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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Convergence Analysis of Quasi-Variational Inclusion and Fixed Point Problems of Finite Family of Certain Nonlinear Mappings in Hilbert Spaces

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Abstract The purpose of this paper is to present a modified Halpern iterative algorithm for finding a common solution of quasi-variational inclusion problem and fixed point problem of a finite family of demimetric mappings and quasi-nonexpansive mapping in the framework of real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating the solution of the aforementioned problems. We give some consequences of our main result, present an application to variational inequality problem and dispaly numerical example to show the behaviour of our result. Our result complements and extends some related results in literature.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H, then the Quasi-Variational Inclusion Problem (in short QVIP) (see [1, 2]) which generalizes the classical Varaiational Inequality Problem (VIP) introduced by Stampacchia [3] is to find $u \in H$ such that

$$\theta \in D(u) + M(u), \tag{1.1}$$

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where $D : H \to H$ is a single-valued nonlinear mapping and $M : H \to 2^H$ is a multivalued mapping.
As applications, a number of problems arising in structural analysis, mechanics and economics can be studied in the framework of this kind of variational inclusion (see [4]). The QVIP can be applied to solve several other optimization problems, (see [5–16]). We denote by $VI(H, D, M)$ the solution set of (1.1).
Let C be a nonempty, closed and convex subset of a real Hilbert space H, a point $p \in C$ is called a fixed point of a mapping $T : C \to C$, if $Tp = p$. We denote by $F(T)$, the set of all fixed points of T.

Definition 1.1. A nonlinear mapping $M : H \to H$ is called
(i) nonexpansive, if

$$||Mx - My|| \leq ||x - y||, \ \forall \ x, y \in H;$$

(ii) quasi-nonexpansive, if $p \in F(T)$ and

$$||Mx - p|| \leq ||x - p||, \ \forall \ x \in H;$$

(iii) monotone, if

$$\langle Mx - My, x - y \rangle \geq 0, \ \forall \ x, y \in H;$$

(iv) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y \rangle \geq \alpha ||x - y||^2, \ \forall \ x, y \in H;$$

(v) α-inverse strongly monotone (ism), if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y \rangle \geq \alpha ||Mx - My||^2, \ \forall \ x, y \in H.$$

A multivalued mapping $M : H \to 2^H$ is called maximal monotone, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y \rangle \geq 0$ for every $(y, v) \in Gra(M)$ (the graph of mapping M) implies $u \in Mx$ and $v \in My$.

Definition 1.2. Let $T : H \to H$ be a mapping with $F(T) \neq \emptyset$, then T is called k-demimetric if there exist $k \in (-\infty, 1)$, for any $x \in H$ and $q \in F(T)$ such that

$$\langle x - q, x - Tx \rangle \geq \frac{1 - k}{2} ||x - Tx||^2. \tag{1.2}$$

Equivalently, T is k-demimetric, if there exists $k \in (-\infty, 1)$ such that

$$||Tx - q||^2 \leq ||x - q||^2 + k||x - Tx||^2, \ \forall \ x \in H \text{ and } q \in F(T).$$

Below is an example of a demimetric mapping in Hilbert spaces.

Example 1.3. Let $H = \mathbb{R}$ (the real line with usual metric). Define $T : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{-9}{2}x, \ 0 \leq x \leq 1, \\ \\ 0, \ \text{otherwise}. \end{cases}$$

Clearly, $F(T) = \{0\}$, and $k = \frac{7}{11}$. Hence, we have that T is $\frac{7}{11}$-demimetric mapping.

Example 1.4. [17] Let H be the real line and $C = [-2, 1]$. Define

$$Tx = \begin{cases} \dfrac{x+9}{10}, & x \in [0,1], \\ \\ \dfrac{3+x}{4}, & x \in [-2,0). \end{cases}$$

Obviously, $F(T) = \{1\}$. We will show that there exists $\delta \in (-\infty, 1)$ such that

$$|Tx - 1|^2 \leq |x-1|^2 + \delta|x - Tx|^2, \qquad \forall x \in [-2, 1].$$

Consider the following two cases:
Case (i): Let $x \in [0, 1]$, then

$$|x - Tx|^2 = \left|x - \frac{x+9}{10}\right|^2 = \left|\frac{9}{10}(x-1)\right|^2 = \frac{81}{100}|x-1|^2.$$

Also

$$\begin{aligned} |Tx-1|^2 &= \left|\frac{x+9}{10} - 1\right|^2 = \frac{1}{100}|x-1|^2 \\ &= |x-1|^2 - \frac{99}{100}|x-1|^2 \\ &= |x-1|^2 - \frac{99}{81} \times \frac{81}{100}|x-1|^2 \\ &\leq |x-1|^2 + \delta_1 . \frac{81}{100}|x-1|^2, \end{aligned}$$

for any $\delta_1 \in [-\frac{99}{81}, 1)$. Hence $|Tx - 1|^2 \leq |x-1|^2 + \delta_1|x - Tx|^2$.
Case (ii): Let $x \in [-2, 0)$, thus

$$|x - Tx|^2 = \left|x - \frac{3+x}{4}\right|^2 = \left|\frac{3(x-1)}{4}\right|^2 = \frac{9}{16}|x-1|^2.$$

