1-\beta_k)\|q_k-u^*\|^2+\beta_k\|p_k-u^*\|^2\\
&\quad -\delta_k(1-\delta_k-\beta_k)\|q_k-T_kq_k\|^2 \qquad (3.7)\\
&\le \|p_k-u^*\|^2-(1-\beta_k)\rho_k(1-\rho_k)\|p_k-T_kp_k\|^2\\
&\quad -\delta_k(1-\delta_k-\beta_k)\|q_k-T_kq_k\|^2\\
&\le \|u_k-u^*\|^2+\theta_k^2\|u_k-u_{k-1}\|^2+2\theta_k\|u_k-u^*\|\|u_k-u_{k-1}\|\\
&\quad -(1-\beta_k)\rho_k(1-\rho_k)\|p_k-T_kp_k\|^2\\
&\quad -\delta_k(1-\delta_k-\beta_k)\|q_k-T_kq_k\|^2.
\end{aligned}
$$

Since $\sum_{k=1}^{\infty}\theta_k\|u_k-u_{k-1}\|<\infty$ and $\lim_{k\to\infty}\|u_k-u^*\|$ exists, it follows that $\lim_{k\to\infty}\|q_k-T_kq_k\|=0$ and $\lim_{k\to\infty}\|p_k-T_kp_k\|=0$. On the other hand,

$$\|u_k-T_ku_k\|\le\|u_k-p_k\|+\|p_k-T_kp_k\|+\|T_kp_k-T_ku_k\|\le 2\|u_k-p_k\|+\|p_k-T_kp_k\|, \qquad (3.8)$$

and

$$
\begin{aligned}
\|s_k-q_k\| &\le \|s_k-p_k\|+\|p_k-q_k\|\\
&\le (1-\delta_k-\beta_k)\|q_k-p_k\|+\delta_k\|T_kq_k-p_k\|\\
&\quad +\beta_k\|T_kp_k-p_k\|+\|p_k-q_k\|\\
&\le (2-\delta_k-\beta_k)\|q_k-p_k\|+\delta_k\|T_kq_k-q_k\| \qquad (3.9)\\
&\quad +\delta_k\|q_k-p_k\|+\beta_k\|T_kp_k-p_k\|\\
&= (2-\beta_k)\|q_k-p_k\|+\delta_k\|T_kq_k-q_k\|+\beta_k\|T_kp_k-p_k\|\\
&\le ((2-\beta_k)\rho_k+\beta_k)\|T_kp_k-p_k\|+\delta_k\|T_kq_k-q_k\|.
\end{aligned}
$$

These imply by MMMA that $\lim_{k\to\infty}\|u_k-T_ku_k\|=0$ and $\lim_{k\to\infty}\|s_k-q_k\|=0$. Using MMMA and nonexpansivity of T_k, we obtain

$$
\begin{aligned}
\|u_{k+1}-u_k\| &= \|T_ks_k-u_k\|\\
&\le \|T_ks_k-T_ku_k\|+\|T_ku_k-u_k\|\\
&\le \|s_k-u_k\|+\|T_ku_k-u_k\| \qquad (3.10)\\
&\le \|s_k-q_k\|+\|q_k-u_k\|+\|T_ku_k-u_k\|\\
&\le \|s_k-q_k\|+\|q_k-p_k\|+\|p_k-u_k\|+\|T_ku_k-u_k\|,
\end{aligned}
$$

which

$$\|q_k-p_k\|=\rho_k\|T_kp_k-p_k\|\to 0 \quad \text{as } k\to\infty,$$

and

$$\|p_k-u_k\|=\theta_k\|u_k-u_{k-1}\|\to 0 \quad \text{as } k\to\infty.$$

These imply that $\lim_{k\to\infty}\|u_k-u_{k+1}\|=0$. Using $(A3)$, we have $\omega_w(u_k)\subset\Upsilon$. Hence, using Lemma 2.4, we conclude that $\{u_k\}$ converges weakly to a point in Υ. ∎

Finally, we apply MMMA for solving the minimization problem (1.3) by setting $T_k = \text{prox}_{\gamma_k\Phi}(I_d - \gamma_k\nabla\Psi)$, the forward-backward operator of Φ and Ψ with respect to γ_k, where $\Phi : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ is proper convex and lower semi-continuous, and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, whose gradient is Lipschitz continuous $(L > 0)$.

