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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Strong Convergence Theorem for Some Nonexpansive-Type Mappings in Certain Banach Spaces

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Abstract Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^*. A new class of *relatively J-nonexpansive* maps, $T : E \to E^*$ is introduced and studied. A strong convergence theorem for approximating a common J-fixed point of a countable family of relatively J-nonexpansive maps is proved. An example of a countable family of relatively J-nonexpansive maps with a non-empty common J-fixed point is constructed. Finally, a numerical example is presented to show that our algorithm is implementable.

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1. Introduction

Let E be a real Banach space with dual space E^*. A map $A : E \to E$ is said to be *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0,$$

where $J : E \to 2^{E^*}$ is the *normalized duality map*. The map A is called *m-accretive* if it is accretive, and in addition, the graph of A is not properly contained in the graph of any other accretive operator. Consider the evolution equation

$$\frac{du}{dt} + Au = 0, \tag{1.1}$$

where $A : E \to E$ is an accretive operator. Observe that at equilibrium, equation (1.1) reduces to

$$Au = 0, \tag{1.2}$$

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whose solution(s) will be the equilibrium state of the system (1.1). Since A is nonlinear, in general, there is no known closed form solution of equation (1.2). Browder [1] defined a map $T : E \to E$ by $T := I - A$; and called T a *pseudocontractive map*. Observe that $Au = 0 \Leftrightarrow Tu = u$. So, zeros of A (which are equilibrium states of (1.1)) correspond to fixed points of T. Hence, approximating fixed points of pseudocontractive maps has become a flourishing area of interest to researchers in nonlinear operator theory (see, for e.g., these monographs of Alber [2], Berinde [3], Chidume [4], Goebel and Reich [5] and the references contained in them).

Let $A : E \to E^*$ be a map. Then, A is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall \ x, y \in E.$$

The map A is called *maximal monotone* if it is monotone and, in addition, the graph of A is not properly contained in the graph of any other monotone map.

Interest in monotone maps stems mainly from their usefulness in applications. For example, monotone maps appear in convex optimization problems. Consider the function $f : E \to \mathbb{R} \cup \{\infty\}$, where f is convex, proper and lower semi-continuous. The *subdifferential* of f, $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) := \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^*\rangle, \ \forall \ y \in E\}$$

is a monotone operator on E. It is easy to see that $0 \in \partial f(u)$ if and only if u is a minimizer of f.

In general, if $A : E \to E^*$ is a monotone map, solutions of (1.2) correspond to either the equilibrium states of an evolution system, or, minimizers of some convex functional defined on E. This is a motivation for studying the equation $Au = 0$, where $A : E \to E^*$ is monotone.

Observe that Browder's fixed point technique for equation (1.2) is not applicable when A is monotone in Banach spaces, since A maps E to E^*, defining $T := I - A$ does not even make sense. We note, however, that for the special case in which E is a real Hilbert space, say H, we have $E = E^* = H$ and the fixed point technique introduced by Browder is still applicable. This, perhaps, explains why virtually all results for approximating solutions of (1.2), when A is of monotone type, have been confined to real Hilbert spaces. However, as has been rightly observed by (a Series Editor of Kluwer Academic Publishers), Hazewinkle, "... many, and probably most, mathematical objects and models do not naturally live in Hilbert space", [6] pg. viii.

Recently, the notion of *J-fixed point* (which has also been called *semi-fixed point*, Zegeye [7], *duality fixed point*, Liu [8]) has been defined and studied by Chidume and Idu [9], for maps from a space, say E, to its dual space E^*.

Let $T : E \to E^*$ be any map. A point $u \in E$ is called a *J-fixed point of T* if $Tu = Ju$, where $J : E \to E^*$ is the single valued normalized duality map on E. We shall denote the set of J fixed points of T by $F_J(T)$.

Consider the map $T : E \to E^*$ defined by $T := J - A$, where $A : E \to E^*$ is monotone. Observe that u is a J-fixed point of T if and only if u is a solution of (1.2). Consequently, approximating solutions of (1.2) is equivalent to approximating J-fixed points of maps

$T : E \to E^*$. This connection is now generating considerable research interest in the study of J-fixed points (see Chidume and Idu [9], Chidume and Monday [10], Chidume *et al.* [11–13], and the references contained in them). This notion turns out to be very useful and applicable in approximating solutions of equation (1.2). For example, Chidume and Idu [9], introduced the concept of *J-pseudocontractive maps* and proved a strong convergence theorem for approximating J-fixed points of a J-pseudocontractive map. As an application of their theorem, they proved a strong convergence theorem for approximating a zero of a maximal monotone map.