Then

$$\begin{aligned} |Tx-1|^2 &= \left|\frac{3+x}{4} - 1\right|^2 = \left|\frac{x-1}{4}\right|^2 = \frac{1}{16}|x-1|^2 \\ &= |x-1|^2 - \frac{15}{16}|x-1|^2 \\ &= |x-1|^2 - \frac{15}{9} . \frac{9}{16}|x-1|^2 \\ &\leq |x-1|^2 + \delta_2 . \frac{9}{16}|x-1|^2, \end{aligned}$$

for any $\delta_2 \in [-\frac{15}{9}, 1)$. Hence $|Tx - 1|^2 \leq |x-1|^2 + \delta_1|x - Tx|^2$. In particular, choose $\delta = \min\{\delta_1, \delta_2\}$. Thus, T is $-\frac{15}{9}$-demimetric.

According to (1.2), it has been proved that a k-strict pseudocontraction T with $F(T) \neq \emptyset$ is k-demimetric and an α-generalized hybrid mapping T with $F(T) \neq \emptyset$ is 0-demimetric, see [18].
In 2018, Chen and Lee [19] proved the following strong convergence theorem for approximating a common solution of quasi-variational inclusion and fixed point problems of nonexpansive mapping in the framework of real Hilbert spaces as follows:

Theorem 1.5. *Let H be a real Hilbert space, $A : H \to H$ be an α-ism mapping, $M : H \to 2^H$ be a maximal monotone mapping and $S : H \to H$ be a nonexpansive mapping. Suppose that the set $F(S) \cap VI(H, A, M) \neq \emptyset$. Suppose $x_0 = x \in H$ and $\{x_n\}$ is the sequence defined by*

$$\begin{cases} x_{n+1} = \alpha_n x + (1 - \alpha_n) S y_n; \\ y_n = J_{M,\lambda}(x_n - \lambda A x_n); \end{cases}$$

where $\lambda \in (0, 2\alpha]$ and α_n is a sequence in [0,1] satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(H,A,M)} x_0$.

Also, recently Takahashi, Wen and Yao [20] introduced a shrinking projection method for approximating solution of a finite family of demimetric mappings together with a variational inequality problems in a real Hilbert space. They proved the following strong convergence theorem:

Theorem 1.6. *Let C be a nonempty closed and covex subset of a real Hilbert space H. Let $\{k_1, ..., k_m\} \subset (-\infty, 1)$ and $\{\mu_1, ..., \mu_m\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^m$ be a finite family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-ism mappings of C into H. Assume that $\cap_{j=1}^m F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset$. Let $x_1 \in C$ and $C_1 = C$, then $\{x_n\}$ is a sequence generated iteratively by*

$$\begin{cases} z_n = \sum_{j=1}^m \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n; \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n; \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}; \\ x_{n+1} = P_{C_{n+1}x_1}, \ \forall\ n \in \mathbb{N}; \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, ..., \xi_m\}$, $\{\sigma_1, ..., \sigma_m\} \subset (0, 1)$ and $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:
(i) $0 < a \leq \lambda_n \leq \min\{1 - k_1, ..., 1 - k_m\}$, $0 < b \leq \eta_n \leq 2\min\{\mu_1, ..., \mu_n\}$;
(ii) $\sum_{j=1}^m \xi_j = 1$ and $\sum_{i=1}^n \sigma_i = 1$;
(iii) $0 \leq c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.
Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^m F(T_j) \cap (\cap_{i=1}^n VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^m F(T_j)\cap(\cap_{i=1}^n VI(C,B_i))} x_1$.

In 2009, Chakkrid and Suantai [12] introduced an iterative algorithm to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the VIP for the ism mapping which solves some VIP. They proved a strong convergence result for approximating solutions of the aforementioned problems.
Motivated by the works of Takahashi et. al [20], Chen and Lee [19], Chakkrid and Suantai [12] and other related works in literature, we introduce an Halpern iteration process for approximating solutions of quasi-variational inclusion and fixed point problems of demimetric and quasi-nonexpansive mappings in the framework of real Hilbert spaces. A strong convergence result for approximating solutions of the aforementioned problems was proved. We gave some consequences of our main result and also present a numerical example to display the applicability of our main result. The result presented in this paper extends and complements some related results in literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.
Let C be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \ \forall\ y \in C.$$

