Dedicated to Prof. Suthep Suantai on the occasion of his $60^{th}$ anniversary

# Best Proximity Results on Condensing Operators via Measure of Noncompactness with Application to Integral Equations

**Moosa Gabeleh**[1]**, Mehdi Asadi**[2] **and Erdal Karapınar**[3,4,5,*]

[1] *Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran*
*e-mail : Gabeleh@abru.ac.ir (M. Gabeleh)*

[2] *Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran*
*e-mail : masadi.azu@gmail.com (M. Asadi)*

[3] *ETSI Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*
*e-mail : erdalkarapinar@tdmu.edu.vn (E. Karapinar)*

[4] *Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey.*
*e-mail : erdal.karapinar@cankaya.edu.tr (E. Karapinar)*

[5] *Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan*
*e-mail : erdalkarapinar@yahoo.com (E. Karapinar)*

**Abstract** We prove the best proximity point results for condensing operators on C-class of functions, by using a concept of measure of noncompactness. The results are applied to show the existence of a solution for certain integral equations. We express also an illsutrative examples to indicate the validity of the observed results.

**MSC:** 47H08; 54H25
**Keywords:** integral equations; best proximity; fixed point; measure of noncompactness

## 1. Introduction and Preliminaries

In 1930 Kuratowski [1], introduced the measure of non-compactness $\alpha(S)$ where $S$ is a bounded subset of a metric space $X$. This notion was used effectively in the definition of a Hausdorff measure of non-compactness, $\chi(S)$, see e.g. [2] and the references therein. One of the main aim of this paper is to derive best proximity point results for certain mappings, by using the concept of a measure of noncompactness.

---

*Corresponding author.

We shall present some definitions, notations and results which will be needed in the sequel. Throughout this paper, the letter $E$ represents an infinite dimensional Banach space. The symbols $\overline{\text{co}}(C)$ denotes the closure of convex hull of $C \subset E$, which is the smallest closed and convex set that contains $C$. Furthermore, the expressions $\mathfrak{M}_E$ and $\mathfrak{N}_E$ indicated the family of nonempty bounded subsets of $E$ and the subfamily consisting of all relatively compact subsets of $E$, respectively.

A function $\psi : [0, \infty) \to [0, \infty)$ is called an *altering distance function* [3] if the following properties are satisfied:

(1) $\psi$ is nondecreasing and continuous;
(2) $\psi^{-1}(0) = 0$;
(3) $\psi(t) < t$ for $t > 0$.

The class of all altering distance functions will be denoted by $\Psi$. Also, by $\Phi$ we denote the class of all continuous and nondecreasing functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for all $t > 0$.

A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called $C$-*class function* [4] if it is continuous and satisfies the following axioms:

(1) $F(s, t) \le s$ ;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

We denote $C$-class functions as $\mathcal{C}$, for short.

**Definition 1.1** ([5]). A mapping $\mu : \mathfrak{M}_E \to [0, \infty)$ is said to be a *measure of noncompactness in $E$* if it satisfies the following conditions:

(A1) $\emptyset \ne \text{Ker}\mu := \{X \in \mathfrak{M}_E : \mu(X) = 0\} \subseteq \mathfrak{N}_E$.
(A2) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.
(A3) $\mu(\overline{X}) = \mu(\text{co}X) = \mu(X)$.
(A4) $\mu(\lambda X + (1 - \lambda)Y) \le \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
(A5) If $(X_n)$ is a sequence of closed sets in $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$, for each positive integer $n$, and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the intersection set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

The family $\text{Ker}\mu$ described in $(A1)$ is said to be *the kernel of the measure of noncompactness $\mu$*. Note that the intersection set lies in $\ne \text{Ker}$, that is, $X_\infty \in \text{Ker}\mu$, since $\mu(X_\infty) \le \mu(X_n)$ for any $n$.

The following is one of the pioneer results in the direction of finding fixed point via the measure of non-compactness and it extend the well-known Schauder fixed point theorem.

**Theorem 1.2** ([5]). *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $T : C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(T(X)) \le k\mu(X),$$

*for any subset $X$ of $C$, then $T$ has a fixed point.*

**Definition 1.3.** Let $X$ be a Banach space. We say that $X$ is *strictly convex* if the following implication holds, for all $x, y, p \in X$ and $R > 0$:

$$\begin{cases} \|x - p\| \le R, \\ \|y - p\| \le R, \\ x \ne y \end{cases} \Rightarrow \|\frac{x + y}{2} - p\| < R.$$

Let $A$ and $B$ be two nonempty subsets of a normed linear space $Y$. The pair $(A, B)$ satisfies a property if both $A$ and $B$ satisfy that property. So, we say that $(A, B)$ is *closed* if and only if both $A$ and $B$ are closed; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,\ B \subseteq D$. From now on, $\mathcal{B}(x; r)$ will mean the closed ball in the Banach space $X$ centered at $x \in X$ with radius $r > 0$. We shall also adopt the following notations

$$\delta_x(A) = \sup\{d(x, y) \colon y \in A\} \text{ for all } x \in X,$$
$$\delta(A, B) = \sup\{d(x, y) \colon x \in A,\ y \in B\},$$
$$\mathrm{diam}(A) = \delta(A, A).$$

We mention that if $A$ is a nonempty and compact subset of a Banach space $X$, then $\overline{\mathrm{co}}(A)$ is compact (see Dunford-Schwartz [6]). In addition, we set

$$\mathrm{dist}(A, B) := \inf\{\|x - y\| : (x, y) \in A \times B\},$$

$$A_0 := \{x \in A : \exists\ y' \in B\ :\ \|x - y'\| = \mathrm{dist}(A, B)\ (y' \text{ is called a proximal point of } x)\},$$

$$B_0 := \{y \in B : \exists\ x' \in A\ :\ \|x' - y\| = \mathrm{dist}(A, B)\ (x' \text{ is called a proximal point of } y)\}.$$

**Definition 1.4.** A nonempty pair $(A, B)$ in a normed linear space $Y$ is said to be *proximinal* if $A = A_0$ and $B = B_0$.

It is remarkable to note that if $(A, B)$ is a nonempty, bounded, closed and convex pair in a reflexive Banach space $X$, then $(A_0, B_0)$ is also nonempty, closed and convex.

A mapping $T : A \cup B \to A \cup B$ is said to be
$(i)$   *relatively nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for any $(x, y) \in A \times B$,
$(ii)$   *relatively u-continuous mapping* if

$$\text{for all } \varepsilon > 0,\ \text{there is } \delta > 0 :\ \text{if } \|x - y\|^* < \delta \text{ then } \|Tx - Ty\|^* < \varepsilon,$$

for all $(x, y) \in A \times B$, where $\|x - y\|^* = \|x - y\| - \mathrm{dist}(A, B)$.
$(iii)$   *cyclic* if $T(A) \subseteq B$ and $T(B) \subseteq A$,
$(iv)$   *noncyclic* if $T(A) \subseteq A$ and $T(B) \subseteq B$,
$(v)$   *compact* if the pair $(\overline{T(A)}, \overline{T(B)})$ is compact (see [7]).