An *inertial-type algorithm* was first introduced and studied by Polyak [14], as an acceleration process in solving smooth convex minimization problems. This algorithm is a *two step* iterative method in which the next iterate is obtained using the previous *two* iterates. Numerical experiments have shown that incorporating an inertial term in an algorithm speeds up the convergence of the sequence generated by the algorithm. Thus, a lot of research effort is now devoted to *inertial-type algorithms* (see, e.g., [15–23] and the references contained in them).

Recently, Chidume *et al.* [15], studied an inertial algorithm for approximating a common fixed point of a countable family of *relatively nonexpansive maps* in a uniformly smooth and uniformly convex real Banach space. They proved that the sequence generated by their algorithm converges strongly to a common fixed point of the family.

It is our purpose in this paper to contribute to the on-going research on iterative methods for approximating J-fixed points of nonlinear maps defined from E to E^*.

2. Preliminaries

Let J be the *normalized duality map* from E to 2^{E^*}. It is well known that if E is a reflexive, strictly convex and smooth real Banach space, then J is single-valued and bijective. In particular, if E is uniformly smooth and uniformly convex, then the dual space E^* is also uniformly smooth and uniformly convex and the normalized duality map J and its inverse, J^{-1}, are both uniformly continuous on bounded sets. (see, e.g., Ibaraki and Takahashi, [24]).

Let E be a smooth real Banach space and $\phi : E \times E \to \mathbb{R}$ be defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall\, x, y \in E. \tag{2.1}$$

The function ϕ was first introduced by Alber and has been extensively studied by many authors (see, for example, Alber [25]; Chidume *et al.* [26–29]; Ofoedu and Shehu [30]; Kamimura and Takahashi [31]; Nilsrakoo and Saejung, [32]; Reich [33]; Xu, [34]; Zegeye [7]; and the references contained in them). It is easy to see from the definition of ϕ that, in a real Hilbert space H, equation (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$.

Furthermore, given $x, y, z \in E$, and $\tau \in (0, 1)$, we have the following properties and definitions (see, e.g., Nilsrakoo and Saejung, [32]):

P1: $(\|u\| - \|v\|)^2 \le \phi(u, v) \le (\|u\| + \|v\|)^2$,
P2: $\phi(u, v) = \phi(u, w) + \phi(w, v) + 2\langle w - u, Jv - Jw\rangle$.

Definition 2.1. Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. The map $\Pi_C : E \to C$ defined by $\tilde{x} = \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf_{y \in C} \phi(y, x)$ is called the *generalized projection* of x onto C. Clearly, in a real Hilbert space H, the generalized projection Π_C coincides with the metric projection P_C from H onto C.

Definition 2.2. Let $T : E \to E$ be a map. Then, T is called *relatively nonexpansive* if the following conditions hold:

(*i*) $F(T) := \{x \in E : Tx = x\} \neq \emptyset$;
(*ii*) $\phi(x, Ty) \leq \phi(x, y), \ \forall \ x \in F(T)$ and $y \in E$;
(*iii*) $(I - T)$ is demi-closed at zero, i.e., whenever a sequence $\{x_n\}$ in C converges weakly to x and $\{x_n - Tx_n\}$ converges strongly 0, then $x \in F(T)$,

Lemma 2.3 ([2]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. Then,*

(1) if $x \in E$ and $y \in C$, then $\tilde{x} = \Pi_C x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$, for all $y \in C$,
(2) $\phi(y, \tilde{x}) + \phi(\tilde{x}, x) \leq \phi(y, x)$, for all $x \in E$, $y \in C$.

Lemma 2.4 ([32]). *Let E be a smooth Banach space. Then,*

$$\phi\left(u, J^{-1}[\beta Jx + (1-\beta)Jy]\right) \leq \beta\phi(u, x) + (1-\beta)\phi(u, y), \ \forall \ \beta \in [0,1], \ u, x, y \in E.$$

Lemma 2.5 ([31]). *Let E be a uniformly convex and smooth real Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. If either $\{u_n\}$ or $\{v_n\}$ is bounded and $\phi(u_n, v_n) \to 0$, then $\|u_n - v_n\| \to 0$.*

Remark 2.6. The converse of Lemma 2.5 is also true whenever $\{u_n\}$ and $\{v_n\}$ are both bounded (see, e.g., [15]).