Algorithm 2 : A forward-backward modified MM-algorithm (FBMMMA)

Initial: Given $u_0, u_1 \in \mathcal{H}$ arbitrarily and $k = 1$.

Step 1. Compute

$$\begin{cases} p_k = u_k + \theta_k(u_k - u_{k-1}), \\ q_k = (1-\rho_k)p_k + \rho_k\text{prox}_{\gamma_k\Phi}(I_d - \gamma_k\nabla\Psi)p_k, \\ s_k = (1-\delta_k-\beta_k)q_k + \delta_k\text{prox}_{\gamma_k\Phi}(I_d-\gamma_k\nabla\Psi)q_k + \beta_k\text{prox}_{\gamma_k\Phi}(I_d-\gamma_k\nabla\Psi)p_k, \\ u_{k+1} = \text{prox}_{\gamma_k\Phi}(I_d-\gamma_k\nabla\Psi)s_k. \end{cases}$$

Set $k =: k+1$ and go back to **Step 1.**

Theorem 3.3. *Let $\{u_k\}$ be a sequence generated by FBMMMA where θ_k, δ_k, β_k, θ_k are the same as in Theorem 3.1, and $\gamma_k \in (0, 2/L)$ such that $\{\gamma_k\}$ converges to γ. Then $\|u_{k+1} - u^*\| \leq \mathcal{M}\prod_{j=1}^{k}(1+2\theta_j)$, where $\mathcal{M} = \max\{\|u_1 - u^*\|, \|u_2 - u^*\|\}$ and $u^* \in \arg\min(\Phi+\Psi)$.*

Proof. Let T_k be the forward-backward operator of Φ and Ψ with respect to γ_k and $T_k = \text{prox}_{\gamma_k\Phi}(I_d - \gamma_k\nabla\Psi)$. Using Proposition 26.1 in [27], T and $\{T_k\}$ are nonexpansive operators for all k, and $Fix(T) = \bigcap_{k=1}^{\infty} Fix(T_k) = \arg\min(\Phi+\Psi)$. Hence, we obtain the required result directly by Theorem 3.1. ∎

Theorem 3.4. *Let $\{u_k\}$ be a sequence generated by FBMMMA where θ_k, δ_k, β_k, θ_k are the same as in Theorem 3.1, and $\gamma_k \in (0, 2/L)$ such that $\{\gamma_k\}$ converges to γ. Then $\{u_k\}$ converges weakly to a point in $\arg\min(\Phi+\Psi)$.*

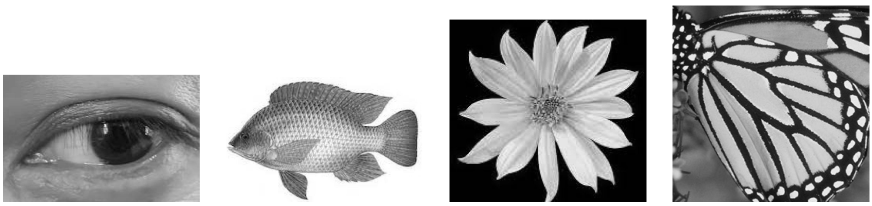
Proof. Using Lemma 2.1, we obtain that $\{T_k\}$ satisfies $(A3)$. Hence, we obtain the required result directly by Theorem 3.2. ∎

4. Image Restoration

In this section, we apply FBMMMA to solving the image restoration problem (1.2) which $\Psi(u) = \frac{1}{2}\|Yu - b\|_2^2$ and $\Phi(u) = \eta\|u\|_1$ compare with FISTA [17] and NAGA [18] by using maximum iteration number is 200 iteration. All codes were written in Matlab 2016b and run on Dell i-5 Core laptop. We illustrate the performance of our proposed algorithm for image restoration problems. The improvement in signal to noise ratio (ISNR) are used to measure the quality of the restored images. They are defined as follows:

$$\text{ISNR} = 10\log\frac{\|u-b\|_2^2}{\|u-u_k\|_2^2}$$

where u, b, k, and u_k are the original image, the observed image, the noise matrix added in the test, and estimated image at iteration k, respectively. The Lipschitz constant L, was computed by the maximum eigenvalues of the matrix $\Psi^T\Psi$. We take $\eta = 10^{-4}$, $\gamma_k = \frac{1}{L}$, $\rho_k = 0.99$, $\delta_k = 0.01$, $\beta_k = 0.99$, $\theta_k = 0.99$. We show numerical comparison in case Gaussian blur "$fspecial('gaussian', 9, 9)$" and Motion blur "$fspecial('motion', 15, 9)$" of FISTA, NAGA and FBMMMA in Table 1 and Table 2. The SNR results in case Gaussian blur show in Figure 2, 3, 4, 5. Image restoration results in case Gaussian blur show in Figure 6. The SNR results in case Motion blur show in Figure 7, 8, 9, 10. Image restoration results in case Gaussian blur show in Figure 11.



(A) (B) (C) (D)

FIGURE 1. Test images: (A) Eye, (B) Fish, (C) Power, and (D) Butterfly.

Experiment	FISTA	NAGA	FBMMMA
	SNR	SNR	SNR
Eye (256 × 165)	21.6976	21.8010	22.0386
Fish (256 × 164)	16.2553	20.3542	20.6041
Power (187 × 172)	17.2094	17.3374	17.6585
Butterfly (256 × 256)	16.1611	16.2553	16.2553

TABLE 1. Numerical comparison in case Gaussian blur.

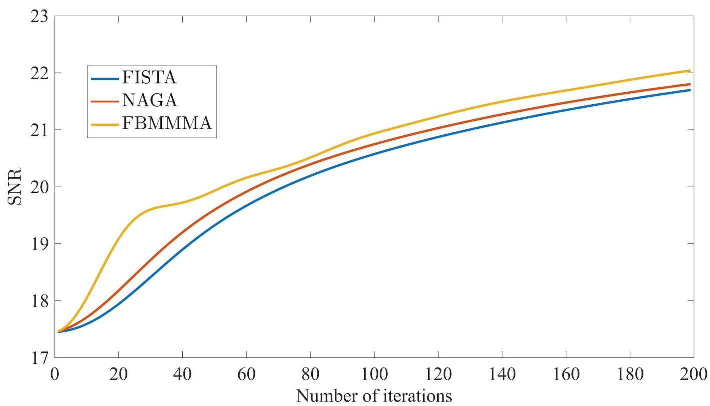


FIGURE 2. SNR results in case Gaussian blur for Eye image.

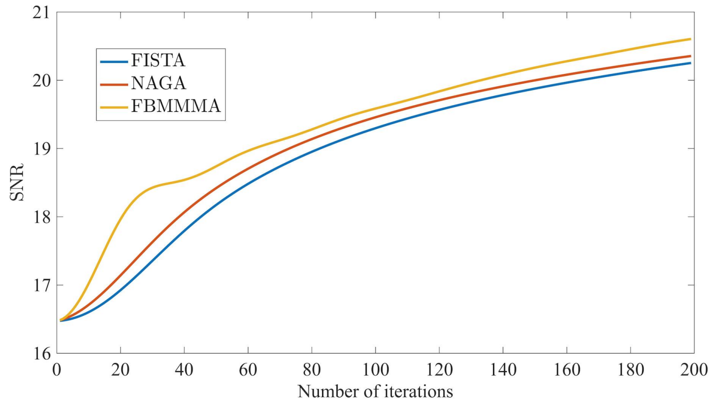


FIGURE 3. SNR results in case Gaussian blur for Fish image.

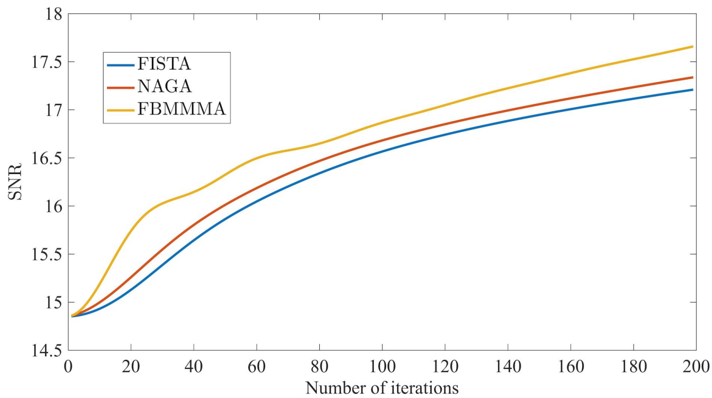


FIGURE 4. SNR results in case Gaussian blur for Power image.