**Definition 1.5.** Let $(A, B)$ be a nonempty pair in a Banach space $X$ and $T : A \cup B \to A \cup B$ be a mapping. If $T$ is cyclic, then a point $p \in A \cup B$ is said to be a *best proximity point* for $T$ provided that

$$\|p - Tp\| = \mathrm{dist}(A, B).$$

Also, if $T$ is noncyclic, then the pair $(p, q) \in A \times B$ is called a best proximity pair for $T$ provided that

$$p = Tp, \quad q = Tq, \quad \|p - q\| = \mathrm{dist}(A, B).$$

Existence of best proximity points (pairs) for cyclic (noncyclic) relatively nonexpansive mappings was first studied by Eldred-Kirk-Veeramani ([8]), under a geometric concept of *proximal normal structure*. Here, we state the following existence results which play important roles in our coming discussions.

**Theorem 1.6** ([7])**.** *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a reflexive Banach space $X$. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If $T$ is compact, then it admits a best proximity point.*

**Theorem 1.7** ([9])**.** *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space $X$. Assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping. If $T$ is compact, then it admits a best proximity pair.*

The cyclic (noncyclic) version of condensing mappings was introduced in [7] in order to study the existence of best proximity points (pairs) and to generalize Theorems 1.6 and 1.7 above.

**Definition 1.8.** Let $(A, B)$ be a nonempty and convex pair in a Banach space $X$ and $\mu$ a measure of non-compactness on $X$. A cyclic (noncyclic) mapping $T : A \cup B \to A \cup B$ is said to be a *condensing operator* if there exists $r \in (0, 1)$ such that for any nonempty, bounded, closed, convex, proximal and $T$-invariant pair $(H_1, H_2) \subseteq (A, B)$ such that $\mathrm{dist}(H_1, H_2) = \mathrm{dist}(A, B)$ we have

$$\mu(T(H_1) \cup T(H_2)) \le r\mu(H_1 \cup H_2).$$

Next results are real extensions of Theorem 1.2 due to Darbo.

**Theorem 1.9** ([7])**.** *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a reflexive Banach space $X$ and $\mu$ an measure of non-compactness on $X$. If $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping which is condensing in the sense of Definition 1.8, then it admits a best proximity point.*

The above theorem holds true for noncyclic relatively nonexpansive mapping whenever we add an additional condition "strict convexity":

**Theorem 1.10** ([7])**.** *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space $X$ and $\mu$ an measure of non-compactness on $X$. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping which is condensing in the sense of Definition 1.8, then it admits a best proximity pair.*

We also refer to Gabeleh-Vetro [10] for the generalizations of Theorems 1.9 and 1.10, by considering a class of cyclic (noncyclic) Meir-Keeler condensing operators.

## 2. CONDENSING OPERATORS ON C-CLASS OF FUNCTIONS

Motivated by the class of condensing operators in Definition 1.8, we introduce the following new classes of cyclic (noncyclic) mappings.

**Definition 2.1.** Let $(A, B)$ be a nonempty and convex pair in a Banach space $X$ and $\mu$ an measure of non-compactness on $X$. A cyclic (noncyclic) mapping $T : A \cup B \to A \cup B$ is said to be a *condensing operator on C-class of functions* if for any nonempty, bounded, closed, convex, proximal and $T$-invariant pair $(H_1, H_2) \subseteq (A, B)$ such that $\mathrm{dist}(H_1, H_2) = \mathrm{dist}(A, B)$ we have

$$\psi\Big(\mu\big(T(H_1) \cup T(H_2)\big)\Big) \le F\Big(\psi\big(\mu(H_1 \cup H_2)\big), \varphi\big(\mu(H_1 \cup H_2)\big)\Big), \tag{2.1}$$

for all $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$.

**Remark 2.2.** If in the above definition $\psi(t) = t$ and $F(s, t) = rs$ for all $s, t \in [0, \infty)$ and for some $r \in (0, 1)$, then $T$ is a condensing operator in the sense of Definition 1.8.

**Remark 2.3.** If in the above definition $\psi(t) = t$ and $F(s, t) = s\beta(s)$ for all $s, t \in [0, \infty)$ where $\beta : [0, \infty) \to [0, 1)$ is a function such that $\beta(t_n) \to 1 \Rightarrow t_n \to 0$, then $T$ is a $\beta$-condensing operator which was recently introduced in [9].

We begin our main results with the next existence theorem.

**Theorem 2.4.** *Let $(A, B)$ be a nonempty, disjoint, bounded, closed and convex pair in a strictly convex Banach space $X$ such that $A_0$ is nonempty and $\mu$ is an measure of non-compactness on $X$. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively u-continuous mapping which is a condensing operator on C-class of functions. Then $T$ has a best proximity pair.*

*Proof.* Notice that $(A_0, B_0)$ is closed, convex and proximinal. Relatively u-continuity of the mapping $T$ ensures that $(A_0, B_0)$ is $T$-invariant. By a similar notations of the proof of [9, Theorem 6], we set $\mathcal{A}^0 = A_0$ and $\mathcal{D}^0 = B_0$ and for all $n \in \mathbb{N}$ define

$$\mathcal{C}^n = \overline{\text{co}}(T(\mathcal{C}^{n-1})), \quad \mathcal{D}^n = \overline{\text{co}}(T(\mathcal{D}^{n-1})).$$

Thus

$$\mathcal{C}^1 = \overline{\text{co}}(T(\mathcal{C}^0)) = \overline{\text{co}}(T(A_0)) \subseteq A_0 = \mathcal{C}^0,$$

and iteratively we have $\mathcal{C}^{n-1} \supseteq \mathcal{C}^n$ for all $n \in \mathbb{N}$. Analogously, we find that $\mathcal{D}^{n-1} \supseteq \mathcal{D}^n$ for all $n \in \mathbb{N}$. On the other hand, we have

$$T(\mathcal{C}^n) \subseteq \overline{\text{co}}(T(\mathcal{C}^n)) = \mathcal{C}^{n+1} \subseteq \mathcal{C}^n.$$

Equivalently, we have $T(\mathcal{D}^n) \subseteq \mathcal{D}^n$. Thus, we conclude, for all $n \in \mathbb{N}$, that each pair $(\mathcal{C}^n, \mathcal{D}^n)$ is $T$ invariant and moreover each mentioned pair is closed and convex. Moreover, by the fact that $T$ relatively u-continuous, if $(x, y) \in \mathcal{C}^0 \times \mathcal{D}^0$ with $\|x - y\| = \text{dist}(A, B)$, then $\|T^n x - T^n y\| = \text{dist}(A, B)$ for all $n \in \mathbb{N}$. Since $(T^n x, T^n y) \in \mathcal{C}^n \times \mathcal{D}^n$, we have

$$\text{dist}(\mathcal{C}^n, \mathcal{D}^n) = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$

On the other hand, if $u \in \mathcal{C}^1 = \overline{\text{co}}(T(\mathcal{C}^0))$, then $u = \sum_{j=1}^m c_j T(u_j)$ where $u_j \in \mathcal{C}^0$ for all $1 \leq j \leq m$ such that $c_j \geq 0$ and $\sum_{j=1}^m c_j = 1$. Since $(\mathcal{C}^0, \mathcal{D}^0)$ is proximinal, for all $1 \leq j \leq m$ there exists $v_j \in \mathcal{D}^0$ such that $\|u_j - v_j\| = \text{dist}(\mathcal{C}^0, \mathcal{D}^0) (= \text{dist}(A, B))$ and so $\|Tu_j - Tv_j\| = \text{dist}(A, B)$. Put $v := \sum_{j=1}^m c_j T(v_j)$. Then $v \in \mathcal{D}^1$ and

$$\|u - v\| = \|\sum_{j=1}^m c_j T(u_j) - \sum_{j=1}^m c_j T(v_j)\| \leq \sum_{j=1}^m \|T(u_j) - T(v_j)\| = \text{dist}(A, B).$$