Lemma 2.7 ([35]). *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E and let $(S_i)_{i=1}^{\infty}$, $S_i : C \to E$, for each $i \geq 1$, be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $(\eta_i)_{i=1}^{\infty} \subset (0,1)$ and $(\mu_i)_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Consider the map $S : C \to E$ defined by*

$$Sx = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)JS_i x\right) \text{ for each } x \in C. \tag{2.2}$$

Then, S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(S_i)$.

2.1. Analytical Representations of Duality Maps in L_p, l_p, and W_m^p, Spaces, $1 < p < \infty$

The analytical representations of duality maps are known in L_p, l_p, and W_m^p, $1 < p < \infty$. Precisely, in the spaces l_p, $L_p(G)$ and $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$\begin{aligned} Jz &= \|z\|_{l_p}^{2-p} y \in l_q,\ y = \{|z_1|^{p-2}z_1, |z_2|^{p-2}z_2, ...\},\ z = \{z_1, z_2, ...\}, \\ J^{-1}z &= \|z\|_{l_q}^{2-q} y \in l_p,\ y = \{|z_1|^{q-2}z_1, |z_2|^{q-2}z_2, ...\},\ z = \{z_1, z_2, ...\}, \\ Jz &= \|z\|_{L_p}^{2-p} |z(s)|^{p-2} z(s) \in L_q(G),\ s \in G, \\ J^{-1}z &= \|z\|_{L_q}^{2-q} |z(s)|^{q-2} z(s) \in L_p(G),\ s \in G,\ \text{and} \\ Jz &= \|z\|_{W_m^p}^{2-p} \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha (|D^\alpha z(s)|^{p-2} D^\alpha z(s)) \in W_{-m}^q(G), m > 0, s \in G, \end{aligned}$$

(see e.g., Alber and Ryazantseva, [25]; p. 36).

3. Main Results

The symbols $\to$ and $\rightharpoonup$ will denote strong and weak convergence, respectively.

Definition 3.1. Let E be a reflexive, strictly convex and smooth real Banach space and let $T : E \to E^*$ be a map. A point $x^* \in E$ is called an *asymptotic J-fixed point of T* if there exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x^*$ and $\|Jx_n - Tx_n\| \to 0$, as $n \to \infty$. We shall denote the set of asymptotic J-fixed points of T by $\widehat{F}_J(T)$.

Definition 3.2. A map $T : E \to E^*$ is said to be *relatively J-nonexpansive* if

(*i*) $\widehat{F}_J(T) = F_J(T) \neq \emptyset$;
(*ii*) $\phi(p, J^{-1}Tx) \le \phi(p, x),\ \forall\ x \in E,\ p \in F_J(T)$;
where $F_J(T) = \{x \in E :\ Tx = Jx\}$.

We first prove the following Lemma which will be central in the sequel.

Lemma 3.3. *Let E be a uniformly convex and uniformly smooth real Banach space. Let $T : E \to E^*$ be a relatively J-nonexpansive map such that $F_J(T) \neq \emptyset$. Define inductively the sequence $\{x_n\}$ by $x_0, x_1 \in E$*

$$\begin{cases} C_0 = E, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}\big((1-\beta)Jw_n + \beta T w_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

$n \ge 0$, where $\alpha_n \in [0, 1)$, $\beta \in (0, 1)$. Then, $\{x_n\}$ converges strongly to $p = \Pi_{F_J(T)} x_0$

Proof. We divide the proof into two steps:

Step 1. We show that $\{x_n\}$ is well-defined, $F_J(T) \subset C_n$, $\forall\, n \geq 0$ and $\{x_n\}$ is bounded. Let $z \in C_{n+1}$, then,

$$\begin{aligned}&\phi(z, y_n) \leq \phi(z, w_n)\\&\Leftrightarrow \|z\|^2 - 2\langle z, Jy_n\rangle + \|y_n\|^2 \leq \|z\|^2 - 2\langle z, Jw_n\rangle + \|w_n\|^2\\&\Leftrightarrow 2\langle z, Jw_n - Jy_n\rangle \leq \|w_n\|^2 - \|y_n\|^2.\end{aligned} \tag{3.2}$$