P_C is called the metric projection of H onto C and it is well known that P_C is a nonexpansive mapping of H onto C that satisfies the inequality:

$$||P_C x - P_C y|| \leq \langle x - y, P_C x - P_C y\rangle.$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0,$$

and

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2, \ \forall\ x \in H, \ y \in C.$$

Definition 2.1. Let Q be a convex subset of a vector space X and $f : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then,
(i) f is convex if for each $\lambda \in [0, 1]$ and $x, y \in Q$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);$$

(ii) f is called proper if there exists at least one $x \in Q$ such that

$$f(x) \neq +\infty;$$

(iii) f is lower semi-continuous at $x_0 \in Q$ if

$$f(x_0) \leq \liminf_{x \rightarrow x_0} f(x).$$

Definition 2.2. A single-valued mapping $A : H \rightarrow H$ is said to be hemi-continuous, if for any $x, y, z \in H$, the function $t \mapsto \langle A(x + ty), z\rangle$ is continuous at 0.

Lemma 2.3 ([19])**.** *Let H be a real Hilbert space, then $u \in H$ is a solution of variational inclusion (i) if and only if $u = J_{M,\rho}(u - \rho Du)$ $\forall\ \rho > 0$, i.e*

$$VI(H, D, M) = F(J_{M,\rho}(I - \rho D)), \ \forall\ \rho > 0.$$

Also, if $\rho \in (0, 2\alpha]$, then $VI(H, D, M)$ is a closed convex subset in H.

Lemma 2.4 ([19])**.** *(i) The resolvent operator $J_{M,\rho}$ associated with M is single-valued and nonexpansive for all $\rho > 0$.*
(ii) The resolvent operator $J_{M,\rho}$ is 1-ism i.e.

$$||J_{M,\rho}(x) - J_{M,\rho}(y)||^2 \leq \langle x - y, J_{M,\rho}(x) - J_{M,\rho}(y)\rangle, \ \forall\ x, y \in H.$$

Lemma 2.5. *Let H be a real Hilbert space, then the following inequalities holds:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle.$
(ii) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2, \ \forall\ x, y \in H.$

Lemma 2.6 ([21]). *Let H be a real Hilbert space, and $T : H \to H$ be a quasi-nonexpansive mapping. Set $T_\alpha = \alpha I + (1-\alpha)T$ for $\alpha \in [0,1)$. Then the following holds, for all $(x,p) \in H \times F(T)$.*
(i) $||T_\alpha x - p||^2 \leq ||x-p||^2 - \alpha(1-\alpha)||Tx - x||^2$.
(ii) $F(T_\alpha) = F(T)$.

Lemma 2.7 ([22]). *Let H be a real Hilbert space and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be an η-demimetric mapping. Then $F(T)$ is closed and convex.*

Definition 2.8. Let $T : H \to H$ be a mapping, then $I - T$ is said to be demiclosed at the origin if for any sequence $\{x_n\}$ in H, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, imply $x = Tx$.

Lemma 2.9 ([23]). *Let X be a real Banach space with X^* its dual. Let $T : X \to 2^{X^*}$ be a maximal monotone mapping and $P : X \to X^*$ be a hemicontinuous bounded monotone mapping with $Dom(T) = X$. Then the mapping $S = T + P : X \to 2^{X^*}$ is a maximal monotone mapping.*

Lemma 2.10 ([19]). *Let H be a real Hilbert space and $A : H \to H$ be an α-ism mapping, then*
(i) A is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping,
(ii) If λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda A$ is nonexpansive, where I is the identity mapping on H.

Lemma 2.11 ([24]). *Let H be a real Hilbert space, and $T : H \to H$ be β-strict pseudo-contractive mapping. Then $I - T$ is demiclosed at the origin.*

Lemma 2.12 ([25]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n \delta_n, \;\; n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence satisfying
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n \delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H, and $D : H \to H$ be an μ-ism. Let $M : H \to 2^H$ be a maximal monotone mapping and $T : H \to H$ be a quasi-nonexpansive mapping. For $\{\xi_i\}_{i=1}^m$, let $\{S_i\}_{i=1}^m : H \to H$ be a finite family of ξ_i-demimetric mappings such that $S_i - I$ is demiclosed at the origin. Suppose $\Delta := \cap_{i=1}^m F(S_i) \cap F(T) \cap VI(H, D, M) \neq \emptyset$, then the sequences $\{y_n\}, \{u_n\}$ and $\{x_n\}$ generated iteratively for $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} y_n = J_{M,\rho}(x_n - \rho D x_n); \\ u_n = y_n + \sum_{i=1}^m \theta_{n,i} \frac{1-\xi_i}{2}(S_i - I)y_n; \\ x_{n+1} = (1-\alpha_n - t_n)u_n + \alpha_n T u_n + t_n u; \end{cases} \tag{3.1}$$

where $\rho \in (0, 2\mu]$, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ is a sequence in $(0, 1-a)$ for some $a > 0$ satisfying the following conditions:

(i) $\sum_{i=1}^{m} \theta_{n,i} = 1$ *and* $\liminf_{n\to\infty} \theta_{n,i} > 0$,
(ii) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=1}^{\infty} t_n = \infty$;
(iii) $0 < \liminf \alpha_{n\to\infty} \leq \limsup \alpha_{n\to\infty} < 1$.

