



Dedicated to Prof. Suthep Suantai on the occasion of his 60th anniversary

On Novel Common Fixed Point Results for Enriched Nonexpansive Semigroups

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Abstract In this paper, we introduce a new semigroup of an enriched nonexpansive mapping in the sense of Berinde, namely, an enriched nonexpansive semigroup. Additionally, we establish some weak and strong convergence results for enriched nonexpansive semigroups to approximate common fixed points using Mann iterative process in uniformly convex Banach spaces. An illustrative example supporting our results is given in the last section.

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1. INTRODUCTION

Let X be a normed space and C be a nonempty subset of X . Recall that a mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$ and, we use $F(T)$ to denote the set of all fixed points of T , that is, $F(T) = \{x \in C : Tx = x\}$. The class of nonexpansive mappings is one of important classes of nonlinear mappings. Moreover, since many problems can be seen as a fixed point problems of nonexpansive mappings, an approximation of fixed points for nonexpansive mappings has a variety of specific applications such as minimization problems, variational inequality problems and nonlinear evolution equations. For instance, the reader can see in [1–3] and references therein. During the last forty years, there are several existence and convergence results of fixed points for nonexpansive mappings.

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Very recently, Berinde [4] introduced a new larger class of nonexpansive mappings, which will be called an enriched nonexpansive mapping. Its definition is given in the case of normed spaces as follows:

Definition 1.1 ([4]). Let $(X, \|\cdot\|)$ be a normed space. A mapping $T : X \rightarrow X$ is said to be an *enriched nonexpansive mapping* if there exists $b \in [0, \infty)$ such that

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\|, \quad (1.1)$$

for all $x, y \in X$. To indicate the constant involved in (1.1), we shall also call T as a *b-enriched nonexpansive mapping*.

It is easily seen that any nonexpansive mapping is a 0-enriched nonexpansive mapping, but the converse is not true. In order to claim this fact, Berinde give the following example.

Example 1.2. Let $X = [\frac{1}{2}, 2]$ be endowed with the usual norm and $T : X \rightarrow X$ be defined by $Tx = \frac{1}{x}$ for all $x \in [\frac{1}{2}, 2]$. Then T is a $\frac{3}{2}$ -enriched nonexpansive mapping but T is not a nonexpansive mapping (see [4] for more details).

Note that any enriched nonexpansive mapping is continuous which is similar to the case of nonexpansive mapping.

On the other hand, one parameter semigroups of nonlinear self-mappings has been studied by many researchers via several types of mappings (see [5–7] and references therein). It is well known that the construction of common fixed points of semigroups is an important subject in the nonlinear operator theory and their applications (see [8–10]).

Inspired by the above results, the purpose of this paper is to introduce a new semigroup of an enriched nonexpansive mapping in the sense of Berinde [4]. Some weak and strong convergence results for enriched nonexpansive semigroups using Mann iterative process in uniformly convex Banach spaces are investigated. To support our results, we also present an illustrative numerical example.

2. PRELIMINARIES

Throughout this paper, let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X , and G be an unbounded subset of $[0, \infty)$ such that for all $s, t \in G$, we have

$$s + t \in G \text{ and if } s > t, \text{ then } s - t \in G, \quad (2.1)$$

e.g. $G = [0, \infty)$, $G = \mathbb{N}$, or $G = \mathbb{N} \cup \{0\}$.

In order to obtain our main results, we need some definitions and lemmas given in the following:

Definition 2.1. Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying (2.1), and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be a family of self mappings. A point $x \in C$ is said to be a *common fixed point* of τ if $T_t x = x$ for all $T_t \in \tau$. Denote by $Fix(\tau)$ the set of all common fixed points of τ .

Definition 2.2. Let C be a nonempty closed convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X . The *asymptotic radius* of $\{x_n\}$ with respect to C is denoted by $r_C(\{x_n\})$, and it is defined by

$$r_C(\{x_n\}) = \inf_{y \in C} \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

The asymptotic center of $\{x_n\}$ with respect to C is denoted by $A_C(\{x_n\})$ and it is defined by

$$A_C(\{x_n\}) = \{y \in C : \limsup_{n \rightarrow \infty} \|x_n - y\| = r_C(\{x_n\})\}.$$

It is well known that in a uniformly convex Banach space, $A_C(\{x_n\})$ consists exactly one point.

Definition 2.3. Let C be a nonempty subset of a Banach space X . A mapping $T : X \rightarrow C$ is said to be *demiclosed with respect to* $y \in X$, if for each sequence $\{x_n\}$ in C which converges weakly to $x \in C$ and $\{Tx_n\}$ converges strongly to y imply that $Tx = y$.

Definition 2.4 ([11]). A Banach space X satisfies the Opial’s condition if for each sequence $\{x_n\}$ in X which converges weakly to $x \in X$, we have

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|,$$

for all $y \in X$ with $y \neq x$.