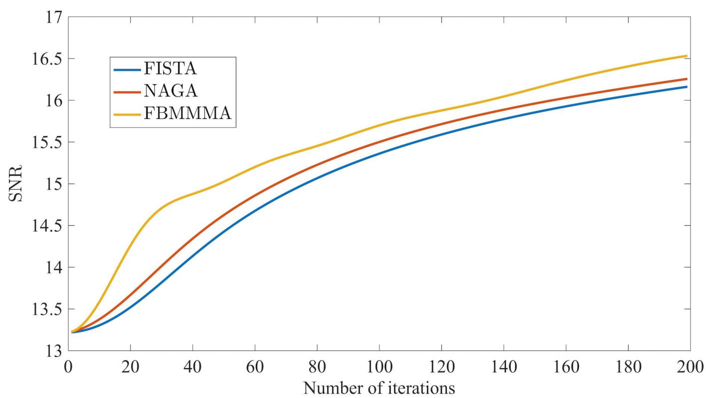


FIGURE 5. SNR results in case Gaussian blur for Butterfly image.

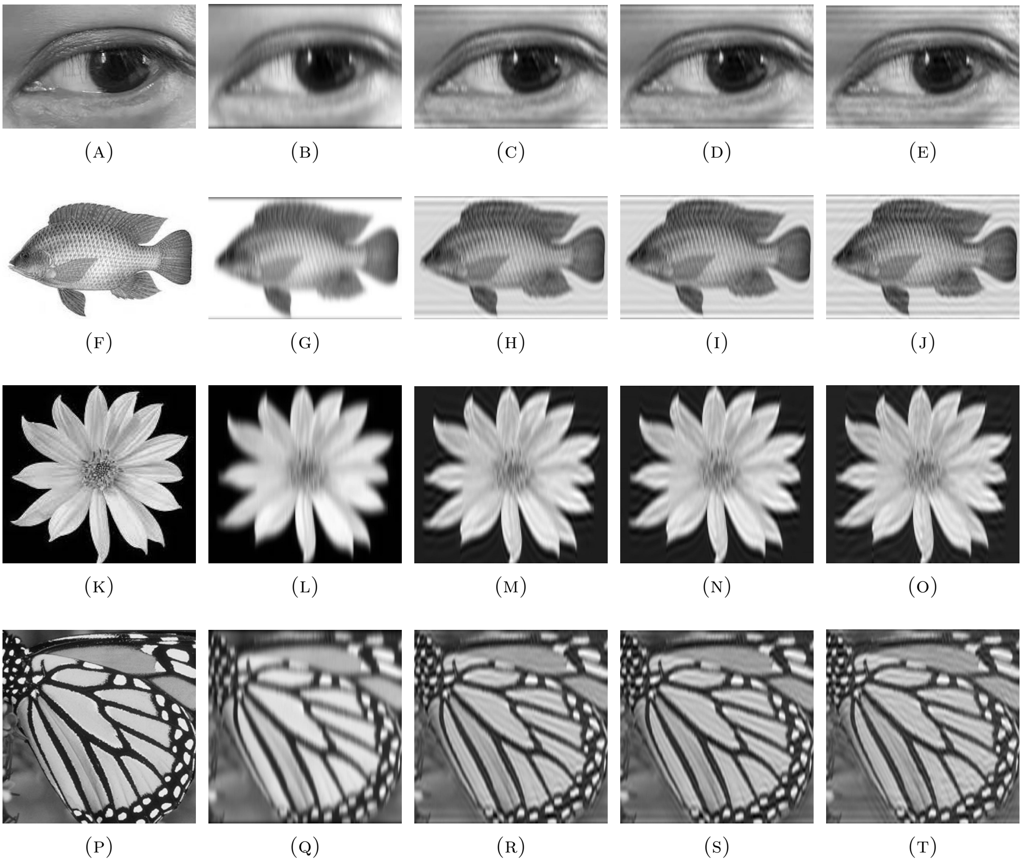


FIGURE 6. Restoration results in case Gaussian blur for test images. (A), (F), (K), (P) Original image; (B), (G), (L), (Q) blurred and noisy; (C), (H), (M), (R) FISTA; (D), (I), (N), (S) NAGA restored result; (E), (J), (O), (T) FBMMMA restored result.