Then, the sequence $\{x_n\}$ *converges strongly to an element* $p = P_\Delta u$, *where* P_Δ *is the metric projection of* H *onto* Δ.

Proof. For any given $p \in \triangle$, $\rho \in (0, 2\mu]$ and Lemma 2.3, we have that $p = J_{M,\rho}(p - \rho Dp)$. Moreso, by applying Lemma 2.10, it is clear that $1 - \rho D : H \to H$ is nonexpansive. Hence, we have that

$$\begin{aligned}||y_n - p|| &= ||J_{M,\rho}(x_n - \rho Dx_n) - J_{M,\rho}(p - \rho Dp)||\\ &\leq ||x_n - \rho Dx_n - (p - \rho Dp)||\\ &\leq ||x_n - p|| \ \forall \ n \geq 0. \end{aligned} \tag{3.2}$$

From (3.1) and applying the convexity of $||.||^2$, we have

$$\begin{aligned}||u_n - p||^2 &= ||y_n + \sum_{i=1}^{m} \theta_{n,i} \frac{1-\xi_i}{2}(S_i - I)y_n - p||^2\\ &\leq \sum_{i=1}^{m} \theta_{n,i}||y_n + \frac{1-\xi_i}{2}(S_i - I)y_n - p||^2\\ &= \sum_{i=1}^{m} \theta_{n,i}\big(||y_n - p||^2 + (\frac{1-\xi_i}{2})^2||(S_i - I)y_n||^2\\ &\quad + 2(\frac{1-\xi_i}{2})\langle y_n - p, (S_i - I)y_n\rangle\big)\\ &= \sum_{i=1}^{m} \theta_{n,i}(||y_n - p||^2 + (\frac{1-\xi_i}{2})^2||(S_i - I)y_n||^2\\ &\quad - 2(\frac{1-\xi_i}{2})(\frac{1-\xi_i}{2})||(S_i - I)y_n||^2\\ &\leq ||y_n - p||^2 - \sum_{i=1}^{m} \theta_{n,i} \frac{(1-\xi)^2}{4}||(S_i - I)y_n||\\ &\leq ||y_n - p||^2. \end{aligned} \tag{3.3}$$

Hence, we have from (3.2) and (3.3) that

$$\begin{aligned}||u_n - p|| &\leq ||y_n - p||\\ &\leq ||x_n - p|| \end{aligned} \tag{3.4}$$

Using (3.1), (3.4) and utilizing the convexity of $||.||^2$, we have

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\alpha_n-t_n)u_n+\alpha_nTu_n+t_nu-p||^2\\
&= ||(1-\alpha_n-t_n)(u_n-p)+\alpha_n(Tu_n-p)+t_n(u-p)||^2\\
&\leq (1-\alpha_n-t_n)||u_n-p||^2+\alpha_n||Tu_n-p||^2+t_n||u-p||^2\\
&\leq (1-\alpha_n-t_n)||u_n-p||^2+\alpha_n||u_n-p||^2+t_n||u-p||^2\\
&= (1-t_n)||u_n-p||^2+t_n||u-p||^2 \qquad (3.5)\\
&= (1-t_n)||x_n-p||^2+t_n||u-p||^2\\
&\leq \max\{||x_n-p||^2,||u-p||^2\}\\
&\ \ \vdots\\
&\leq \max\{||x_1-p||^2,||u-p||^2\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{y_n\},\{u_n\}$ and $\{Tu_n\}$ are all bounded. Now using (3.3) and (3.5), we have that

$$\begin{aligned}
||x_{n+1}-p||^2 &= (1-t_n)||u_n-p||^2+t_n||u-p||^2\\
&\leq (1-t_n)||y_n-p||^2-(1-t_n)\sum_{i=1}^{m}\theta_{n,i}\frac{(1-\xi_i)^2}{4}||(S_i-I)y_n||^2\\
&\quad +t_n||u-p||^2\\
&\leq t_n||u-p||^2+(1-t_n)||x_n-\rho Dx_n-(p-\rho Dp)||^2\\
&\quad -(1-t_n)\sum_{i=1}^{m}\theta_{n,i}\frac{(1-\xi_i)^2}{4}||(S_i-I)y_n||^2\\
&\leq t_n||u-p||^2+(1-t_n)\{||x_n-p||^2+\rho(\rho-2\mu)||Dx_n-Dp||^2\}\\
&\quad -(1-t_n)\sum_{i=1}^{m}\theta_{n,i}\frac{(1-\xi_i)^2}{4}||(S_i-I)y_n||^2\\
&= t_n||u-p||^2+(1-t_n)||x_n-p||^2+(1-t_n)\rho(\rho-2\mu)||Dx_n-Dp||^2\\
&\quad -(1-t_n)\sum_{i=1}^{m}\theta_{n,i}\frac{(1-\xi_i)^2}{4}||(S_i-I)y_n||^2. \qquad (3.6)
\end{aligned}$$