Hence, the pair $(\mathcal{C}^1, \mathcal{D}^1)$ is proximinal. By a similar argument we conclude that the $(\mathcal{C}^n, \mathcal{D}^n)$ is proximinal for all $n \in \mathbb{N} \cup \{0\}$. Notice that if there exists $k \in \mathbb{N}$ for which $\max\{\mu(\mathcal{C}^k), \mu(\mathcal{D}^k)\} = 0$, then $(\mathcal{C}^k, \mathcal{D}^k)$ is a compact pair and the result follows from Theorem 1.7. Thus we suppose that $\max\{\mu(\mathcal{C}^n), \mu(\mathcal{D}^n)\} > 0$ for all $n \in \mathbb{N}$. In view of the fact that $T$ is a condensing operator on C-class of functions, we obtain

$$\psi\big(\mu(\mathcal{C}^{n+1} \cup \mathcal{D}^{n+1})\big) = \psi\big(\max\{\mu(\mathcal{C}^{n+1}), \mu(\mathcal{D}^{n+1})\}\big)$$
$$= \psi\big(\max\{\mu(\overline{\text{co}}(T(\mathcal{C}^n))), \mu(\overline{\text{co}}(T(\mathcal{D}^n)))\}\big)$$
$$= \psi\big(\max\{\mu((T(\mathcal{C}^n))), \mu((T(\mathcal{D}^n)))\}\big)$$
$$\leq \psi\big(\mu(T(\mathcal{C}^n) \cup T(\mathcal{D}^n))\big) \qquad (*)$$

$$\leq F\Big(\psi\big(\mu(\mathcal{C}^n \cup \mathcal{D}^n)\big), \varphi\big(\mu(\mathcal{C}^n \cup \mathcal{D}^n)\big)\Big) \leq \psi\big(\mu(\mathcal{C}^n \cup \mathcal{D}^n)\big). \qquad (2.2)$$

Since $\{\mu(\mathcal{C}^n \cup \mathcal{D}^n)\}$ is a decreasing sequence we may assume that

$$\lim_{n\to\infty} \mu(\mathcal{C}^n \cup \mathcal{D}^n) = r$$

for some $r \geq 0$. Now from (2) and the continuity of the $\psi, F$ we must have

$$\psi(r) \leq F\big(\psi(r), \varphi(r)\big) \leq \psi(r),$$

and so by the property of the function $F$ we conclude that either $\psi(r) = 0$ or $\varphi(r) = 0$. In both cases, we must have $r = 0$. Thereby,

$$\lim_{n\to\infty} \mu(\mathcal{C}^n \cup \mathcal{D}^n) = \max\{\lim_{n\to\infty} \mu(\mathcal{C}^n), \lim_{n\to\infty} \mu(\mathcal{D}^n)\} = 0.$$

It now follows from the condition (A5) of Definition 1.1 that the pair $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is nonempty, closed and convex which is $T$-invariant, where $\mathcal{C}_\infty = \bigcap_{n=0}^\infty \mathcal{C}^n$ and $\mathcal{D}_\infty = \bigcap_{n=0}^\infty \mathcal{D}^n$. Also $\mathrm{dist}(\mathcal{C}_\infty, \mathcal{D}_\infty) = \mathrm{dist}(A, B)$ and clearly, $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is proximal. On the other hand,

$$\max\{\mu(\mathcal{C}_\infty), \mu((\mathcal{D}_\infty))\} = 0,$$

which ensures that the pair $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is compact. Now the result follows from Theorem 1.7. ∎

The cyclic version of Theorem 2.4 can be constructed in order to study the existence of best proximity points in the setting of Banach spaces which are not strictly convex, necessarily.

**Theorem 2.5.** *Let $(A, B)$ be a nonempty, disjoint, bounded, closed and convex pair in a Banach space $X$ such that $A_0$ is nonempty and $\mu$ is an measure of non-compactness on $X$. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping which is a condensing operator on C-class of functions. Then $T$ has a best proximity point.*

*Proof.* Using a similar argument of Theorem 2.4 we have that $(A_0, B_0)$ is closed, convex, proximal and $T$-invariant, that is, $T(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$. By a similar argument of the proof of [9, Theorem 8] we define the sequence $\{(\mathcal{C}^n, \mathcal{D}^n)\}$ as below:

$$\mathcal{C}^n = \overline{\mathrm{co}}(T(\mathcal{C}^{n-1})), \quad \mathcal{D}^n = \overline{\mathrm{co}}(T(\mathcal{D}^{n-1})),$$

where, $\mathcal{C}^0 := A_0$ and $\mathcal{D}^0 := B_0$, then we have

$$\mathcal{C}^1 = \overline{\mathrm{co}}(T(\mathcal{C}^0)) = \overline{\mathrm{co}}(T(A_0)) \subseteq B_0 = \mathcal{D}^0,$$

and so, $T(\mathcal{C}^1) \subseteq T(\mathcal{D}^0)$ which ensures that $\mathcal{C}^2 = \overline{\mathrm{co}}(T(\mathcal{C}^1)) \subseteq \overline{\mathrm{co}}(T(\mathcal{D}^0)) = \mathcal{D}^1$. Iteratively, we obtain $\mathcal{C}^{n+1} \subseteq \mathcal{D}^n$ which is equivalent to say that $\mathcal{D}^n \subseteq \mathcal{C}^{n-1}$ for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{C}^{n+2} \subseteq \mathcal{D}^{n+1} \subseteq \mathcal{C}^n \subseteq \mathcal{D}^{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

This concludes that $\{(\mathcal{C}^{2n}, \mathcal{D}^{2n})\}_{n\geq 0}$ is a decreasing sequence consisting of closed and convex pairs in $A_0 \times B_0$. Besides,

$$T(\mathcal{D}^{2n}) \subseteq T(\mathcal{C}^{2n-1}) \subseteq \overline{\mathrm{co}}(T(\mathcal{C}^{2n-1})) = \mathcal{C}^{2n},$$

$$T(\mathcal{C}^{2n}) \subseteq T(\mathcal{D}^{2n-1}) \subseteq \overline{\mathrm{co}}(T(\mathcal{D}^{2n-1})) = \mathcal{D}^{2n}.$$