Using inequality (3.2), it is easy to see that C_n is closed and convex, $\forall\, n \geq 0$. We now show that $F_J(T) \subset C_n$, $\forall\, n \geq 0$. We show this by induction. For $n = 0$, $F_J(T) \subset E = C_0$. Assume $F_J(T) \subset C_n$, for some $n \geq 0$, let $p \in F_J(T)$. Then, by Lemma 2.4 and the fact that T is relatively J-nonexpansive, we have that

$$\begin{aligned}\phi(p, y_n) &= \phi\big(p, J^{-1}((1-\beta)Jw_n + \beta Tw_n)\big)\\&= \phi\big(p, J^{-1}((1-\beta)Jw_n + \beta J(J^{-1}Tw_n))\big)\\&\leq (1-\beta)\phi(p, w_n) + \beta\phi(p, J^{-1}Tw_n)\\&\leq (1-\beta)\phi(p, w_n) + \beta\phi(p, w_n) = \phi(p, w_n),\end{aligned}$$

which implies that $p \in C_{n+1}$. Thus, $F_J(T) \subset C_n$, for all $n \geq 0$. Hence, $\{x_n\}$ is well-defined.

Next, we show that $\{x_n\}, \{y_n\}$ and $\{w_n\}$ are bounded. By definition, observe that $x_n = \Pi_{C_n} x_0$ and $C_{n+1} \subset C_n$, for all $n \geq 0$. So, by Lemma 2.3 (2), we have that $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$, which implies that $\{\phi(x_n, x_0)\}$ is nondecreasing. Furthermore,

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0),$$

implies that $\{\phi(x_n, x_0)\}$ is bounded and consequently, $\{\phi(x_n, x_0)\}$ is convergent. By P1, $\{x_n\}$ is bounded. From Lemma 2.3 (2), we have that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(x_n, x_0) \to 0 \ as\ n, m \ \to \infty.$$

Hence, $\{x_n\}$ is Cauchy and this implies that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. Using the definition of w_n, we have that $\|x_n - w_n\| = \|\alpha_n(x_{n-1} - x_n)\| \leq \|x_{n-1} - x_n\| \to 0$, as $n \to \infty$. Since $\{w_n\}$ and $\{x_n\}$ are bounded, by Remark 2.6 we have that $\phi(x_n, w_n) \to 0$, as $n \to \infty$. Observe that $x_{n+1} \in C_n$, so $0 \leq \phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, w_n) \to 0$, as $n \to \infty$. By Lemma 2.5, we have $\|x_n - y_n\| \to 0$, as $n \to \infty$ and thus, $\{y_n\}$ is bounded.

Step 2. We show that $x_n \to \Pi_{F_J(T)} x_0$. First, establish that $\|Jw_n - Tw_n\| \to 0$, as $n \to \infty$. By Remark 2.6, since $\{x_n\}$ and $\{y_n\}$ are bounded, we have that $\phi(x_n, y_n) \to 0$, as $n \to \infty$. By P2, and uniform continuity of J on bounded sets, we have that

$$\begin{aligned}\phi(w_n, y_n) &= \phi(w_n, x_n) + \phi(x_n, y_n) + 2\langle x_n - w_n, Jy_n - Jx_n\rangle\\&\leq \phi(w_n, x_n) + \phi(x_n, y_n) + 2\|x_n - w_n\|\|Jy_n - Jx_n\| \to 0,\ as\ n \to \infty.\end{aligned}$$

Next, from the definition of y_n, we observe that $\|Jy_n - Jw_n\| = \beta\|Jw_n - Tw_n\|$. Since $\|y_n - w_n\| \to 0$, by uniform continuity of J on bounded sets, we have that $\|Jy_n - Jw_n\| \to 0$, and so $\beta\|Jw_n - Tw_n\| \to 0$. Hence, $\|Jw_n - Tw_n\| \to 0$, as $n \to \infty$.