Lemma 2.5 ([12]). Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the following assertions hold:

- (1) $\lim_{n \rightarrow \infty} a_n$ exists;
- (2) $\lim_{n \rightarrow \infty} a_n = 0$ whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 2.6 ([13]). Let $\{a_n\} \subset \mathbb{R}^+, \{b_n\} \subset \mathbb{R}$ and $\{\delta_n\} \subset (0, 1)$ be sequences satisfying the inequality

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n, \quad n \geq 1.$$

If (i) $\sum_{n=1}^{\infty} \delta_n = \infty$ and (ii) $\limsup_{n \rightarrow \infty} \frac{b_n}{\delta_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < \infty$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7 ([14]). Suppose that X is a uniformly convex Banach space and $0 < \alpha_n < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq d$,

$\limsup_{n \rightarrow \infty} \|y_n\| \leq d$ and $\lim_{n \rightarrow \infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = d$ for some $d \geq 0$.

Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2.8 ([15]). Let C be a closed convex subset of a uniformly convex Banach space X satisfying the Opial’s condition. If a sequence $\{x_n\} \subset C$ converges weakly to a point x , then x is the asymptotic center of $\{x_n\}$ in C .

3. ENRICHED NONEXPANSIVE SEMIGROUPS

In this section, we introduce a new semigroup of an enriched nonexpansive mapping in the sense of Berinde, namely, an enriched nonexpansive semigroup.

Definition 3.1. Let C be a nonempty closed convex subset of a uniformly convex Banach space X and G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1). Then the family $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ is said to be an *enriched nonexpansive semigroup* on C if the following conditions are satisfied:

- (B₁) $T_{s+t}x = T_sT_tx$ for all $s, t \in G$ and $x \in C$;
- (B₂) for all $x \in C$, the mapping $G \ni t \mapsto T_tx$ is continuous;
- (B₃) for each $t \in G$, $T_t : C \rightarrow C$ is an enriched nonexpansive mapping on C , i.e., there is a constant $b_t \in [0, \infty)$ such that

$$\|b_t(x - y) + T_tx - T_ty\| \leq (b_t + 1)\|x - y\|, \tag{3.1}$$

for all $x, y \in C$.

In the case of $b_t = b \in [0, \infty)$ for all $t \in G$ in (3.1), the family $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ is said to be a *b-enriched nonexpansive semigroup* on C . It is easy to see that any *b-enriched nonexpansive semigroup* is an enriched nonexpansive semigroup.

Now, we give the following example, which is an enriched nonexpansive semigroup.

Example 3.2. Let $X = \mathbb{R}$ (the set of real numbers) be equipped with the usual norm, $C = [-1, \frac{1}{2}]$ be a nonempty closed convex subset of X and let $\tau = \{T_t : C \rightarrow C \mid t \in \mathbb{N}\}$ be defined by

$$T_tx = \begin{cases} xe^{-t} & \text{if } x \in [-1, 0], \\ -3xe^{-t} & \text{if } x \in (0, \frac{1}{2}], \end{cases} \tag{3.2}$$

for all $t \in \mathbb{N}$.

Now we claim that τ is an enriched nonexpansive semigroup on C , i.e., the family $\tau = \{T_t : C \rightarrow C \mid t \in \mathbb{N}\}$ satisfies all conditions in Definition 3.1. It is easy to see that the family τ satisfies (B₂), i.e., for all $x \in C$, the mapping $\mathbb{N} \ni t \mapsto T_tx$ is continuous.

Next, we will show that the family τ satisfies (B₁), i.e., $T_{s+t}x = T_sT_tx$ for all $s, t \in \mathbb{N}$ and $x \in C$. Let $x \in C$ and let $s, t \in \mathbb{N}$. Then there are two cases as follows:

- Case I: If $x \in [-1, 0]$, we get

$$T_{s+t}x = xe^{-(s+t)} = e^{-s}(xe^{-t}) = e^{-s}(T_tx) = T_sT_tx.$$

- Case II: Assume that $x \in (0, \frac{1}{2}]$. From (3.2), we have

$$T_{s+t}x = -3xe^{-(s+t)}$$

and

$$T_sT_tx = T_s(-3xe^{-t}) = (-3xe^{-t})e^{-s} = -3xe^{-(s+t)}.$$

Therefore, the family τ satisfies (B₁), i.e.,

$$T_{s+t}x = T_sT_tx, \text{ for all } s, t \in \mathbb{N} \text{ and } x \in C.$$

Finally, we verify that for each $t \in \mathbb{N}$, T_t is an enriched nonexpansive mapping satisfying (3.1) with $b_t = 1.2$. For each $x, y \in C$ with $x = y$, it is clearly that $\|b_t(x - y) + T_tx - T_ty\| = 0$ and so (3.1) holds.

Now, we assume that $x \neq y$ and consider the following cases.