Thus $(\mathcal{C}^{2n}, \mathcal{D}^{2n})$ is $T$-invariant. We also can see that by a similar approach of the proof of Theorem 2.4,

$$\mathrm{dist}(\mathcal{C}^{2n}, \mathcal{D}^{2n}) \leq \|T^{2n}x - T^{2n}y\| \leq \|x - y\| = \mathrm{dist}(A, B),$$

and that $(\mathcal{C}^{2n}, \mathcal{D}^{2n})$ is also proximinal for all $n \in \mathbb{N}$. Notice that if

$$\max\{\{\mu(\mathcal{C}^{2k}), \mu(\mathcal{D}^{2k})\} = 0$$

for some $k \in \mathbb{N}$, then the result follows from Theorem 1.9. Let

$$\max\{\{\mu(\mathcal{C}^{2n}), \mu(\mathcal{D}^{2n})\} > 0$$

for all $n \in \mathbb{N}$. By the fact that $T$ is a condensing operator on C-class of functions,

$$
\begin{aligned}
\psi\big(\mu(\mathcal{C}^{2n+2} \cup \mathcal{D}^{2n+2})\big) &= \psi\big(\max\{\mu(\mathcal{C}^{2n+2}), \mu(\mathcal{D}^{2n+2})\}\big) \\
&\leq \psi\big(\max\{\mu(\mathcal{D}^{2n+1}), \mu(\mathcal{C}^{2n+1})\}\big) \\
&= \psi\big(\max\{\mu(\overline{\mathrm{co}}(T(\mathcal{D}^{2n}))), \mu(\overline{\mathrm{co}}(T(\mathcal{C}^{2n})))\}\big) \\
&= \psi\big(\max\{\mu((T(\mathcal{C}^{2n}))), \mu((T(\mathcal{D}^{2n})))\}\big) \\
&\leq \psi\big(\mu(T(\mathcal{C}^{2n}) \cup T(\mathcal{D}^{2n}))\big) \\
&\leq F\Big((\mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n})), \mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n})\Big) \qquad (**) \\
&\leq \psi\big(\mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n})\big).
\end{aligned}
$$

It now follows from the conditions on C-class of functions that

$$\lim_{n\to\infty} \mu(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}) = \max\{\lim_{n\to\infty} \mu(\mathcal{C}^{2n}), \lim_{n\to\infty} \mu(\mathcal{D}^{2n})\} = 0.$$

Now if we set $\mathcal{C}_\infty = \bigcap_{n=0}^{\infty} \mathcal{C}^{2n}$, and $\mathcal{D}_\infty = \bigcap_{n=0}^{\infty} \mathcal{D}^{2n}$ then $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is nonempty, closed, convex, and $T$-invariant with $\mathrm{dist}(A, B) = \mathrm{dist}(\mathcal{C}_\infty, \mathcal{D}_\infty)$ for which we have $\max\{\mu(\mathcal{C}_\infty), \mu((\mathcal{D}_\infty))\} = 0$. Again by using Theorem 1.9 the result follows. ∎

It is worth noticing that if in Theorem 2.4 $A = B$, then the existence of fixed points will be concluded as follows.

**Corollary 2.6** ([11])**.** *Let $A$ be a nonempty, bounded, closed, and convex subset of a Banach space $X$ and let $T : A \to A$ be a nonexpansive mapping such that*

$$\psi(\mu(T(H))) \leq F(\psi(\mu(H)), \varphi(\mu(H))), \qquad (2.3)$$

*for any subset $H \subseteq A$ and where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then $T$ has a fixed point.*

**Remark 2.7.** It is remarkable to note that the considered mapping $T$ in Corollary 2.6 need to be nonexpansive and if that is continuous, then the result still holds (see Theorem 2.1 of [11] for more details).

## 3. Application to a Class of Functional Integral Equations

Let $a > 0$ and $\mathcal{C}([0, a])$ be the family of all continuous real valued functions defined on interval $[0, a]$. It is known that $\mathcal{C}([0, a])$ is a Banach space with the standard norm

$$\|x\| = \max\{|x(t)| : t \in [0, a]\}.$$

Let $X$ be a subset of $\mathfrak{M}_{\mathbb{C}([0,a])}$. For $\varepsilon > 0$ and $x \in X$, we denote by $\omega(x, \varepsilon)$ the modulus of continuity of $x$ defined by

$$\omega(x, \varepsilon) = \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, a], |t_1 - t_2| \leq \varepsilon\}. \qquad (3.1)$$

Furthermore, let $\omega(X, \varepsilon)$ and $\omega_0(X)$ are defined by

$$
\begin{aligned}
\omega(X, \varepsilon) &= \sup\{\omega(x, \varepsilon) : x \in X\}, \\
\omega_0(X) &= \lim_{\varepsilon \to 0} \omega(X, \varepsilon).
\end{aligned}
$$

It was announced in [5] that above function $\omega_0$ is a measure of non-compactness in space $\mathcal{C}[0, a]$.

Let $I = [0, a], J = [0, C]$ and let $\varphi \in \Phi, \psi \in \Psi, F \in \mathcal{C}$. Assume that $\alpha_i, \beta_j, : I \to I$, $\gamma_k : J \to I$, $\phi : I \to \mathbb{R}^+$, are continuous functions, where $1 \leq i \leq m$, $1 \leq j \leq l$, and $1 \leq k \leq n$. Moreover, motivated by the results of [12], we consider the following continuous functions

$$
g : I \times \mathbb{R}^l \to \mathbb{R}, \ f : I \times \mathbb{R}^m \to \mathbb{R}, u : I \times J \times \mathbb{R}^n \to \mathbb{R},
$$

so that

(1) $\exists a_i$ for $1 \leq i \leq m$ such that

$$
|f(t, x_1, \cdots, x_m) - f(t, y_1, \cdots, y_m)| \leq F(\psi(\sum_1^m a_i|x_i - y_i|), \varphi(\sum_1^m a_i|x_i - y_i|)), \ (3.2)
$$

(2) $\exists b_i \ 1 \leq i \leq l$ such that

$$
|g(t, x_1, \cdots, x_l) - g(t, y_1, \cdots, y_l)| \leq F(\psi(\sum_1^l b_i|x_i - y_i|), \varphi(\sum_1^l b_i|x_i - y_i|)), \ (3.3)
$$

(3) $\exists h_i : \mathbb{R}^+ \to \mathbb{R}^+$ for which $h_i$ is nondecreasing for any $1 \leq i \leq n$ and

$$
|u(t, \tau, x_1, \cdots, x_n)| \leq \sum_1^n h_i(|x_i|), \tag{3.4}
$$

where $t \in I, \tau \in J, x_i, y_i \in \mathbb{R}$.

(4) There exists a positive solution $r_0$ of the inequality

$$
Blr + M + C(Amr + N)(\sum_1^n h_i(r)) \leq r, \tag{3.5}
$$

where $B = \max\{b_i : 1 \leq i \leq l\}, A = \max\{a_i : 1 \leq i \leq m\}$ and $M, N$, and $C$ are the positive constants such that

$$
|g(t, 0, 0, \cdots, 0)| \leq M, \quad |f(t, 0, 0, \cdots, 0)| \leq N, \quad \text{and} \quad \phi(t) \leq C, \quad \forall t \in I.
$$

(5) By definition (3.1)

$$
\sup_{t, t' \in I, |t - t'| \leq \varepsilon} \{F(|\psi(x(t) - x(t'))|, |\varphi(x(t) - x(t'))|)\} \leq F(\psi(\omega(x, \varepsilon)), \varphi(\omega(x, \varepsilon)))
$$

$$
\tag{3.6}
$$

Set

$$
\begin{aligned}
x_\beta(t) &:= (x(\beta_1(t)), x(\beta_2(t)), \cdots, x(\beta_l(t))), \\
x_\alpha(t) &:= (x(\alpha_1(t)), x(\alpha_2(t)), \cdots, x(\alpha_m(t))), \\
x_\gamma(t) &:= (x(\gamma_1(t)), x(\gamma_2(t)), \cdots, x(\gamma_n(t))), \quad \forall t \in I.
\end{aligned}
$$

Consider

$$x(t) = g(t, x_\beta(t)) + f(t, x_\alpha(t)) \int_0^{\phi(t)} u(t, \tau, x_\gamma(\tau)) d\tau. \qquad (3.7)$$

$$Tx(t) := g(t, x_\beta(t)) + f(t, x_\alpha(t)) \int_0^{\phi(t)} u(t, \tau, x_\gamma(\tau)) d\tau. \qquad (3.8)$$

**Theorem 3.1** ([11])**.** *Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $T : C \to C$ be continuous mapping, such that*

$$\psi(\mu(T(M))) \le F(\psi(\mu(M)), \varphi(\mu(M))), \qquad (3.9)$$

*for any subset $M$ of $C$ and where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then $T$ has a fixed point.*

**Theorem 3.2.** *Under the assumptions* (3.2),(3.3),(3.4),(3.5) *and* (3.6) *the nonlinear integral equation* (3.8) *has at least a solution.*

*Proof.* By the above conditions, we shall prove the measure of noncompactness $\omega_0(X)$ is satisfying the contraction (3.9). To do this we have some claims:

Claim 1. $Tx \in B_{r_0}$; $B_{r_0}$ is a ball.