Now, since $\{w_n\}$ is bounded, there exists $\{w_{n_k}\}$, a subsequence of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^*$, as $k \to \infty$. Thus, $\|Jw_{n_k} - Tw_{n_k}\| \to 0$, as $k \to \infty$. Since T is relatively J-nonexpansive, we have that $x^* \in F_J(T)$. Furthermore, there exists $\{x_{n_k}\} \subset \{x_n\}$, such that $x_{n_k} \rightharpoonup x^*$, as $k \to \infty$. We now show that $x^* = \Pi_{F_J(T)} x_0$. Set $y = \Pi_{F_J(T)} x_0$. Since

$x_n = \Pi_{C_n} x_0$ and $F_J(T) \subset C_n, \ \forall \ n \ \geq 0$, we have $\phi(x_n, x_0) \leq \phi(y, x_0)$. By the weak lower semi-continuity of the norm, we obtain

$$\begin{aligned} \phi(x^*, x_0) &= \|x^*\|^2 - 2\langle x^*, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(y, x_0). \end{aligned} \tag{3.3}$$

But $\phi(y, x_0) \leq \phi(z, x_0), \ \forall \ z \in F_J(T)$

$$\Rightarrow \phi(y, x_0) \leq \phi(x^*, x_0) \leq \phi(y, x_0) \tag{3.4}$$

$$\Rightarrow \phi(y, x_0) = \phi(x^*, x_0).$$

By uniqueness of $\Pi_{F_J(T)} x_0$, $y = x^*$. So, we deduce that $x^* = \Pi_{F_J(T)} x_0$. Next, we show that $x_{n_k} \to x^*$, as $k \to \infty$. Using inequalities (3.3) and (3.4), we obtain that $\phi(x_{n_k}, x_0) \to \phi(x^*, x_0)$, as $k \to \infty$. Thus, $\|x_{n_k}\| \to \|x^*\|$, as $k \to \infty$. By the Kadec-Klee property of E, we conclude that $x_{n_k} \to x^*$ as $k \to \infty$. Therefore, $x_n \to \Pi_{F_J(T)} x_0$. This completes the proof. ■

Using Lemma 3.3, we now prove our main theorem.

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth real Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of relatively J-nonexpansive maps such that $\bigcap_{i=1}^{\infty} F_J(T_i) \neq \emptyset$, where $T_i : E \to E^*$, $\forall \ i$. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Define inductively the sequence $\{x_n\}$ by $x_0, x_1 \in E$*

$$\begin{cases} C_0 = E, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}\big((1-\beta)Jw_n + \beta Tw_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.5}$$

$n \geq 0$, where $Tx = \sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)T_i x)$, for each $x \in E$, $\alpha_n \in [0,1)$, $\beta \in (0,1)$. Then, $\{x_n\}$ converges strongly to $p = \Pi_{F_J(T)} x_0$

Proof. Observe that since T_i is relatively J-nonexpansive for each i, it is easy to see that $S_i = J^{-1}T_i$, is relatively nonexpansive for each i. So, using Lemma 2.7, setting $G = J^{-1}T$, we obtain that

$$Gx = J^{-1}\Big(\sum_{i=1}^{\infty} \eta_i\big(\mu_i Jx + (1-\mu_i)J(S_i x)\big)\Big)$$

is relatively nonexpansive. Furthermore, it is easy to see that T is relatively J-nonexpansive and $F(J^{-1}T) = F_J(T)$. Using Lemma 3.3, since T is relatively J-nonexpansive, $\{x_n\}$ converges strongly to $x^* \in F_J(T)$. ■

Corollary 3.5. *Let $E = L_p$(or l_p or $W_p^M(\Omega)$), $1 < p < \infty$. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of relatively J-nonexpansive maps such that $\bigcap_{i=1}^{\infty} F_J(T_i) \neq \emptyset$, where $T_i : E \to E^*$,*

$\forall\ i$. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Define inductively the sequence $\{x_n\}$ by $x_0, x_1 \in E$

$$\begin{cases} C_0 = E, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}\big((1-\beta)Jw_n + \beta T w_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.6}$$

$n \geq 0$, where $Tx = \sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)T_i x)$, for each $x \in E$, $\alpha_n \in [0,1)$, $\beta \in (0,1)$. Then, $\{x_n\}$ converges strongly to $p = \Pi_{F_J(T)} x_0$.

Example 3.6. Let $E = l_p$. Then $E^* = l_q$, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$f_i : l_p \to l_p \text{ by } \ f_i(x) := f_i(x_1, x_2, \cdots) = \frac{1}{2^i}(0, x_1, x_2, \cdots), \text{ and } \ T_i : l_p \to l_q \text{ by}$$

$$T_i x := (J \circ f_i)(x) = \frac{1}{2^i} J(0, x_1, x_2, \cdots).$$

Clearly, T_i is weakly sequentially continuous for each $i = 1, 2, 3, \cdots$.

Claim: T_i is relatively J-nonexpansive for each i.

Proof of claim.