We divide our proof into two cases.
Case 1: Assume that $\{||x_n-p||^2\}$ is a monotonically non-increasing sequence. It then follows that $\{||x_n-p||^2\}$ is convergent and hence

$$||x_n-p||-||x_{n+1}-p||\to 0,\ n\to\infty.$$

From (3.6), conditions (ii) and (iii) of (3.1), we have that

$$(1-t_n)\rho(2\mu-\rho)\|Dx_n-Dp\|^2\leq t_n\|u-p\|^2+(1-t_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2,$$

which implies that

$$\lim_{n\to\infty}\|Dx_n-Dp\|=0. \qquad (3.7)$$

Also, using (3.6), conditions (i), (ii) and (iii) of (3.1), we have that

$$\lim_{n\to\infty} \|(S_i - I)y_n\| = 0. \tag{3.8}$$

Indeed from Lemma 2.4, we obtain that

$$\begin{aligned}
\|y_n - p\|^2 &= \|J_{M,\rho}(x_n - \rho Dx_n) - J_{M,\rho}(p - \rho Dp)\|^2 \\
&\le \langle x_n - \rho Dx_n - (p - \rho Dp), y_n - p\rangle \\
&= \frac{1}{2}\{\|x_n - \rho Dx_n - (p - \rho Dp)\|^2 + \|y_n - p\|^2 \\
&\quad - \|x_n - \rho Dx_n - (p - \rho Dp) - (y_n - p)\|^2\} \\
&\le \frac{1}{2}\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n - \rho(Dx_n - Dp)\|^2\} \\
&= \frac{1}{2}\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 \\
&\quad + 2\rho\langle x_n - y_n, Dx_n - Dp\rangle - \rho^2\|Dx_n - Dp\|^2\}.
\end{aligned} \tag{3.9}$$

Hence,

$$\begin{aligned}
\|y_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\rho\langle x_n - y_n, Dx_n - Dp\rangle \\
&\quad - \rho^2\|Dx_n - Dp\|^2.
\end{aligned} \tag{3.10}$$

From (3.1) and (3.10), it is clear that

$$\begin{aligned}
||x_{n+1} - p||^2 &\le (1 - t_n)||y_n - p||^2 + t_n||u - p||^2 \\
&\le (1 - t_n)[||x_n - p||^2 - ||x_n - y_n||^2 \\
&\quad + 2\rho\langle x_n - y_n, Dx_n - Dp\rangle - \rho^2||Dx_n - Dp||^2] + t_n||u - p||^2 \\
&= (1 - t_n)||x_n - p||^2 - (1 - t_n)||x_n - y_n||^2 \\
&\quad + 2(1 - t_n)\rho\langle x_n - y_n, Dx_n - Dp\rangle \\
&\quad - \rho^2(1 - t_n)||Dx_n - Dp||^2 + t_n||u - p||^2.
\end{aligned} \tag{3.11}$$

We have from (3.11) that

$$\begin{aligned}
(1 - t_n)||x_n - y_n||^2 &\le (1 - t_n)||x_n - p||^2 - ||x_{n+1} - p||^2 - \rho^2(1 - t_n)||Dx_n - Dp||^2 \\
&\quad + 2(1 - t_n)\rho\langle x_n - y_n, Dx_n - Dp\rangle + t_n||u - p||^2.
\end{aligned} \tag{3.12}$$

Using (3.12) and condition (ii) of (3.1), we obtain that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.13}$$

By applying Lemma 2.6 (i) and Lemma 2.5, we get

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1 - \alpha_n)(u_n - p) + \alpha_n(Tu_n - p) + t_n(p - u_n)||^2 \\
&= ||(1 - \alpha_n)(u_n - p) + \alpha_n(Tu_n - p)||^2 + t_n^2||u_n - p||^2 \\
&\quad + 2t_n\langle p - u_n, (1 - \alpha_n)(u_n - p) + \alpha_n(Tu_n - p)\rangle \\
&\le ||u_n - p||^2 - \alpha_n(1 - \alpha_n)||Tu_n - u_n||^2 + t_n^2||u_n - p||^2 \\
&\quad + 2t_n\langle p - u_n, (1 - \alpha_n)(u_n - p) + \alpha_n(Tu_n - p)\rangle \\
&\le ||x_n - p||^2 - \alpha_n(1 - \alpha_n)||Tu_n - u_n||^2 \\
&\quad + t_n\big[t_n||x_n - p||^2 + 2\langle p - u_n, (1 - \alpha_n)(u_n - p) + \alpha_n(Tu_n - p)\rangle\big].
\end{aligned} \tag{3.14}$$