Claim 2. Operator $T : B_{r_0} \to B_{r_0}$ is continuous.

Claim 3. Operator $T$ satisfies (3.9) with respect to measure of noncompactness $\omega_0$ in $B_{r_0}$.

To prove Claim 1, we have

$$
\begin{aligned}
|Tx(t)| \quad & \le |g(t, x_\beta(t)) - g(t, \mathbf{0_1})| + |g(t, \mathbf{0_1})| + |f(t, x_\alpha(t)) - f(t, \mathbf{0}_m) + f(t, \mathbf{0}_m)| \\
& \times \int_0^{\phi(t)} |u(t, \tau, x_\gamma(\tau))| d\tau \\[2mm]
& \le F(\psi(\textstyle\sum_1^l b_i |x_{\beta_i}(t)|), \varphi(\textstyle\sum_1^l b_i |x_{\beta_i}(t)|)) + M \\[2mm]
& + C \left( F(\psi(\textstyle\sum_1^m a_i |x_{\alpha_i}(t)|), \varphi(\textstyle\sum_1^m a_i |x_{\alpha_i}(t)|)) + N \right) \left( \textstyle\sum_1^n h(|x_{\gamma_i}(t)|) \right) \\
& \hspace{6cm} \text{by (3.2), (3.3),(3.4)} \\[2mm]
& \le \textstyle\sum_1^l b_i |x_{\beta_i}(t)| + M + C \left( \textstyle\sum_1^m a_i |x_{\alpha_i}(t)| + N \right) \left( \textstyle\sum_1^n h(|x_{\gamma_i}(t)|) \right) \\
& \hspace{3cm} \text{by the definition of } C\text{-class function (1) and (3.4)} \\[2mm]
& \le Bl\|x\| + M + C(Am\|x\| + N) \left( \textstyle\sum_1^n h(\|x\|) \right) \\[2mm]
& \le Blr_0 + M + C(Amr_0 + N) \left( \textstyle\sum_1^n h(r_0) \right) \text{ by (3.5)} \\[2mm]
& \le r_0,
\end{aligned}
$$

where $\mathbf{0}_m = \underbrace{(0, \cdots, 0)}_{m \ times}$ and $\mathbf{0}_m = \underbrace{(0, \cdots, 0)}_{m \ times}$. This result shows that $Tx \in B_{r_0}$.

To prove Claim 2; we prove that operator $T : B_{r_0} \to B_{r_0}$ is continuous. To do this, consider $\varepsilon > 0$ and any $x, y \in B_{r_0}$ such that $|x_i - y_i| \leq \varepsilon$. Then we obtain the following inequalities by using conditions of Theorem

$$
\begin{aligned}
|Tx(t) - Ty(t)| \leq{} & |g(t, x_\beta(t)) - g(t, y_\beta(t))| \\
& + |f(t, x_\alpha(t)) - f(t, y_\alpha(t))| \\
& \times \int_0^{\phi(t)} |u(t, \tau, x_\gamma(\tau))| d\tau \\
& + |f(t, y_\alpha(t)) - f(t, \mathbf{0}_m)| \qquad\qquad (\mathbf{0}_m = \underbrace{(0, \cdots, 0)}_{m\ times}) \\
& + |f(t, \mathbf{0}_m)| \int_0^{\phi(t)} |u(t, \tau, x_\gamma(\tau)) - u(t, \tau, y_\gamma(\tau))| d\tau \\
\leq{} & F(\psi(\sum_1^l a_i |x_{\beta_i}(t) - y_{\beta_i}(t)|), \varphi(\sum_1^l a_i |x_{\beta_i}(t) - y_{\beta_i}(t)|)) \quad \text{(by (3.9))} \\
& + C\left( F(\psi(\sum_1^m b_i |x_{\alpha_i}(t) - y_{\alpha_i}(t)|), \varphi(\sum_1^m b_i |x_{\alpha_i}(t) - y_{\alpha_i}(t)|)) \right) \\
& \times \left( \sum_1^n h_i(|x_{\gamma_i}(t)|) \right) \\
& + \left( \sum_1^n h_i(|y_{\gamma_i}(t)| + N) \right) \int_0^{\phi(t)} |u(t, \tau, x_\gamma(\tau)) - u(t, \tau, y_\gamma(\tau))| d\tau \\
\leq{} & \sum_1^l a_i \psi(|x_{\beta_i}(t) - y_{\beta_i}(t)|) + C\left( \sum_1^m b_i \psi(|x_{\alpha_i}(t) - y_{\alpha_i}(t)|) \right) \\
& \times \left( \sum_1^n h(|x_{\gamma_i}(t)|) \right) \\
& + \left( \sum_1^n h(|y_{\gamma_i}(t)| + N) \right) \int_0^{\phi(t)} |u(t, \tau, x_\gamma(\tau)) - u(t, \tau, y_\gamma(\tau))| d\tau \\
\leq{} & \sum_1^l a_i \|x - y\| + C\left( \sum_1^m b_i \|x - y\| \right) \left( \sum_1^n h_i(\|x\|) \right) \\
& + (An\|y\| + N) C\omega_u(I, \varepsilon) \\
\leq{} & mB\varepsilon + CAm\varepsilon \left( \sum_1^n h(r_0) \right) + (Anr_0 + N) C\omega_u(I, \varepsilon),
\end{aligned}
$$

where

$$
\omega_u(I, \varepsilon) = \sup_{t \in I, \tau \in J, x_i, y_i \in R_0, 1 \leq i \leq m, |x_i - y_i| \leq \varepsilon} \{|u(t, \tau, x_1, \cdots, x_m) - u(t, \tau, y_1, \cdots, y_m)|\}
$$

where $J := [0, C]$ and $R_0 = [-r_0, r_0]$. $u$ is uniformly continuous on $I \times J \times R_0^m$ and $\omega_u(I, \varepsilon) \to 0$ as $\varepsilon \to 0$. So $T$ is continuous on $B_{r_0}$.

To prove Claim 3; we show that operator $T$ satisfies (3.9) with respect to measure of noncompactness $\omega_0$ in $B_{r_0}$.

