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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

The General Intermixed Iteration for Equilibrium Problems and Variational Inequality Problems in Hilbert Spaces

Sarawut Suwannaut

Department of Mathematics, Faculty of Science, Lampang Rajabhat University, Lampang, Thailand
e-mail : sarawut-suwan@hotmail.co.th

Abstract In this paper, the general intermixed iteration for two nonlinear mappings is introduced. A strong convergence theorem for approximating a common solution of two finite families of equilibrium problems and variational inequality problems in a Hilbert space is proved. Furthermore, a numerical example for main theorem is given to support the result.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C\times C \to \mathbb{R}$ be a bifunction. *The equilibrium problem* for F is to determine its equilibrium point, i.e., the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \tag{1.1}$$

Equilibrium problems were introduced by [1] in 1994 where such problems have had a significant impact and influence in the development of several branches of pure and applied sciences. Various problems in physics, optimization, and economics are related to seeking some elements of $EP(F)$, see [1, 2]. Many authors have been investigated iterative algorithms for the equilibrium problems, see, for example, [2, 3].

If we take $F(x, y) = \langle y - x, Ax\rangle$, where $A : C \to H$ is a nonlinear mapping, then the classical equilibrium problem is equivalent to finding an element $x \in C$ such that

$$\langle y - x, Ax\rangle \geq 0, \forall y \in C, \tag{1.2}$$

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which is well-known as *the classical variational inequality problem.* The solution set of the problem (1.2) is denoted by $VI(C, A)$.

Variational inequalities were introduced and investigated by Stampacchia [4] in 1964. It is now well known that variational inequalities cover as diverse disciplines as optimal control, optimization, mathematical programming, mechanics and finance, see [5–7]. There are several techniques to analyze various iterative methods for solving variational inequality problem and the related optimization problems, see [8–10] and the references therein.

In 2013, Suwannaut and Kangtunyakarn [3] introduced *the combination of equilibrium problem* which is to find $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x, y) \geq 0, \forall y \in C, \tag{1.3}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \ldots, N$. The set of solution (1.3) is denoted by $EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i)$. If $F_i = F, \forall i = 1, 2, \ldots, N$, then the combination of equilibrium problem (1.3) reduces to the equilibrium problem (1.1).

The fixed point problem for the mapping $T : C \to C$ is to find $x \in C$ such that $x = Tx$. We denote the fixed point set of a mapping T by $Fix(T)$.

Definition 1.1. Let $T : C \to C$ be a mapping. Then

(i). a mapping T is called *contractive* if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in C;$$

(ii). a mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C;$$

(iii). T is said to be κ-*strictly pseudo-contractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in C.$$

Note that the class of κ-strictly pseudo-contractions strictly includes the class of nonexpansive mappings, that is, a nonexpansive mapping is a 0-strictly pseudo-contractive mapping.

For the last decades, many researchers have studied fixed point theorems associated with various types of nonlinear mappings, see, for instance, [11–18].

In 2015, Yao *et al.* [19] proposed the intermixed algorithm for two strictly pseudo-contractive mappings S and T as follows:

For arbitrarily given $x_0 \in C, y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$x_{n+1} = (1 - \beta_n) x_n + \beta_n P_C [\alpha_n f(y_n) + (1 - k - \alpha_n) x_n + kTx_n], n \geq 0,$$
$$y_{n+1} = (1 - \beta_n) y_n + \beta_n P_C [\alpha_n g(x_n) + (1 - k - \alpha_n) y_n + kSy_n], n \geq 0, \tag{1.4}$$

where $T : C \to C$ is a λ-strictly pseudo-contraction, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1 - \lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $(0, 1)$.

Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.4) converge independently to $P_{Fix(T)}f(y^*)$ and $P_{Fix(S)}g(x^*)$, respectively, where $x^* \in Fix(T)$ and $y^* \in Fix(S)$.

In 2018, Suwannaut [20] introduced *the S-intermixed iteration* for two finite families of nonlinear mappings as in the following algorithm:

Starting with $x_1, y_1, z_1 \in C$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined by

$$\begin{aligned} x_{n+1} &= (1-\beta_n)\, x_n + \beta_n \left(\alpha_n f_1(y_n) + (1-\alpha_n)\, Sx_n\right), \\ y_{n+1} &= (1-\beta_n)\, y_n + \beta_n \left(\alpha_n f_2(x_n) + (1-\alpha_n)\, Ty_n\right), n \geq 1, \end{aligned} \tag{1.5}$$

where $S, T : C \to C$ are nonlinear mappings with $Fix(S) \cap Fix(T) \neq \emptyset$, $f_i : C \to C$ is a contractive mapping with coefficients $\rho_i; i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0, 1)$, $\forall n \geq 1$.