From (3.14), we have that

$$\alpha_n(1-\alpha_n)||Tu_n - u_n||^2 \le ||x_n - p||^2 - ||x_{n+1} - p||^2 + t_n\big[t_n||x_n - p||^2 + 2\langle p - u_n, (1-\alpha_n)(u_n - p) + \alpha_n(Tu_n - p)\rangle\big].$$

From condition (ii) and (iii) of (3.1), we have that

$$\lim_{n\to\infty} ||Tu_n - u_n|| = 0. \tag{3.15}$$

From (3.1) and (3.8), we have that

$$||u_n - y_n|| \le \sum_{i=1}^{m} \theta_{n,i} \frac{1-\xi_i}{2} ||S_i y_n - y_n|| \to 0, \text{as } n \to \infty. \tag{3.16}$$

Using (3.13) and (3.16), we have that

$$||u_n - x_n|| \le ||u_n - y_n|| + ||x_n - y_n|| \to 0, \text{ as } n \to \infty. \tag{3.17}$$

Also, using condition (i) of (3.1) and (3.17), we have that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.18}$$

Since $\{x_n\}$ is bounded, there exists a subsequences $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $x^* \in H$. Consequently, from (3.13) and (3.17), we also have subsequence $\{y_{n_j}\}$ and $\{u_{n_j}\}$ which converges weakly to $x^* \in H$. By (3.16) and the demiclosedness principle of $T - I$, we get that $x^* \in F(T)$. Now, from (3.8) and the demiclosedness of $S_i - I$ at the origin, we also get that $x^* \in \cap_{i=1}^m F(S_i)$.
Next, we prove that $x^* \in VI(H, D, M)$.
Since D is μ-ism, it follows from Lemma 2.10 that D is an $\frac{1}{\alpha}$-Lipschitz continuous mapping and $Dom(D) = H$. By applying Lemma 2.9, we have that $M + D$ is maximal monotone. Let $(a, f) \in Gra(M + D), i.e. f - Da \in M(a)$. Now, since we have that $y_{n_j} = J_{M,\rho}(x_{n_j} - \rho D x_{n_j})$, then $x_{n_j} - \rho D x_{n_j} \in (I + \rho M)(y_{n_j})$ which implies that

$$\frac{1}{\rho}(x_{n_j} - y_{n_j} - \rho D x_{n_j}) \in M(y_{n_j}).$$

Using the fact that $M + D$ is maximal monotone, we have

$$\langle a - y_{n_j}, f - Aa - \frac{1}{\rho}\{x_{n_j} - y_{n_j} - \rho D x_{n_j}\}\rangle \ge 0,$$

and so

$$\begin{aligned}\langle a - y_{n_j}, f\rangle &\ge \langle a - y_{n_j}, Da + \frac{1}{\rho}\{x_{n_j} - y_{n_j} - \rho D x_{n_j}\}\rangle\\ &= \langle a - y_{n_j}, Da - Dy_{n_j} + Dy_{n_j} - Dx_{n_j} + \frac{1}{\rho}\{x_{n_j} - y_{n_j}\}\rangle\\ &\ge 0 + \langle a - y_{n_j}, Dy_{n_j} - Dx_{n_j}\rangle\\ &+ \langle a - y_{n_j}, \frac{1}{\rho}\{x_{n_j} - y_{n_j}\}\rangle.\end{aligned} \tag{3.19}$$

From (3.13), we have that

$$\lim_{n\to\infty} ||Dx_{n_j} - Dy_{n_j}|| = 0. \tag{3.20}$$

Using the fact that $y_{n_j} \rightharpoonup x^*$ and substituting (3.20) into (3.19), we have

$$\lim_{j\to\infty} \langle a - y_{n_j}, f\rangle = \langle a - x^*, f\rangle \geq 0.$$