Fix arbitrary $\varepsilon > 0$. Let us consider $x \in X$ and $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \varepsilon$, for any nonempty subset $X$ of $B_{r_0}$:

$$
\begin{aligned}
|Tx(t_1) - Tx(t_2)| &\leq |g(t_1, x_\beta(t_1)) - g(t_1, x_\beta(t_2))| + |g(t_1, x_\beta(t_2)) - g(t_2, x_\beta(t_2))| \\
&\quad + [|f(t_1, x_\alpha(t_1)) - f(t_1, x_\alpha(t_2))| + |f(t_1, x_\alpha(t_2)) - f(t_2, x_\alpha(t_2))|] \\
&\quad \times \int_0^{\phi(t_1)} |u(t_1, \tau, x_\gamma(\tau))| d\tau \\
&\quad + |f(t_2, x_\alpha(t_2))| \int_0^{\phi(t_1)} |u(t_1, \tau, x_\gamma(\tau)) - u(t_2, \tau, x_\gamma(\tau))| d\tau \\
&\quad + |f(t_2, x_\alpha(t_2))| \int_{\phi(t_1)}^{\phi(t_2)} |u(t_2, \tau, x_\gamma(\tau))| d\tau \\
&\leq F(\psi(\sum_1^l b_i |x_{\beta_i}(t_1) - x_{\beta_i}(t_2)|), \varphi(\sum_1^l b_i |x_{\beta_i}(t_1) - x_{\beta_i}(t_2)|)) \\
&\quad + \omega_g(I, \varepsilon) \\
&\quad + C \left( F(\psi(\sum_1^m a_i |x_{\alpha_i}(t_1) - x_{\alpha_i}(t_1)|), \varphi(\sum_1^m a_i |x_{\alpha_i}(t_1) - x_{\alpha_i}(t_2)|)) \right. \\
&\quad + \omega_f(I, \varepsilon)) \\
&\quad \times \left( \sum_1^n h_i(|x_{\gamma_i}(\tau)|) \right) + [|f(t_2, x_{\alpha_i}(t_2)) - f(t, \mathbf{0})| \\
&\quad + |f(t, \mathbf{0})| \left( C\omega_u(I, \varepsilon) + \omega(\phi, \varepsilon) \left( \sum_1^n h_i(|x_{\gamma_i}(\tau)|) \right) \right) \\
&\leq F(\psi(B\sum_1^l \omega(x, \omega(\beta_i, \varepsilon))), \varphi(B\sum_1^l \omega(x, \omega(\beta_i, \varepsilon)))) \\
&\quad + \omega_g(I, \varepsilon) \hspace{4cm} \text{(by (3.6))} \\
&\quad + C \left( F(\psi((A\sum_m^l \omega(x, \omega(\alpha_i, \varepsilon))), \varphi((A\sum_m^l \omega(x, \omega(\alpha_i, \varepsilon)))) + \omega_f(I, \varepsilon) \right) \\
&\quad \times \left( \sum_1^n h_i(\|x\|) \right) \\
&\quad + \left( \sum_1^n a_i \|x\| + N \right) \left( C\omega(x, \omega_u(I, \varepsilon)) + \omega(\phi, \varepsilon) \left( \sum_1^n h_i(\|x\|) \right) \right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|Tx(t_1) - Tx(t_2)| \;\leq\; & F(\psi(Bm\omega(X,\varepsilon), \varphi(Bm\omega(X,\varepsilon)))) + \omega_g(I,\varepsilon) \\
& + C\left(F(\psi((Al\omega(X,\varepsilon))), \varphi((Al\omega(X,\varepsilon))))) + \omega_f(I,\varepsilon)\right)\left(\sum_1^n h_i(\|x\|)\right) \\
& + \left(\sum_1^n a_i\|x\| + N\right)\left(C\omega(x,\omega_u(I,\varepsilon)) + \omega(\phi,\varepsilon)\left(\sum_1^n h_i(\|x\|)\right)\right)
\end{aligned}
$$

(3.10)

$$
\begin{aligned}
\;\leq\; & B\sum_1^m \omega(x,\omega(\beta_i,\varepsilon)) + \omega_g(I,\varepsilon) \\
& + C\left(A\sum_1^l \omega(x,\omega(\beta_i,\varepsilon)) + \omega_g(I,\varepsilon)\right)\left(\sum_1^n h_i(\|x\|)\right) \\
& + \left(\sum_1^n a_i\|x\| + N\right)\left(C\omega(x,\omega_u(I,\varepsilon)) + \omega(\phi,\varepsilon)\left(\sum_1^n h_i(\|x\|)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\;\leq\; & B\sum_1^m \omega(x,\omega(\beta_i,\varepsilon)) + \omega_g(I,\varepsilon) \\
& + C\left(A\sum_1^l \omega(x,\omega(\beta_i,\varepsilon)) + \omega_g(I,\varepsilon)\right)\left(\sum_1^n h_i(r_0)\right) \\
& + (Anr_0 + N)\left(C\omega_u(I,\varepsilon) + \omega(\phi,\varepsilon)\left(\sum_1^n h_i(r_0)\right)\right),
\end{aligned}
$$

where

$$\omega_g(I,\varepsilon) = \sup\{|g(t,x_1,...,x_l) - g(t',x_1,...,x_l)| : t,t' \in I, x_i \in R_0, 1 \leq i \leq l, |t-t'| \leq \varepsilon\}$$
$$\omega_f(I,\varepsilon) = \sup\{|f(t,x_1,...,x_m) - g(t',x_1,...,x_m)| : t,t' \in I, x_i \in R_0, 1 \leq i \leq m, |t-t'| \leq \varepsilon\}$$
$$\omega_u(I,\varepsilon) = \sup\{|u(t,\tau,x_1,...,x_m) - u(t,\tau,y_1,...,y_n)| : t \in I, \tau \in J, x_i,y_i \in R_0, 1 \leq i \leq n,$$
$$|x_i - y_i| \leq \varepsilon\},$$

also

$$\omega_{\alpha_i}(I,\varepsilon) = \sup\{|\alpha_i(t) - \alpha_i(t')| : t,t' \in I, |t-t'| \leq \varepsilon\},$$
$$\omega_{\beta_i}(I,\varepsilon) = \sup\{|\beta_i(t) - \beta_i(t')| : t,t' \in I, |t-t'| \leq \varepsilon\},$$
$$\omega(\phi,\varepsilon) = \sup\{|\phi(t) - \phi(t')| : t,t' \in I, |t-t'| \leq \varepsilon\},$$

By (3.10),

$$
\begin{aligned}
\psi(\omega(TX,\varepsilon)) \;\leq\; & \omega(TX,\varepsilon) \\
\;\leq\; & F(\psi(Bm\omega(X,\varepsilon), \varphi(Bm\omega(X,\varepsilon)))) + \omega_g(I,\varepsilon) \\
& + C\left(F(\psi((Al\omega(X,\varepsilon))), \varphi((Al\omega(X,\varepsilon))))) + \omega_f(I,\varepsilon)\right)\left(\sum_1^n h_i(r_0)\right) \\
& + \left(\sum_1^n a_i\|x\| + N\right)\left(C\omega(x,\omega_u(I,\varepsilon)) + \omega(\phi,\varepsilon)\left(\sum_1^n h_i(r_0)\right)\right)
\end{aligned}
$$

so we obtain $\omega(\alpha_i, \varepsilon) \to 0, \omega(\beta_i, \varepsilon) \to 0$ and $\omega(\phi, \varepsilon) \to 0$ as $\varepsilon \to 0$, by uniformly continuous of $\alpha_i, \beta_i$ on $I$. And similarly $\omega_f(I, \varepsilon) \to 0, \omega_g(I, \varepsilon) \to 0$ and $\omega_u(I, \varepsilon) \to 0$ as $\varepsilon \to 0$, by uniformly continuous of $f, g, u$ on $I \times R_0^m, I \times R_0^l$ and $I \times [0, C] \times R_0^n$, respectively. Hence

$$\psi(\omega_0(T(X))) \le F(\psi(\omega_0(X)), \varphi(\omega_0(X))).$$

Therefore, by Theorem 3.1 we get that $T$ has at least one fixed point in $B_{r_0}$. Consequently, nonlinear functional integral equation (3.8) has at least one continuous solution in $B_{r_0} \subseteq C(I)$. This completes the proof. ∎

In what follows, we present an example to illustrate Theorem 3.2.