Experiment	FISTA	NAGA	FBMMMA
	SNR	SNR	SNR
Eye (256 × 165)	23.0907	23.4269	24.0904
Fish (256 × 164)	21.3795	21.6952	22.5701
Power (178 × 172)	17.3409	17.7694	18.8316
Butterfly (256 × 256)	15.7281	16.1250	17.2098

TABLE 2. Numerical comparison in case Motion blur.

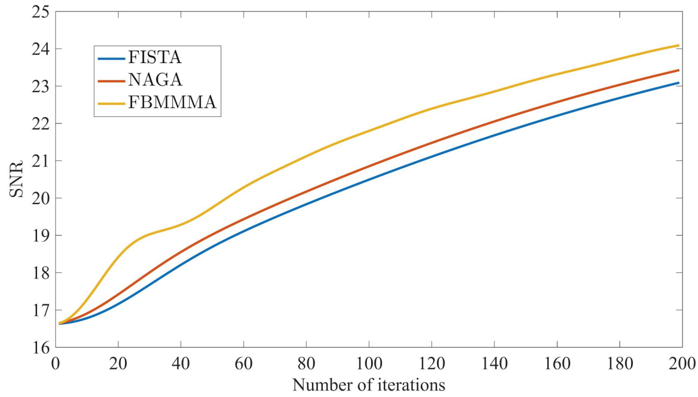


FIGURE 7. SNR results in case Motion blur for Eye image.

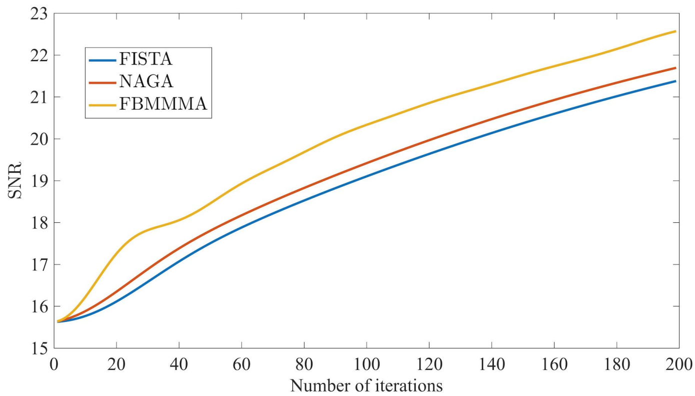


FIGURE 8. SNR results in case Motion blur for Fish image.

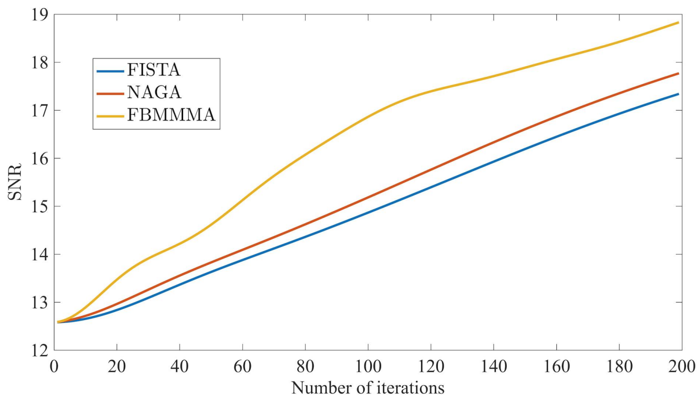


FIGURE 9. SNR results in case Motion blur for Power image.