- Case I: If $x, y \in [-1, 0]$, we get

$$\begin{aligned}
 \|b_t(x - y) + T_t x - T_t y\| &= |1.2(x - y) + xe^{-t} - ye^{-t}| \\
 &= |(1.2 + e^{-t})(x - y)| \\
 &= |1.2 + e^{-t}| |x - y| \\
 &\leq \left|1.2 + \frac{1}{e}\right| |x - y| \\
 &\leq (1.2 + 1) |x - y| \\
 &= (b_t + 1) \|x - y\|.
 \end{aligned}$$

- Case II: If $x, y \in (0, \frac{1}{2}]$, we get

$$\begin{aligned}
 \|b_t(x - y) + T_t x - T_t y\| &= |1.2(x - y) + (-3)xe^{-t} - (-3)ye^{-t}| \\
 &= |(1.2 - 3e^{-t})(x - y)| \\
 &= |1.2 - 3e^{-t}| |x - y| \\
 &\leq (1.2 + 1) |x - y| \\
 &= (b_t + 1) \|x - y\|.
 \end{aligned}$$

- Case III: Let $(x, y) \in ([-1, 0] \times (0, \frac{1}{2}]) \cup ((0, \frac{1}{2}] \times [-1, 0])$. Without loss of generality, we may assume that $x \in [-1, 0]$ and $y \in (0, \frac{1}{2}]$. First, we will show that

$$|(1.2 - 3e^{-t})(x - y) + 4xe^{-t}| \leq 2.2(y - x).$$

Since $x \in [-1, 0]$ and $y > 0$, we get

$$\begin{aligned}
 \left(\frac{3.4 + e^{-t}}{3.4 - 3e^{-t}}\right) x &\leq y, \\
 (3.4 + e^{-t}) x &\leq (3.4 - 3e^{-t})y, \\
 (3.4 + 4e^{-t} - 3e^{-t}) x &\leq (3.4 - 3e^{-t})y, \\
 (3.4 - 3e^{-t}) x + 4e^{-t} x &\leq (3.4 - 3e^{-t})y, \\
 4e^{-t} x &\leq (3.4 - 3e^{-t})(y - x).
 \end{aligned} \tag{3.3}$$

Similarly, we get

$$\begin{aligned}
 \left(\frac{1 - e^{-t}}{1 + 3e^{-t}}\right) x &\leq y, \\
 (1 - e^{-t}) x &\leq (1 + 3e^{-t})y, \\
 (-1 - 3e^{-t})y &\leq (-1 + e^{-t})x, \\
 (-1 - 3e^{-t})y &\leq (-1 - 3e^{-t} + 4e^{-t})x, \\
 (-1 - 3e^{-t})(y - x) &\leq 4e^{-t}x.
 \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned}
 (-1 - 3e^{-t})(y - x) &\leq 4xe^{-t} \leq (3.4 - 3e^{-t})(y - x) \\
 \implies (-2.2 - 3e^{-t} + 1.2)(y - x) &\leq 4xe^{-t} \leq (2.2 - 3e^{-t} + 1.2)(y - x) \\
 \implies -2.2(y - x) &\leq (3e^{-t} - 1.2)(y - x) + 4xe^{-t} \leq 2.2(y - x) \\
 \implies -2.2(y - x) &\leq (1.2 - 3e^{-t})(x - y) + 4xe^{-t} \leq 2.2(y - x),
 \end{aligned} \tag{3.5}$$

that is,

$$|(1.2 - 3e^{-t})(x - y) + 4xe^{-t}| \leq 2.2(y - x). \tag{3.6}$$

Then, by (3.6) we have

$$\begin{aligned}
 \|b_t(x - y) + T_t x - T_t y\| &= |1.2(x - y) + xe^{-t} - (-3)ye^{-t}| \\
 &= |1.2(x - y) + 3ye^{-t} - 3xe^{-t} + 4xe^{-t}| \\
 &= |(1.2 - 3e^{-t})(x - y) + 4xe^{-t}| \\
 &\leq 2.2(y - x) \\
 &= (1.2 + 1)|x - y| \\
 &= (b_t + 1)\|x - y\|.
 \end{aligned}$$

Thus, for all possible cases, it has been proven that for all $t \in \mathbb{N}$, T_t is an enriched nonexpansive mapping on C satisfying (3.1) with $b_t = 1.2$. Therefore, the family $\tau = \{T_t : C \rightarrow C | t \in \mathbb{N}\}$ is an enriched nonexpansive semigroup on C .

Remark 3.3. Example 3.2 shows that for all $t \in \mathbb{N}$, T_t is an enriched nonexpansive mapping on C , but T_t is not a nonexpansive mapping on C for some $t \in \mathbb{N}$. Indeed, for $t = 1, x = 0$ and $y = 0.1$, we have

$$\|T_1 x - T_1 y\| = |0 - (-0.3)e^{-1}| > 0.11 > 0.1 = \|0 - 0.1\| = \|x - y\|.$$

Hence T_1 is not a nonexpansive mapping on C .

4. CONVERGENCE THEOREMS FOR ENRICHED NONEXPANSIVE SEMIGROUPS

In this section, we obtain weak and strong convergence theorems for enriched nonexpansive semigroups using the Mann iterative process in uniformly convex Banach spaces.

Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C | t \in G\}$ be an enriched nonexpansive semigroup on C . For each $t \in G$, we have T_t is an enriched nonexpansive mapping on C , and so there is a constant $b_t \in [0, \infty)$ such that

$$\|b_t(x - y) + T_t x - T_t y\| \leq (b_t + 1)\|x - y\|,$$

for all $x, y \in C$. If we let $\mu_t := \frac{1}{1+b_t}$, then $\mu_t \in (0, 1]$. Putting $b_t = \frac{1}{\mu_t} - 1$ in the above inequality, we have

$$\|(1 - \mu_t)(x - y) + \mu_t T_t x - \mu_t T_t y\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote $\bar{T}_t^{\mu_t} x = (1 - \mu_t)x + \mu_t T_t x$, that is, the averaged operator of T_t and then

$$\|\bar{T}_t^{\mu_t} x - \bar{T}_t^{\mu_t} y\| \leq \|x - y\|,$$

for all $x, y \in C$. Hence, the averaged operator $\bar{T}_t^{\mu_t}$ is a nonexpansive mapping for all $t \in G$. Consequently, if $\bar{\tau} = \{\bar{T}_t^{\mu_t} : C \rightarrow C \mid t \in G\}$ is a family of the averaged mapping associated to T_t for all $t \in G$, given by

$$\bar{T}_t^{\mu_t} = (1 - \mu_t)I + \mu_t T_t, \tag{4.1}$$

for all $t \in G$, where I is the identity mapping and $\mu_t = \frac{1}{b_t+1} \in (0, 1]$, then $Fix(\tau) = Fix(\bar{\tau})$ since T_t and $\bar{T}_t^{\mu_t}$ have the same fixed points for all $t \in G$.

Next, we introduce a Mann iterative process $\{x_n\}$ in C in the setting of a semigroup τ as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T_{t_n} x_n, \end{cases} \tag{4.2}$$

for all $n \in \mathbb{N}$, where $\{t_n\} \subset G$, $T_{t_n} \in \tau$ for all $t_n \in G$ and $\{\lambda_n\} \subset [0, 1]$. Let $\{\beta_n\} \subset [0, 1]$ and $x_0 \in C$. If we use the same defining in (4.1) with $\bar{T}_{t_n}^{\mu_{t_n}} \in \bar{\tau}$ corresponding $T_{t_n} \in \tau$, we get

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)x_n + \beta_n \bar{T}_{t_n}^{\mu_{t_n}} x_n \\ &= (1 - \beta_n)x_n + \beta_n((1 - \mu_{t_n})x_n + \mu_{t_n} T_{t_n} x_n) \\ &= (1 - \mu_{t_n} \beta_n)x_n + \mu_{t_n} \beta_n T_{t_n} x_n \\ &= (1 - \lambda_n)x_n + \lambda_n T_{t_n} x_n, \end{aligned} \tag{4.3}$$

for all $n \in \mathbb{N}$, where $\{t_n\} \subset G$, $\lambda_n := \mu_{t_n} \beta_n \in [0, 1]$.

Now we define the condition as follows:

Definition 4.1. Let C be a nonempty subset of a normed space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1), $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be an enriched nonexpansive semigroup on C , and $\{x_n\}$ be a sequence defined by the iterative scheme (4.2). A sequence $\{t_n\} \subset G$ in (4.2) is said to satisfy the condition (T) if $\mu_{t_n} := \frac{1}{1+b_{t_n}} \rightarrow \mu$ as $n \rightarrow \infty$ for some $\mu \in (0, 1]$, where b_{t_n} is a constant involving a b_{t_n} -enriched nonexpansive mapping.

We first prove the following lemmas which will be used in our main results.

Lemma 4.2. Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be an enriched nonexpansive semigroup on C with $Fix(\tau) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (4.2). Then the following assertions hold:

- (i) $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists for all $z \in Fix(\tau)$;
- (ii) $\{x_n\}$ is bounded;
- (iii) if the sequence $\{t_n\} \subset G$ in (4.2) satisfies the condition (T), then $\lim_{n \rightarrow \infty} \|x_n - T_{t_n} x_n\| = 0$.

Proof. (i) Let $n \in \mathbb{N}$. For a fixed point $z \in \text{Fix}(\tau) = \text{Fix}(\bar{\tau})$, we have $T_t z = z$ for all $t \in G$, and by (4.3) hence

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1 - \lambda_n)x_n + \lambda_n T_{t_n} x_n - z\| \\ &= \|(1 - \beta_n)x_n + \beta_n \bar{T}_{t_n}^{\mu_{t_n}} x_n - z\| \\ &= \|(1 - \beta_n)(x_n - z) + \beta_n (\bar{T}_{t_n}^{\mu_{t_n}} x_n - z)\| \\ &\leq (1 - \beta_n)\|x_n - z\| + \beta_n \|\bar{T}_{t_n}^{\mu_{t_n}} x_n - z\| \\ &\leq (1 - \beta_n)\|x_n - z\| + \beta_n \|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{4.4}$$

Since the sequence $\{\|x_n - z\|\}$ is non-increasing and bounded below, we have $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. This means that (i) holds.

(ii) It can be obtained from (i).