Since $A + M$ is maximal monotone, this implies that $\theta \in (M + D)(x^*)$ *i.e.* $x^* \in VI(H, M, D)$. Hence, we conclude that $x^* \in \Omega = \cap_{i=1}^m F(S_i) \cap F(T) \cap VI(H, M, D)$.
Now, we prove that $\{x_n\}$ converges strongly to x^*. From (3.1), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||(1 - \alpha_n - t_n)u_n + \alpha_n T u_n + t_n u - x^*||^2 \\
&= ||(1 - \alpha_n - t_n)(u_n - x^*) + \alpha_n(Tu_n - x^*) + t_n(u - x^*)||^2 \\
&\leq ||(1 - \alpha_n - t_n)(u_n - x^*) + \alpha_n(Tu_n - x^*)||^2 \\
&\quad + 2t_n\langle x_{n+1} - x^*, u - x^*\rangle \\
&\leq \left[(1 - \alpha_n - t_n)||u_n - x^*|| + \alpha_n||Tu_n - x^*||\right]^2 \\
&\quad + 2t_n\langle x_{n+1} - x^*, u - x^*\rangle \\
&\leq (1 - t_n)^2||u_n - x^*||^2 + 2t_n\langle x_{n+1} - x^*, u - x^*\rangle \\
&\leq (1 - t_n)||x_n - x^*||^2 + 2t_n\langle x_{n+1} - x^*, u - x^*\rangle. \qquad (3.21)
\end{aligned}$$

Since $x_n \to x^*$, then $\langle x_{n+1} - x^*, u - x^*\rangle \to 0$. Using Lemma 2.12 in (3.21), we obtain that $||x_n - x^*|| \to 0,$ as $n \to \infty$. Therefore, $\{x_n\} \to x^* \in \Omega$.
Case II: Assume that $\{||x_n - x^*||^2\}$ is not a monotone decreasing sequence. Set $\Gamma_n := ||x_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large enough n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, $\{\tau(n)\}$ is a non decreasing sequence such that $\tau(n) \to \infty$. Hence,

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \text{ for } n \geq n_0.$$

It follows from (3.6) that

$$\begin{aligned}
&(1 - t_{\tau(n)}) \sum_{i=1}^m \theta_{\tau(n),i} \frac{(1 - \xi_i)^2}{4} ||(S_i - I)y_{\tau(n)}||^2 \\
&\leq t_{\tau(n)}||u - x^*||^2 + (1 - t_{\tau(n)})||x_{\tau(n)} - x^*||^2 - ||x_{\tau(n)+1} - x^*||^2.
\end{aligned}$$

Hence, we have from conditions (i) and (ii) of (3.1) that

$$\lim_{\tau(n)\to\infty} ||(S_i - I)y_{\tau(n)}|| = 0.$$

Following the same manner as in case 1, we can show that

$$\lim_{\tau(n)\to\infty} ||Dx_{\tau(n)} - Dp|| = 0. \qquad (3.22)$$

and

$$\lim_{\tau(n)\to\infty} ||Tu_{\tau(n)} - u_{\tau(n)}|| = 0. \qquad (3.23)$$

Now, for all $n \geq n_0$, we have from (3.21) that

$$\begin{aligned}
0 &\leq ||x_{\tau(n)+1} - x^*||^2 - ||x_{\tau(n)} - x^*||^2 \\
&\leq (1 - t_{\tau(n)})^2 ||u_{\tau(n)} - x^*||^2 + 2t_{\tau(n)} \langle x_{\tau(n)+1} - x^*, u - x^* \rangle - ||x_{\tau(n)} - x^*||^2 \\
&\leq (1 - t_{\tau(n)}) ||x_{\tau(n)} - x^*||^2 + 2t_{\tau(n)} \langle x_{\tau(n)+1} - x^*, u - x^* \rangle - ||x_{\tau(n)} - x^*||^2.
\end{aligned}$$

Thus,

$$||x_{\tau(n)} - x^*||^2 \leq 2\langle x_{n+1} - x^*, u - x^* \rangle \to 0.$$

Hence,

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)} - x^*|| = 0.$$

Therefore,

$$\lim_{\tau(n)\to\infty} \Gamma_{\tau(n)} = \lim_{\tau(n)\to\infty} \Gamma_{\tau(n)+1} = 0.$$

More so, for $n \geq n_0$, it can be seen that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_k > \Gamma_{k+1}$ for $\{\tau(n) + 1\} \leq k < n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

So, $\lim_{n\to\infty} \Gamma_n = 0$. Hence, $\{x_n\}$ converges strongly to $x^* \in \Omega$. ∎

Remark 3.2. The problem discussed in this paper extends work of Takahashi, Wen and Yao [20] in the sense that variational inequality problem is a special case of problem (1.1) discussed in this paper. Moreso, the mappings considered in this paper generalizes the one considered by Lee and Chan [19].

Remark 3.3. The iterative scheme considered in this article has an advantage over the one considered in [20] in the sense that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in a practical calculation of the iterative sequence. The Halpern iteration considered in this article provides more flexibility in defining the algorithm parameters which is important for the numerical implementation perspective.