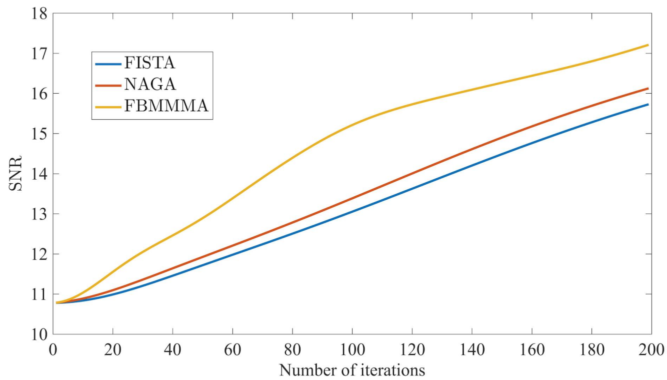


FIGURE 10. SNR results in case Motion blur for Butterfly image.

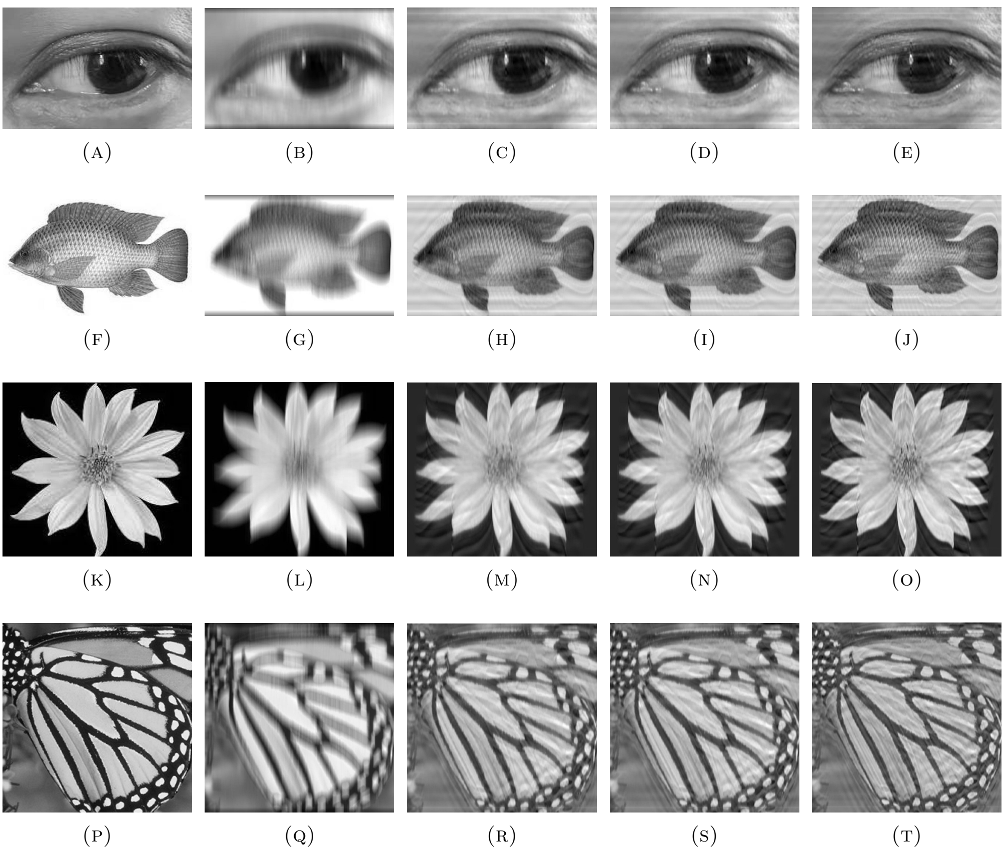


FIGURE 11. Restoration results in case Motion blur for test images. (A), (F), (K), (P) Original image; (B), (G), (L), (Q) blurred and noisy; (C), (H), (M), (R) FISTA; (D), (I), (N), (S) NAGA restored result; (E), (J), (O), (T) FBMMMA restored result.

5. Conclusions

In this paper, we propose a MM-iteration with inertial extrapolation term for approximation of fixed point of nonexpansive mapping and convergence result in real Hilbert spaces. Applying our results in the image restoration problem comparing the proposed methods with the FISTA and the NAGA. Our proposed algorithm has a better performance (SNR) than two algorithms above.

Acknowledgments

The first author thanks Rambhai Barni Rajabhat University for the support. The second author was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation research Cluster (CLASSIC), Faculty of Science, KMUTT.

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