(iii) Suppose that the sequence $\{t_n\} \subset G$ in (4.2) satisfies the condition (T). Let $z \in \text{Fix}(\tau)$. By (i), there is $c \geq 0$ such that

$$\lim_{n \rightarrow \infty} \|x_n - z\| = c. \tag{4.5}$$

By (4.3) and (4.5), we have

$$c = \lim_{n \rightarrow \infty} \|x_{n+1} - z\| = \lim_{n \rightarrow \infty} \|(1 - \beta_n)(x_n - z) + \beta_n (\bar{T}_{t_n}^{\mu_{t_n}} x_n - z)\|. \tag{4.6}$$

Since $\bar{T}_{t_n}^{\mu_{t_n}}$ is a nonexpansive mapping, by (4.5), we get

$$\limsup_{n \rightarrow \infty} \|\bar{T}_{t_n}^{\mu_{t_n}} x_n - z\| \leq \limsup_{n \rightarrow \infty} \|x_n - z\| = c. \tag{4.7}$$

From (4.5), (4.6), (4.7) and using Lemma 2.7, we get

$$\lim_{n \rightarrow \infty} \|x_n - \bar{T}_{t_n}^{\mu_{t_n}} x_n\| = 0. \tag{4.8}$$

From (4.1), we have $\bar{T}_{t_n}^{\mu_{t_n}} x = (1 - \mu_{t_n})x + \mu_{t_n} T_{t_n} x$ for all $x \in C$. Then

$$\begin{aligned} \|x_n - \bar{T}_{t_n}^{\mu_{t_n}} x_n\| &= \|x_n - (1 - \mu_{t_n})x_n - \mu_{t_n} T_{t_n} x_n\| \\ &= \|\mu_{t_n} x_n - \mu_{t_n} T_{t_n} x_n\| \\ &= \mu_{t_n} \|x_n - T_{t_n} x_n\|, \end{aligned}$$

for all $n \in \mathbb{N}$. It follows from the condition (T) and (4.8) together with the above equality that

$$\lim_{n \rightarrow \infty} \|x_n - T_{t_n} x_n\| = 0.$$

■

Lemma 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be a b -enriched nonexpansive semigroup on C . Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (4.2). Then $\text{Fix}(\tau) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$.*

Proof. Suppose that $Fix(\tau) \neq \emptyset$. Since τ is a b -enriched nonexpansive semigroup on C , the sequence $\{t_n\} \subset G$ in (4.2) satisfies the condition (\mathbb{T}) . By Lemma 4.2, we have $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - T_{t_n}x_n\| = 0$, for all $\{t_n\} \subset G$. Next, we will show that $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$.

For each given $t \in G$, we have

$$\begin{aligned} \|x_n - T_t x_n\| &\leq \|x_n - T_{t_n} x_n\| + \|T_{t_n} x_n - T_t x_n\| \\ &\leq \|x_n - T_{t_n} x_n\| + (2b + 1) \max \{\|x_n - T_{t-t_n} x_n\|, \|x_n - T_{t_n-t} x_n\|\}, \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ in the above inequality, we obtain $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$.

Conversely, we assume that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$. Since $\bar{T}_t^{\mu_t} x = (1 - \mu_t)x + \mu_t T_t x$, for all $x \in C$, we have $\lim_{n \rightarrow \infty} \|x_n - \bar{T}_t^{\mu_t} x_n\| = 0$, for all $t \in G$. Let $z_0 \in A_C(\{x_n\})$. For each $t \in G$, we have

$$\begin{aligned} \|\bar{T}_t^{\mu_t} z_0 - x_n\| &\leq \|\bar{T}_t^{\mu_t} z_0 - \bar{T}_t^{\mu_t} x_n\| + \|\bar{T}_t^{\mu_t} x_n - x_n\| \\ &\leq \|z_0 - x_n\| + \|\bar{T}_t^{\mu_t} x_n - x_n\|, \end{aligned} \tag{4.9}$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \rightarrow \infty$ in (4.9), we obtain

$$\limsup_{n \rightarrow \infty} \|\bar{T}_t^{\mu_t} z_0 - x_n\| \leq \limsup_{n \rightarrow \infty} \|z_0 - x_n\|.$$

Since $z_0 \in A_C(\{x_n\})$ and $\bar{T}_t^{\mu_t} z_0 \in C$, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\bar{T}_t^{\mu_t} z_0 - x_n\| &= \limsup_{n \rightarrow \infty} \|z_0 - x_n\| \\ &= r_C(\{x_n\}). \end{aligned}$$

It follows that $\bar{T}_t^{\mu_t} z_0 \in A_C(\{x_n\})$. Since X is uniformly convex, $A_C(\{x_n\})$ is singleton. Hence $\bar{T}_t^{\mu_t} z_0 = z_0$ for all $t \in G$, that is, $z_0 \in Fix(\bar{\tau}) = Fix(\tau)$. Therefore, $Fix(\tau) \neq \emptyset$. This completes the proof. ■