**Theorem 3.3** ([2]). *Let $T$ be the self-operator on $BC([0, \infty))$ in (3.8). If*

*(i) the function $t \to g(t, \mathbf{0})$ is a member of the space $BC([0, \infty))$;*

*(ii) there exists $\delta \in [1, +\infty)$ such that, for each $t \in [0, \infty)$, we have*

$$|g(t, x_\beta(t)) - g(t, y_\beta(t))| \le 2e^\delta \|x_\beta(t) - y_\beta(t)\|$$

*(iii) there are continuous $c_0, c_1 : [0, \infty) \to [0, \infty)$ such that*

$$\lim_{t \to \infty} c_0(t) \int_0^t c_1(s) ds = 0$$

*and $c_0(t)c_1(s) \ge |G(t, s, u)|$ for all $t, s \in [0, \infty)$ such that $t \ge s$, and for each $u \in \mathbb{R}$;*

*(iv) there exists a positive $r_0$ such that $(e^\alpha - 1)r_0 \ge e^\alpha m$, where $m$ is given by*

$$m^* = \sup_{t \ge 0}\{|g(t, \mathbf{0})| + c_0(t) \int_0^t c_1(s) ds\},$$

*then $T$ admits a fixed point in $BC([0, \infty))$.*

Fixed $t \in [0, \infty)$, we get $C(t) = \{u(t) : u \in C\}$ and hence we consider the measure of noncompactness $\mu$ on the family of all nonempty bounded, closed and convex subsets of $BC([0, \infty))$, say $B(BC([0, \infty)))$, as follows

$$\mu(C) = \omega_0(C) + \limsup_{t \to \infty} \operatorname{diam} C(t), \qquad (3.11)$$

where $\operatorname{diam} C(t) = \sup\{|u(t) - v(t)| : u, v \in C\}$.

**Example 3.4.** Put

$$f(t, x_\alpha(t)) = \frac{1}{re^\alpha}\left(\frac{1 + \sum_{i=1}^m |x_i|}{1 + t + \sum_{i=1}^m |x_i|}\right),$$

$$g(t, x_\beta(t)) = \frac{1}{r}\left(\frac{1 + t^2}{2 + t^2} \frac{\ln(1 + \sum_{i=1}^l |x_i|)}{2\sqrt{e^\alpha} + \ln(1 + \sum_{i=1}^l |x_i|)} + 2e^{-t}\right), \quad t \in [0, 1],$$

$$|u(t, \tau, x_\gamma(\tau))| \le \frac{\cos\|x_\gamma(\tau)\|}{1 + t^2} e^{-t} e^{\tau/2},$$

$$\varphi(t) = \sqrt{t},$$

$$\psi(t) = \frac{t}{1 + t},$$

$$F(s, t) = \frac{s}{2e^\alpha},$$

$$h_1 = h_2 = \cdots = h_n = 2.$$

Consider the following functional integral equation

$$
\begin{aligned}
x(t) \;=\; \tfrac{1}{r}&\left(\frac{1+t^2}{2+t^2}\frac{\ln(1+\sum_{i=1}^{l}|x_i|)}{2\sqrt{e^\alpha}+\ln(1+\sum_{i=1}^{l}|x_i|)}+2e^{-t}\right)\\
&+\frac{1}{2re^\alpha}\left(\frac{1+\sum_{i=1}^{m}|x_i|}{1+t+\sum_{i=1}^{m}|x_i|}\right)\int_0^{\varphi(t)}u(t,\tau,x_\gamma(\tau))d\tau,
\end{aligned}
$$

in the space $BC([0,1])$.

We have

$$
M = N = A = B = \frac{1}{r} \quad \text{and} \quad \mathrm{C} = 1,
$$

so

$$
Blr + M + C(Amr + N)(\sum_{1}^{n} h_i(r)) \le l + \frac{1}{r} + (m + \frac{1}{r})n \le r, \tag{3.12}
$$

inequality (3.12) holds for some $r := l + mn + n + 1 > 1$. Clearly, $g$ is continuous and is such that the function $t \to g(t,\mathbf{0})$ is an element of $BC([0,1])$.

We have

$$
\begin{aligned}
0 &\le |f(t,x_\alpha(t)) - f(t,y_\alpha(t))| \\
&\le \frac{1}{2re^\alpha}\left(\frac{1+\sum_{i=1}^{m}|x_i|}{1+t+\sum_{i=1}^{m}|x_i|} - \frac{1+\sum_{i=1}^{m}|y_i|}{1+t+\sum_{i=1}^{m}|y_i|}\right) \\
&\le \frac{1}{2re^\alpha}\left(\frac{t\sum_{i=1}^{m}(|x_i|-|y_i|)}{1+t+\sum_{i=1}^{m}(|x_i|-|y_i|)}\right) \\
&\le \frac{1}{2re^\alpha}\left(\frac{\sum_{i=1}^{m}(|x_i-y_i|)}{1+\sum_{i=1}^{m}(|x_i-y_i|)}\right) \\
&\le \frac{1}{2re^\alpha}\left(\psi(\sum_{i=1}^{m}(|x_i-y_i|))\right) \\
&\le \frac{1}{r}\left(F(\psi(\sum_{i=1}^{m}(|x_i-y_i|)),\varphi((\sum_{i=1}^{m}(|x_i-y_i|))))\right) \\
&\le F\left(\psi(\sum_{i=1}^{m}(|x_i-y_i|)),\varphi((\sum_{i=1}^{m}(|x_i-y_i|)))\right),
\end{aligned}
$$

(according to (3.2) of Theorem 3.2)

and likewise (3.3) of Theorem 3.2 holds.

$$
\begin{aligned}
0 &\le |g(t,x_\beta(t)) - g(t,y_\beta(t))| \\
&\le \frac{1}{2e^\alpha}\sum_{i=1}^{l}(|x_i-y_i|) \quad \text{for all} \quad \alpha \in [1,\infty) \\
&\le \frac{1}{2e^\alpha}\sum_{i=1}^{l}(|x_i-y_i|) \le 2e^\delta\|x_\beta(t)-y_\beta(t)\|
\end{aligned}
$$

(for some $\delta \in [1,\infty)$; (ii) of Theorem 3.3)

and proof of Theorem 3.3. And

$$|u(t, \tau, x_\gamma(\tau))| \leq \frac{\cos \|x_\gamma(\tau)\|}{1 + t^2} e^{-t} e^{\tau/2}$$

$$\leq \frac{1}{1 + t^2} e^{-t} e^{\tau/2} \leq \frac{2}{1 + t^2} \leq 2n = \sum_{i=1}^{n} h_i(|x_i|).$$

(according to (3.4) of Theorem 3.2)