Moreover, a strong convergence theorem for finding a common solution of two finite families of equilbrium problems is proved.

Inspired by previous research described above, we introduce the new iterative method called *the general intermixed iteration* for two finite families of nonlinear mappings as in the following algorithm:

Starting with $x_1, y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{aligned} x_{n+1} &= (1-\beta_n)\, x_n + \beta_n P_C \left(\alpha_n \gamma f_1(y_n) + (I-\alpha_n A)\, Sx_n\right), \\ y_{n+1} &= (1-\beta_n)\, y_n + \beta_n P_C \left(\alpha_n \gamma f_2(x_n) + (I-\alpha_n A)\, Ty_n\right), n \geq 1, \end{aligned} \tag{1.6}$$

where $S, T : C \to C$ are nonlinear mappings with $Fix(S) \cap Fix(T) \neq \emptyset$, $f_i : C \to C$ is a contractive mapping with coefficients $\rho_i; i = 1, 2$ and $\rho = \min_{i \in \{1,2\}}\{\rho_i\}$, A is a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\rho}$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0, 1)$, $\forall n \geq 1$.

Remark 1.2. The general intermixed iteration (1.6) is a modification and extension of several iterations as follows:

(1) If $\gamma = 1$ and $A = I$, then the general intermixed iteration (1.6) reduces to the S-intermixed iteration (1.5). Hence, the S-intermixed iteration is a special case of the general intermixed iteration.

(2) The general intermixed iteration (1.6) can be seen as a modification and extension of the intermixed algorithm (1.4) in sense that the constant k is not considered.

Motivated by the related research, we introduce the general intermixed iteration for two nonlinear mappings. Under some control conditions, a strong convergence theorem for finding a common solution of a finite family of equilibrium problems and a finite family of variational inequality problems is proved in Hilbert spaces. Finally a numerical example is given in a space of real numbers.

2. Preliminaries

Let H be a real Hilbert space and C a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $''\rightharpoonup''$ and $''\to''$, respectively. In a real Hilbert space H, it is well known that

$$\|\alpha x+(1-\alpha)y\|^2=\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2,$$

for all $x,y\in H$ and $\alpha\in[0,1]$.

For every $x\in H$, there is a unique nearest point P_Cx in C such that

$$\|x-P_Cx\|\leq\|x-y\|,\forall y\in C.$$

Such an operator P_C is called the metric projection of H onto C.

Recall that H satisfies *Opial's condition* [21], i.e., for any sequence $\{x_n\}$ with $x_n\rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|$$

holds for every $y\in H$ with $y\neq x$.

To prove the main results in this paper, the following lemmas and remark are used:

Definition 2.1. Let $T:H\to H$ be a mapping. Then T is called

(i). a *strongly positive operator on* H if there exists a constant $\bar{\gamma}>0$ with property

$$\langle Tx,x\rangle\geq\bar{\gamma}\|x\|^2,\forall x\in H.$$

(ii). ξ*-inverse-strongly monotone* if there exists a positive real number ξ such that

$$\langle x-y,Tx-Ty\rangle\geq\xi\|Tx-Ty\|^2,\forall x,y\in H.$$

Lemma 2.2. *Let H be a real Hilbert space. Then, the following inequality holds:*

$$\|x+y\|^2\leq\|x\|^2+2\langle y,x+y\rangle,$$

for all $x,y\in H$.

Lemma 2.3 ([22])**.** *For a given $z\in H$ and $u\in C$,*

$$u=P_Cz\Leftrightarrow\langle u-z,v-u\rangle\geq 0,\forall v\in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_Cx-P_Cy\|^2\leq\langle P_Cx-P_Cy,x-y\rangle,\forall x,y\in H.$$

Lemma 2.4 ([23])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1}\leq(1-\alpha_n)s_n+\delta_n,\forall n\geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i). $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii). $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

Then, $\lim_{n\to\infty}s_n=0$.