(i) We show that $F_J(T_i) = \widehat{F_J}(T_i) \neq \emptyset$, $\forall\ i \geq 1$. Now,

$$T_i x = Jx \ \Leftrightarrow \ \frac{1}{2^i} J(0, x_1, x_2, \cdots) = J(x_1, x_2, \cdots).$$

This implies $\frac{1}{2^i}(0, x_1, x_2, \cdots) = (x_1, x_2, \cdots)$, since J is one-to-one.

Hence, $x = (0, 0, \cdots)$. Thus, $F_J(T_i) = \{\mathbf{0}\}$, $\forall i \geq 1$. We show that $\widehat{F_J}(T_i) = F_J(T_i)$. Consider the constant sequence $\{x_n\} \subset l_p$, where $x_n = (0, 0, \cdots)$, $\forall\ n$. Clearly, $x_n \rightharpoonup \mathbf{0}$ and $\|Jx_n - T_i x_n\| \to 0$. Hence, $\mathbf{0} \in \widehat{F_J}(T_i)$. Let $x^* \in \widehat{F_J}(T_i)$. Then, there exists $\{x_n\} \subset l_p$ such that

$$x_n \rightharpoonup x^* \text{ and } \ \|Jx_n - T_i x_n\| \to 0. \tag{$*$}$$

Now, by the weak sequential continuity of J and T_i, we obtain

$$x_n \rightharpoonup x^* \ \Rightarrow \ Jx_n \rightharpoonup Jx^* \text{ and } \ x_n \rightharpoonup x^* \ \Rightarrow \ T_i x_n \rightharpoonup T_i x^*.$$

Furthermore, from $(*)$,

$$\|Jx_n - T_i x_n\| \to 0, \text{ so we obtain } \ (Jx_n - T_i x_n) \rightharpoonup \mathbf{0}.$$

Thus, by the uniqueness of the weak limit, we obtain that

$$Jx^* - T_i x^* = \mathbf{0},$$

i.e., $T_i x^* = Jx^*$ so that $x^* \in F_J(T_i) = \{\mathbf{0}\}$.

Hence, $x^* = (0, 0, \cdots)$,

which yields $F_J(T_i) = \widehat{F_J}(T_i) = \{\mathbf{0}\}$, $\forall\ i$.

(ii) We show $\phi(u, J^{-1}T_i x) \leq \phi(u, x)$, $\forall u \in F_J(T_i)$, $\forall x \in l_p$.
Let $u \in F_J(T_i)$ and $x \in l_p$. Then,

$$\begin{aligned}\phi(u, J^{-1}T_i x) &= \phi\big(\mathbf{0}, \frac{1}{2^i}(0, x_1, x_2, \cdots)\big) \\ &= \|x\|^2_{l_p} = \phi(\mathbf{0}, x) = \phi(u, x).\end{aligned}$$

Hence, T_i is relatively J-nonexpansive for each i.

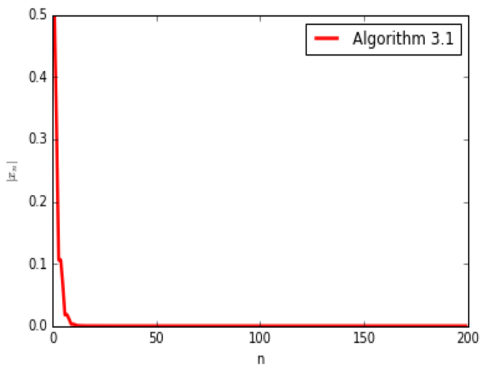
Next we show how to compute $Tx = \sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)T_i x)$ in Theorem 3.4 and Corollary 3.5. Let T_i be as defined above. Then, $\bigcap_{i=1}^{\infty} F\big(T_i(x)\big) = \{0\}$. Define $\{\mu_i\} \subset (0,1)$ and $\{\eta_i\} \subset (0,1)$ by $\eta_i = \frac{1}{2^i}$, $\mu_i = \frac{1}{2^i}$, $i \geq 1$. Clearly, $\displaystyle\sum_{i=1}^{\infty} \eta_i = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. Furthermore,