Lemma 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be an enriched nonexpansive semigroup on C with $Fix(\tau) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (4.2). Assume that X satisfies the Opial’s condition. If $\{x_n\}$ converges weakly to $z \in C$ and $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$, then $T_t z = z$, for all $t \in G$. That is $I - T_t$ is demiclosed at zero, for all $t \in G$.*

Proof. Suppose that $\{x_n\}$ converges weakly to $z \in C$ and $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$. This implies that $\lim_{n \rightarrow \infty} \|x_n - \bar{T}_t^{\mu_t} x_n\| = 0$, for all $t \in G$. By the assumption and Lemma 2.8, we have $z \in A_C(\{x_n\})$. For each $t \in G$, we have

$$\begin{aligned} \|\bar{T}_t^{\mu_t} z - x_n\| &\leq \|\bar{T}_t^{\mu_t} z - \bar{T}_t^{\mu_t} x_n\| + \|\bar{T}_t^{\mu_t} x_n - x_n\| \\ &\leq \|z - x_n\| + \|\bar{T}_t^{\mu_t} x_n - x_n\|, \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \rightarrow \infty$ in the above inequality, we obtain

$$\limsup_{n \rightarrow \infty} \|\bar{T}_t^{\mu_t} z - x_n\| \leq \limsup_{n \rightarrow \infty} \|z - x_n\|.$$

Since $z \in A_C(\{x_n\})$ and $\overline{T}_t^{\mu_t} z \in C$, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\overline{T}_t^{\mu_t} z - x_n\| &= \limsup_{n \rightarrow \infty} \|z - x_n\| \\ &= r_C(\{x_n\}). \end{aligned}$$

It follows that $\overline{T}_t^{\mu_t} z \in A_C(\{x_n\})$. Since X is uniformly convex, $A_C(\{x_n\})$ is singleton. Hence, $\overline{T}_t^{\mu_t} z = z$, for all $t \in G$. It follows from $Fix(\overline{\tau}) = Fix(\tau)$ that $T_t z = z$, for all $t \in G$, that is, $I - T_t$ is demiclosed at zero, for all $t \in G$. This completes the proof. ■

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be a b -enriched nonexpansive semigroup on C with $Fix(\tau) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (4.2). Assume that X satisfies the Opial's condition, then $\{x_n\}$ converges weakly to the common fixed point of τ .*

Proof. Let $z \in Fix(\tau)$. By Lemma 4.2, we get $\{x_n\}$ is bounded. Since X is uniformly convex, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to q for some $q \in X$. Now, we will show that $\{x_n\}$ has a unique weak sub-sequential limit in $Fix(\tau)$. Let q_1 and q_2 be weak limits of subsequences $\{x_{n_{k_1}}\}$ and $\{x_{n_{k_2}}\}$ of $\{x_n\}$, respectively. From Lemma 4.3, we have $\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0$, for all $t \in G$ and by Lemma 4.4, we get $T_t q_1 = q_1$ and $T_t q_2 = q_2$, for all $t \in G$. It follows that $q_1, q_2 \in Fix(\tau)$.

Next we will prove that $q_1 = q_2$. Assume that $q_1 \neq q_2$. From the Opial's condition, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q_1\| &= \lim_{n_{k_1} \rightarrow \infty} \|x_{n_{k_1}} - q_1\| \\ &< \lim_{n_{k_1} \rightarrow \infty} \|x_{n_{k_1}} - q_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - q_2\| \\ &= \lim_{n_{k_2} \rightarrow \infty} \|x_{n_{k_2}} - q_2\| \\ &< \lim_{n_{k_2} \rightarrow \infty} \|x_{n_{k_2}} - q_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - q_1\|, \end{aligned}$$

which is a contradiction. Hence, $q_1 = q_2$. Consequently, $\{x_n\}$ converges weakly to an element of $Fix(\tau)$. ■

Now, we obtain strong convergence theorems for enriched nonexpansive semigroups using the Mann iterative process in uniformly convex Banach spaces. First, we establish the following useful lemma:

Lemma 4.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be an enriched nonexpansive semigroup on C with $Fix(\tau) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence defined by iterative scheme (4.2). If $\liminf_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$ then $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$, where $d(x_n, Fix(\tau)) = \inf\{\|x_n - z\| : z \in Fix(\tau)\}$.*

Proof. Suppose that $\liminf_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$ and $z \in Fix(\tau)$. Now, we will show that $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau))$ exists. For each $n \in \mathbb{N}$, by (4.4), we get

$$\|x_{n+1} - z\| \leq \|x_n - z\|.$$

Taking the infimum all over $z \in Fix(\tau)$ on both sides, we obtain

$$d(x_{n+1}, Fix(\tau)) \leq d(x_n, Fix(\tau)). \tag{4.10}$$

So the sequence $\{d(x_n, Fix(\tau))\}$ is non-increasing and bounded below. This implies that $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau))$ exists. It follows that $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$. Similarly, we can prove that if $\limsup_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$, then $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$. ■

Theorem 4.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be a b -enriched nonexpansive semigroup on C with $Fix(\tau) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (4.2). Then the sequence $\{x_n\}$ converges strongly to the common fixed point of τ if and only if $\liminf_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$, where $d(x_n, Fix(\tau)) = \inf\{\|x_n - z\| : z \in Fix(\tau)\}$.*

Proof. Let $z \in Fix(\tau)$. Suppose that the sequence $\{x_n\}$ converges strongly to $z \in Fix(\tau)$. For a given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\|x_n - z\| < \epsilon,$$

for all $n \geq n_0$. Taking the infimum all over $z \in Fix(\tau)$ in the above inequality, we get

$$d(x_n, Fix(\tau)) < \epsilon,$$

for all $n \geq n_0$. This implies that $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$, that is, $\liminf_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$ and $\limsup_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$.

Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$. By Lemma 4.6, we have $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$. Now, we will show that $\{x_n\}$ is a Cauchy sequence in X . Since $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$, for a given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, Fix(\tau)) < \frac{\epsilon}{4}.$$

It implies that

$$\inf\{\|x_n - z\| : z \in Fix(\tau)\} < \frac{\epsilon}{4},$$

for all $n \geq n_0$. Then

$$\inf\{\|x_{n_0} - z\| : z \in Fix(\tau)\} < \frac{\epsilon}{4},$$

and so there exists $z \in Fix(\tau)$ such that

$$\|x_{n_0} - z\| < \frac{\epsilon}{2}. \tag{4.11}$$

For each $m, n \in \mathbb{N}$, it can be obtained that, when $n \geq n_0$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - z\| + \|x_n - z\| \\ &\leq \|x_{n_0} - z\| + \|x_{n_0} - z\| \\ &= 2\|x_{n_0} - z\| \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C . By the completeness of C , the sequence $\{x_n\}$ converges strongly to a point $z_0 \in C$. Next, we will show that $z_0 \in \text{Fix}(\tau)$. For each $n \in \mathbb{N}$ and $t \in G$, we have

$$\begin{aligned} \|\bar{T}_t^{\mu_t} z_0 - z_0\| &\leq \|\bar{T}_t^{\mu_t} z_0 - \bar{T}_t^{\mu_t} x_n\| + \|\bar{T}_t^{\mu_t} x_n - x_n\| + \|x_n - z_0\| \\ &\leq 2\|z_0 - x_n\| + \|\bar{T}_t^{\mu_t} x_n - x_n\|. \end{aligned} \tag{4.12}$$

Taking the limit as $n \rightarrow \infty$ in (4.12), by Lemma 4.3, we have $\bar{T}_t^{\mu_t} z_0 = z_0$, for all $t \in G$, that is $z_0 \in \text{Fix}(\tau) = \text{Fix}(\bar{\tau})$. Therefore, the sequence $\{x_n\}$ converges strongly to the common fixed point of τ . ■

Next, we define the condition as follows:

Definition 4.8. Let C be a nonempty subset of a normed space X and G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1). A family $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ is said to satisfy the condition (B) if there exists a non-decreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r > 0$ such that for all $x \in C$, $g(d(x, \text{Fix}(\tau))) \leq \|x - T_t x\|$, for all $t \in G$.

Theorem 4.9. Let C be a nonempty closed convex subset of a uniformly convex Banach space X , G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : C \rightarrow C \mid t \in G\}$ be a b -enriched nonexpansive semigroup on C satisfying the condition (B) with $\text{Fix}(\tau) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (4.2). Then the sequence $\{x_n\}$ converges strongly to the common fixed point of τ .

Proof. Let $z \in \text{Fix}(\tau)$. It follows from Lemma 4.3 that

$$\lim_{n \rightarrow \infty} \|x_n - T_t x_n\| = 0, \tag{4.13}$$

for all $t \in G$. For each $n \in \mathbb{N}$, from the condition (B), we get

$$g(d(x_n, \text{Fix}(\tau))) \leq \|x_n - T_t x_n\|,$$

for all $t \in G$. Taking the limit as $n \rightarrow \infty$ in the above inequality, by (4.13), we have

$$\lim_{n \rightarrow \infty} g(d(x_n, \text{Fix}(\tau))) = 0.$$

As $g : [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing function with $g(0) = 0$ and $g(r) > 0$ for all $r > 0$, then we have

$$\lim_{n \rightarrow \infty} d(x_n, \text{Fix}(\tau)) = 0.$$

Now, all conditions of Theorem 4.7 are satisfied. Therefore, we can conclude that the sequence $\{x_n\}$ converges strongly to the common fixed point of τ . ■

5. NUMERICAL EXAMPLE

In this section, we present a numerical example to illustrate our main theorems.

Example 5.1. Let $X = \mathbb{R}$ be equipped with the usual norm, $C = [-1, \frac{1}{2}]$ be nonempty closed convex subset of \mathbb{R} , and $\tau = \{T_t : C \rightarrow C \mid t \in \mathbb{N}\}$ such that

$$T_t x = \begin{cases} x e^{-t} & \text{if } x \in [-1, 0], \\ -3x e^{-t} & \text{if } x \in (0, \frac{1}{2}], \end{cases} \tag{5.1}$$

for all $t \in \mathbb{N}$.