Lemma 2.5 ([22]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.6 ([24]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \emptyset$. Then,*

$$VI(C, aA + (1 - a)B) = VI(C, A) \cap VI(C, B), \ \forall a \in (0, 1).$$

Furthermore, if $0 < \lambda < \min\{2\alpha, 2\beta\}$, then we have $I - \lambda\,(aA + (1 - a)B)$ is nonexpansive mapping.

From concept of Lemma 2.6, we can prove the following result.

Lemma 2.7. *For $i = 1, 2, \ldots, N$, let $A_i : C \to H$ be an α_i-inverse strongly monotone with $0 < \lambda < 2\alpha_i$ and $\bigcap_{1=1}^{N} VI(C, A_i) \neq \emptyset$. Then we have the following statements:*

(i). $VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{1=1}^{N} VI(C, A_i)$,

(ii). $I - \lambda \sum_{i=1}^{N} a_i A_i$ is a nonexpansive mapping,

where $a_i \in (0, 1)$, for $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$

Proof. To prove *(i)*, we will show that $\frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i$ is $\frac{\lambda}{2}$-inverse strongly monotone. For every $x, y \in C$, we obtain

$$\begin{aligned}
\langle \frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i x - \frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i y, x - y\rangle &= \langle \sum_{i=2}^{N} \frac{a_i}{1-a_1}\left(A_i x - A_i y\right), x - y\rangle \\
&= \sum_{i=2}^{N} \frac{a_i}{1-a_1}\langle A_i x - A_i y, x - y\rangle \\
&\geq \sum_{i=2}^{N} \frac{\alpha_i a_i}{1-a_1}\|A_i x - A_i y\|^2 \\
&\geq \sum_{i=2}^{N} \frac{\lambda}{2}\left(\frac{a_i}{1-a_1}\right)\|A_i x - A_i y\|^2 \\
&= \frac{\lambda}{2}\sum_{i=2}^{N}\left(\frac{a_i}{1-a_1}\right)\|A_i x - A_i y\|^2 \\
&\geq \frac{\lambda}{2}\left\|\sum_{i=2}^{N}\left(\frac{a_i}{1-a_1}\right)(A_i x - A_i y)\right\|^2 \\
&= \frac{\lambda}{2}\left\|\frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i x - \frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i y\right\|^2.
\end{aligned}$$

Therefore, $\frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i$ is $\frac{\lambda}{2}$-inverse strongly monotone.
Using the same argument mentioned above, we also obtain $\frac{1}{1-\sum_{i=1}^{j-1} a_i}\sum_{i=j}^{N} a_i A_i$ is $\frac{\lambda}{2}$-inverse strongly monotone, for any $j = 2, 3, \ldots, N - 1$.