Let $c_1, c_2 : [0, \infty) \to [0, \infty)$ be defined by

$$c_1(t) = e^{-t}, \quad c_2(\tau) = e^{\tau/2} \quad \text{for all} \quad \text{t}, \tau \in [0, \infty),$$

which means condition (ii) of Theorem 3.3 holds.
By

$$|u(t, \tau, x_\gamma(\tau))| \leq \frac{\cos \|x_\gamma(\tau)\|}{1 + t^2} e^{-t} e^{s/2} \leq e^{-t} e^{s/2} \quad \text{for all} \quad \text{t}, s \in [0, \infty).$$

Clearly,

$$\lim_{t \to \infty} e^{-t} \int_0^t e^{s/2} ds = \lim_{t \to \infty} 2e^{-t}(e^{t/2} - 1) = 0,$$

the condition (iii) form Theorem 3.3 is also holds. Also,

$$m^* = \sup_{t \geq 0}\{|g(t, \mathbf{0})| + c_0(t) \int_0^t c_1(s)ds\} = \sup_{t \geq 0}\{2e^{-t} + 2e^{-t}(2e^{t/2} - 1)\} = 2.$$

Fixed $t \in [0, \infty)$, we get $C(t) = \{u(t) : u \in C\}$ and hence we consider the measure of noncompactness $\mu$ on the family of all nonempty bounded, closed and convex subsets of $BC([0, \infty))$, say $B(BC([0, \infty)))$, as follows

$$\mu(C) = \omega_0(C) + \limsup_{t \to \infty} \operatorname{diam} C(t), \tag{3.13}$$

where $\operatorname{diam} C(t) = \sup\{|u(t) - v(t)| : u, v \in C\}$. and so we get

$$\limsup_{t \to \infty} \operatorname{diam}(T(C))(t) \leq \frac{1}{2e^\delta} \limsup_{t \to \infty} \operatorname{diam}(C)(t). \tag{3.14}$$

By (3.13) and (3.14), we deduce that

$$\psi(\mu(T(C))) = \frac{\mu(T(C))}{1 + \mu(T(C))}$$
$$\leq \mu(T(C))$$
$$\leq \frac{1}{2e^\delta} \mu(C)$$
$$= F(\psi(\mu(C)), \varphi(\mu(C))).$$

If we put $r_0 = 3$ in the condition (iv) of Theorem 3.3 will be hold. So, Theorem 3.3 confirms that the operator

$$Tx(t) := g(t, x_\beta(t)) + f(t, x_\alpha(t)) \int_0^{\varphi(t)} u(t, \tau, x_\gamma(s))ds, \tag{3.15}$$

has solution.

## References

[1] K. Kuratowski, Sur les espaces complets, Fund. Math 5 (1930) 301–309.

[2] M. Asadi, M. Gabeleh, C. Vetro, A new approach to the generalization of Darbo's fixed point problem by using simulation functions with application to integral equations, Results Math. (2019) 74–86.

[3] M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorem by altering distances between the points, Bull. Aust. Math. Soc. 30 (1) (1984) 1–9.

[4] A.H. Ansari, Note on $(\psi,\phi)$-contractive type mappings and related fixed point, Conference: The 2nd Regional Conference on Mathematics and Applications, Payame Noor University (2014) 377–380.

[5] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics, Vol. 60. Dekker, New York, 1980.

[6] N. Dunford, J.T. Schwartz, Linear Operators, 3, Interscience, New York, 1971.

[7] M. Gabeleh, J. Markin, Optimum solutions for a system of differential equations via measure of noncompactness, Indag. Math. (N.S.) 29 (2018) 895–906.

[8] A.A. Eldred, W.A. Kirk, P. Veeramani, Proximal normal structure and relatively nonexpansive mappings, Studia Math. 171 (2005) 283–293.

[9] M. Gabeleh, S.P. Moshokoa, C. Vetro, Cyclic (noncyclic) $\varphi$- condensing operator and its application to a system of differential equations, Nonlinear Analysis: Modelling and Control, (to appear).

[10] M. Gabeleh, C. Vetro, A new extension of Darbo's fixed point theorem using relatively Meir-Keeler condensing operators, Bull. Aust. Math. Soc. 98 (2018) 286–297.

[11] F. Nikbakhtsarvestani, S.M. Vaezpour, M. Asadi, $F(\psi,\phi)$-contraction in terms of measure of noncompactness with application for nonlinear integral equations, J. Inequal. Appl. 2017 (1) (2017).

[12] C.M. Chen, E. Karapinar, G.T. Chen, On the Meir-Keeler-Khan set contractions, J. Nonlinear Sci. Appl. 9 (8) (2016) 5271–5280.

[13] A. Almeida, E. Karapinar, K. Sadarangani, A note on best proximity point theorems under weak P-property, Abstract and Applied Analysis (2014) Article ID 716825.

[14] A. Aghajani, N. Sabzali, Existence of coupled fixed points via measure of noncompactness and applications, J. Nonlinear Convex Anal. 15 (2014) 941–952.

[15] J.M. Ayerbe Toledano, T.D. Benavides, G.L. Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Operator Theory: Advances and Applications, Vol. 99, Birkhauser, Basel, 1997.

[16] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, Measures of Noncompactness and Condensing Operators, Vol. 55, Birkhauser, Basel, 1992.

[17] M.S. Brodskii, D.P. Milman, On the center of a convex set, Dokl. Akad. Nauk SSSR 59 (1948) 837–840.

[18] G. Darbo, Punti uniti in transformazioni a codominio non compatto, Rend. Semin. Mat. Univ. Padova 24 (1955) 84–92.

[19] M. Gabeleh, Remarks on minimal sets for cyclic mappings in uniformly convex Banach spaces, Numer. Funct. Anal. Optim. 38 (2017) 360–375.

[20] M. Gabeleh, A characterization of proximal normal structure via proximal diametral sequences, J. Fixed Point Theory Appl. 19 (2017) 2909–2925.

[21] M. Gabeleh, Best Proximity Points for Cyclic Mappings, Ph.D Thesis, 2012.

[22] M.U. Ali, B.Alqahtani, T. Kamran, E. Karapinar, Best Proximity Point Results for Controlled Proximal Contraction, IEEE Access 7 (2019) 128009–128013.

[23] A. Kostic, E. Karapinar, V. Rakocevic, Best proximity points and fixed points with $R$-functions in the framework of $w$-distances, Bull. Aust. Math. Soc. 99 (3) (2019).

[24] E. Karapinar, C.M. Chen, C.T. Lee, Best proximity point theorems for two weak cyclic contractions on metric-like spaces, Mathematics 7 (2019).

[25] A. Samadi, M.B. Ghaemi, An extension of Darbo fixed point theorem and its applications to coupled fixed point and integral equations, Filomat 28 (2014) 879–886.

[26] E. Karapinar, S. Karpagam, P. Magadevan, B. Zlatanov, On $\Omega$ class of mappings in a $p$-cyclic complete metric space, Symmetry 11 (4) (2019).

[27] E. Karapinar, Fixed point theory for cyclic weak $\phi$-contraction, Appl. Math. Lett. 24 (6) (2011) 822–825.

[28] E. Karapinar, Best proximity points of Kannan type cylic weak phi-contractions in ordered metric spaces, Analele Stiintifice ale Universitatii Ovidius Constanta 20 (3) (2012) 51–64.

[29] J. Schauder, Der Fixpunktsatz in Funktionalriiumen, Studia Math. 2 (1930) 171–180.