From Example 3.2, the family τ is a b -enriched nonexpansive semigroup on C with $b = 1.2$. It is easy to see that $Fix(\tau) = \{0\}$, that is, $T_t(0) = 0$, for all $t \in \mathbb{N}$. Suppose that $\{\lambda_n\} \subset [0, 1]$ satisfying $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n \rightarrow \infty} t_n = \infty$, we get $\lim_{n \rightarrow \infty} d(x_n, Fix(\tau)) = 0$, for all $\{x_n\} \subset C$. Hence, all the conditions of Theorem 4.7 are satisfied.

By Theorem 4.7, we obtain for each $x_0 \in C$ and $\{t_n\} \subset \mathbb{N}$, the Mann iterative process $\{x_n\}$ defining in (4.2) with $\{\lambda_n\} \subset [0, 1]$ satisfying $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n \rightarrow \infty} t_n = \infty$, converges strongly to 0.

Finally, we present some numerical experiments for the convergence behaviors. In the first set of experiments, there are three initial numbers $x_0 = -0.9$, $x_0 = 0.4$, and $x_0 = 0.1$ for testing. Each initial number x_0 is tested on the Mann iterative process $\{x_n\}$ with $\lambda_n = (n + 1)^{-\frac{1}{2}}$ and $t_n = n$. The experiments are performed to see the the convergence behavior of $\{x_n\}$ and the results are shown in Figure 1(a).

A second set of experiments, we choose $x_0 = 0.25$, $\lambda_n = (n + 1)^{-\frac{1}{2}}$ and test different choices of the sequence $\{t_n\} \subset \mathbb{N}$. The convergence behavior of $\{x_n\}$ with $t_n = 2n + 5$, $t_n = n$, and $t_n = n^3$ are shown in Figure 1(b).

In the final set of experiments, we choose $x_0 = -1$, $t_n = 2n + 5$ and test different choices of the sequence $\{\lambda_n\} \subset [0, 1]$ satisfying $\sum_{n=0}^{\infty} \lambda_n = \infty$. Figure 1(c) illustrates the convergence behavior of $\{x_n\}$ with $\lambda_n = (n + 1)^{-\frac{1}{2}}$, $\lambda_n = 0.5$, and $\lambda_n = 0.9$. From Figure 1(a)-(c), it can be seen that the Mann iterative process $\{x_n\}$ converges to 0.

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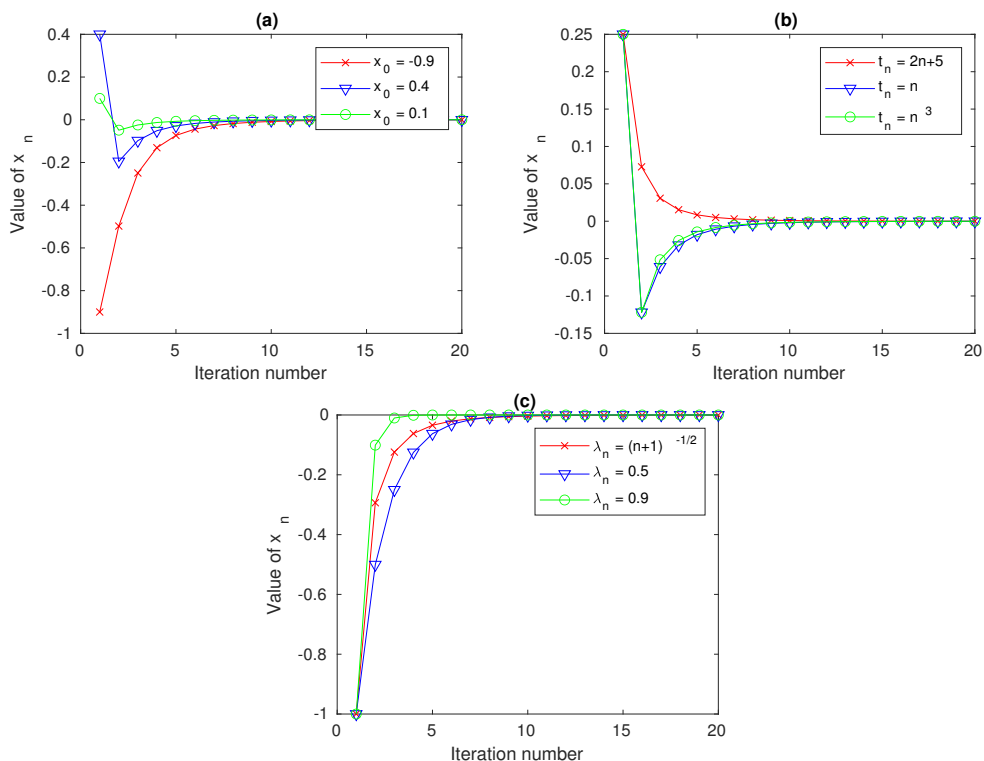


FIGURE 1. The convergence behavior of the Mann iterative process $\{x_n\}$

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