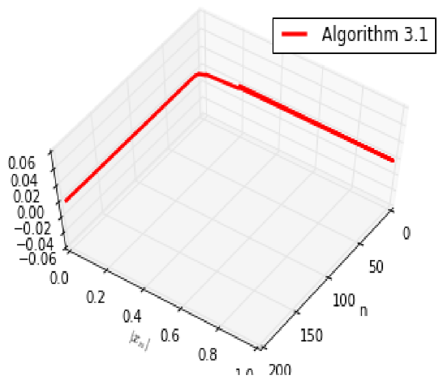
$$\begin{aligned}Tx &= \sum_{i=1}^{\infty} \eta_i\big(\mu_i Jx + (1-\mu_i)T_i x\big) \\ &= \sum_{i=1}^{\infty}\Big[\frac{1}{2^i}\Big(\frac{1}{2^i}J(x_1, x_2, \cdots) + \Big(1 - \frac{1}{2^i}\Big)\frac{1}{2^i}J(0, x_1, x_2, \cdots)\Big)\Big] \\ &= \sum_{i=1}^{\infty}\Big[\frac{1}{2^i}\Big(\frac{1}{2^i}\|x\|^{2-p}(|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots) \\ &\quad + \Big(1 - \frac{1}{2^i}\Big)\frac{1}{2^i}\|x\|^{2-p}(0, |x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots)\Big)\Big] \\ &= \sum_{i=1}^{\infty}\Big[\frac{1}{2^i}\Big(\frac{1}{2^i}\|x\|^{2-p}\big(|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots\big) \\ &\quad + \frac{1}{2^i}\|x\|^{2-p}\big(0, |x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots\big) \\ &\quad - \frac{1}{2^{2i}}\|x\|^{2-p}\big(0, |x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots\big)\Big)\Big] \\ &= \sum_{i=1}^{\infty}\Big[\frac{1}{2^i}\Big(\frac{1}{2^i}\|x\|^{2-p}\big(|x_1|^{p-2}x_1, |x_1|^{p-2}x_1 + |x_2|^{p-2}x_2, |x_2|^{p-2}x_2 \\ &\qquad + |x_3|^{p-2}x_3, \cdots\big) - \frac{1}{2^{2i}}\|x\|^{2-p}\big(0, |x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots\big)\Big)\Big] \\ &= \|x\|^{2-p}\big(|x_1|^{p-2}x_1, |x_1|^{p-2}x_1 + |x_2|^{p-2}x_2, |x_2|^{p-2}x_2 \\ &\quad + |x_3|^{p-2}x_3, \cdots\big)\Big(\sum_{i=1}^{\infty}\frac{1}{2^{2i}}\Big) - \|x\|^{2-p}\big(0, |x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots\big)\Big(\sum_{i=1}^{\infty}\frac{1}{2^{3i}}\Big) \\ &= \frac{1}{3}\|x\|^{2-p}\big(|x_1|^{p-2}x_1, |x_1|^{p-2}x_1 + |x_2|^{p-2}x_2, |x_2|^{p-2}x_2 + |x_3|^{p-2}x_3, \cdots\big) \\ &\quad - \frac{1}{7}\|x\|^{2-p}\big(0, |x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \cdots\big).\end{aligned}$$

4. Numerical Illustration

In the last section, we give an example of a family of relatively J-nonexpansive maps in l_p spaces, $1 < p < \infty$, that has a unique common J-fixed point. For the purpose of illustrating the convergence of our algorithm, we give two examples here where the duality map J is the identity.

Example 4.1. Let $E = \mathbb{R}$ and $Tx = \frac{1}{2}(x - \sin x)$, $C_0 = \mathbb{R}$ in Lemma 3.3. It is easy to see that T is relatively J-nonexpansive with 0 as its unique fixed point. Here J is the identity map on $\mathbb{R}$. Set $\alpha_n = \frac{4n}{4n+5}, \beta = \frac{1}{4}$, $x_0 = x_1 = \frac{1}{2}$. Then, by Lemma 3.3, the sequence generated by algorithm (3.1) converges to zero. The numerical results are sketched in the figures below, where the y-axis represents the value of $|x_n - 0|$ while the x-axis represents the number of iterations n.





5. Conclusion

In this paper, a new class of *relatively J-nonexpansive* maps is introduced and studied. An *inertial* J-fixed point algorithm to approximate a common J-fixed point of a countable family of relatively J-nonexpansive maps is proposed, in a uniformly convex and uniformly smooth real Banach space. Furthermore, strong convergence is established and also an example of the map introduced is given. Finally, a numerical example is presented to show that our algorithm is implementable.

6. Declarations

6.1. Competing Interest

The authors declare that they have no conflict of interest.

6.2. Funding

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