Corollary 3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H, and $D : H \to H$ be an μ-ism. Let $M : H \to 2^H$ be a maximal monotone mapping and $T : H \to H$ be a quasi-nonexpansive mapping. Suppose $\Delta := F(T) \cap VI(H, D, M) \neq \emptyset$, then the sequences $\{y_n\}$ and $\{x_n\}$ generated iteratively for $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} y_n = J_{M,\rho}(x_n - \rho D x_n); \\ x_{n+1} = (1 - \alpha_n - t_n) y_n + \alpha_n T y_n + t_n u; \end{cases} \tag{3.24}$$

where $\rho \in (0, 2\mu]$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ is a sequence in $(0, 1 - a)$ for some $a > 0$ satisfying the following conditions:

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=1}^{\infty} t_n = \infty$;
(ii) $0 < \liminf \alpha_n \leq \limsup \alpha_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $p = P_\Delta u$.

4. Application and Numerical Example

4.1. Variational Inequality Problem

Let C be a nonempty closed and convex subset of H and $D : C \to H$ be a mapping. Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Dx, y - x\rangle \geq 0, \ \forall \ y \in C. \tag{4.1}$$

We denote by $VI(C, D)$, the solution set of (4.1). It is known that x is a solution set of (4.1) if and only if x is a fixed point of the mapping $P_C(I - \lambda D)$, where I denotes the identity on H. Let i_C be a function defined by $i_C(x) = 0, \ x \in C$ and $i_C(x) = \infty, \ x \notin C$. It is easy to see that i_C is a proper, convex and lower semicontinuous function on H, and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent $J_{i_C,\rho} = (I + \rho\partial_{i_C})^{-1}$ of the subdifferential operator ∂_{i_C}. Letting $x = J_{i_C,\rho}y$, we have that

$$y \in x + \rho\partial_{i_C}x \Leftrightarrow y \in x + \rho N_C x \Leftrightarrow x = P_C y,$$

where $N_C x := \{e \in H : \langle e, v - x\rangle \ \forall \ x \in C\}$. On substituting $M = \partial_{i_C}$ in Theorem 3.1, we have the following result.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H, and $D : H \to H$ be an μ-ism. Let $T : H \to H$ be a quasi-nonexpansive mapping. For $\{\xi_i\}_{i=1}^m$, let $\{S_i\}_{i=1}^m : H \to H$ be a finite family of ξ_i-demimetric mappings such that $S_i - I$ is demiclosed at the origin. Suppose $\Delta := \cap_{i=1}^m F(S_i) \cap F(T) \cap VI(C, D) \neq \emptyset$, then the sequences $\{y_n\}, \{u_n\}$ and $\{x_n\}$ generated iteratively for $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} y_n = P_C(x_n - \rho D x_n); \\ u_n = y_n + \sum_{i=1}^m \theta_{n,i} \frac{1-\xi_i}{2}(S_i - I)y_n; \\ x_{n+1} = (1 - \alpha_n - t_n)u_n + \alpha_n T u_n + t_n u; \end{cases} \tag{4.2}$$

where $\rho \in (0, 2\mu]$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ is a sequence in $(0, 1 - a)$ for some $a > 0$ satisfying the following conditions:

(i) $\sum_{i=1}^m \theta_{n,i} = 1$ *and* $\liminf_{n\to\infty} \theta_{n,i} > 0$,
(ii) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=1}^\infty t_n = \infty$;
(iii) $0 < \liminf \alpha_{n\to\infty} \leq \limsup \alpha_{n\to\infty} < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $p = P_\Delta u$.

4.2. Numerical Example

Example 4.2. Let $H = \mathbb{R}$, the set of all real numbers, with inner product defined by $\langle x, y\rangle = xy \ \forall \ x, y \in \mathbb{R}$, and induced usual norm $|.|$. Let $M(x) = \{3x\} \ \forall \ x \in \mathbb{R}$ and define $D : \mathbb{R} \to \mathbb{R}$ by $D(x) = x + 6$ with $\mu = \frac{1}{4}$. Suppose $S(x) = \frac{-9}{2}x, \ \forall \ x \in \mathbb{R}$ with $\xi = \frac{7}{11}$ and $T(x) = \frac{2}{3}x$, with $F(T) = \{0\}, \forall \ x \in \mathbb{R}$, then let $\alpha_n = \frac{n+1}{2(n+1)}$ and $t_n = \frac{1}{2(n+2)}$, then (3.1) becomes

$$\begin{cases} u_n = \frac{16-3x_n}{4}; \\ u_n = y_n + \sum_{i=1}^m \theta_{n,i} \frac{1-\xi_i}{2}(S_i - I)y_n; \\ x_{n+1} = \frac{n+1}{2(n+2)} + \frac{n+1}{3(n+2)} + \frac{t_n}{2(n+2)}. \end{cases}$$

Case 1: $x_1 = (2, 1)^T, \ u = (-1, 3)^T$.
Case 2: $x_1 = (-0.2, -1)^T, \ u = (-1, 1)^T$.

Case 3: $x_1 = (1, 2)^T,\ u = (4, 1)^T$.
Case 4: $x_1 = (400, 100)^T,\ u = (300, 100)^T$.

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