From Lemma 2.6, we get

$$
\begin{aligned}
&VI\left(C, \sum_{i=1}^{N} a_i A_i\right)\\
&= VI\left(C, a_1 A_1 + (1-a_1)\left(\frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i\right)\right)\\
&= VI(C, A_1) \cap VI\left(C, \frac{1}{1-a_1}\sum_{i=2}^{N} a_i A_i\right)\\
&= VI(C, A_1) \cap VI\left(C, \frac{a_2}{1-a_1}A_2 + \frac{1}{1-a_1}\sum_{i=3}^{N} a_i A_i\right)\\
&= VI(C, A_1) \cap VI\left(C, \frac{a_2}{1-a_1}A_2 + \left(\frac{1-\sum_{i=1}^{2} a_i}{1-a_1}\right)\sum_{i=3}^{N} \frac{a_i}{1-\sum_{i=1}^{2} a_i} A_i\right)\\
&= VI(C, A_1) \cap VI\left(C, \frac{a_2}{1-a_1}A_2 + \left(1-\frac{a_2}{1-a_1}\right)\sum_{i=3}^{N} \frac{a_i}{1-\sum_{i=1}^{2} a_i} A_i\right)\\
&= VI(C, A_1) \cap VI(C, A_2) \cap VI\left(C, \sum_{i=3}^{N} \frac{a_i}{1-\sum_{i=1}^{2} a_i} A_i\right)\\
&= \bigcap_{i=1}^{2} VI(C, A_i) \cap VI\left(C, \frac{a_3}{1-\sum_{i=1}^{2} a_i}A_3 + \sum_{i=4}^{N} \frac{a_i}{1-\sum_{i=1}^{2} a_i} A_i\right)\\
&= \bigcap_{i=1}^{2} VI(C, A_i) \cap VI\left(C, \frac{a_3}{1-\sum_{i=1}^{2} a_i}A_3 + \left(\frac{1-\sum_{i=1}^{3} a_i}{1-\sum_{i=1}^{2} a_i}\right)\sum_{i=4}^{N} \frac{a_i}{1-\sum_{i=1}^{3} a_i} A_i\right)\\
&= \bigcap_{i=1}^{2} VI(C, A_i) \cap VI\left(C, \frac{a_3}{1-\sum_{i=1}^{2} a_i}A_3 + \left(1-\frac{a_3}{1-\sum_{i=1}^{2} a_i}\right)\sum_{i=4}^{N} \frac{a_i}{1-\sum_{i=1}^{3} a_i} A_i\right)\\
&= \bigcap_{i=1}^{3} VI(C, A_i) \cap VI\left(C, \sum_{i=4}^{N} \frac{a_i}{1-\sum_{i=1}^{3} a_i} A_i\right)\\
&\vdots\\
&= \bigcap_{i=1}^{N-2} VI(C, A_i) \cap VI\left(C, \sum_{i=N-1}^{N} \frac{a_i}{1-\sum_{i=1}^{N-2} a_i} A_i\right)\\
&= \bigcap_{i=1}^{N-2} VI(C, A_i) \cap VI\left(C, \frac{a_{N-1}}{1-\sum_{i=1}^{N-2} a_i}A_{N-1} + \frac{a_N}{1-\sum_{i=1}^{N-2} a_i}A_N\right)\\
&= \bigcap_{i=1}^{N-2} VI(C, A_i) \cap VI\left(C, \frac{a_{N-1}}{1-\sum_{i=1}^{N-2} a_i}A_{N-1} + \left(\frac{1-\sum_{i=1}^{N-1} a_i}{1-\sum_{i=1}^{N-2} a_i}\right)\frac{a_N}{1-\sum_{i=1}^{N-1} a_i}A_N\right)\\
&= \bigcap_{i=1}^{N-2} VI(C, A_i) \cap VI\left(C, \frac{a_{N-1}}{1-\sum_{i=1}^{N-2} a_i}A_{N-1} + \left(1-\frac{a_{N-1}}{1-\sum_{i=1}^{N-2} a_i}\right)A_N\right)
\end{aligned}
$$

$$= \bigcap_{i=1}^{N-2} VI\left(C, A_i\right) \cap VI\left(C, A_{N-1}\right) \cap VI\left(C, A_N\right)$$
$$= \bigcap_{i=1}^{N} VI\left(C, A_i\right).$$

To prove *(ii)*, let $x, y \in C$. Then, we get

$$\begin{aligned}
&\left\| \left(I - \lambda \sum_{i=1}^{N} a_i A_i\right) x - \left(I - \lambda \sum_{i=1}^{N} a_i A_i\right) y \right\|^2 \\
&\quad = \left\| (x-y) - \lambda \left(\sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y \right) \right\|^2 \\
&\quad = \left\| (x-y) - \lambda \sum_{i=1}^{N} a_i \left(A_i x - A_i y\right) \right\|^2 \\
&\quad = \|x-y\|^2 - 2\lambda \left\langle x - y, \sum_{i=1}^{N} a_i \left(A_i x - A_i y\right) \right\rangle + \lambda^2 \left\| \sum_{i=1}^{N} a_i \left(A_i x - A_i y\right) \right\|^2 \\
&\quad \leq \|x-y\|^2 - 2\lambda \sum_{i=1}^{N} a_i \left\langle x-y, A_i x - A_i y \right\rangle + \lambda^2 \sum_{i=1}^{N} a_i \left\|A_i x - A_i y\right\|^2 \\
&\quad \leq \|x-y\|^2 - 2\lambda \sum_{i=1}^{N} a_i \alpha_i \left\|A_i x - A_i y\right\|^2 + \lambda^2 \sum_{i=1}^{N} a_i \left\|A_i x - A_i y\right\|^2 \\
&\quad = \|x-y\|^2 - \sum_{i=1}^{N} \lambda a_i \left(2\alpha_i - \lambda\right) \left\|A_i x - A_i y\right\|^2 \\
&\quad \leq \|x-y\|^2.
\end{aligned}$$

Hence, $I - \lambda \sum_{i=1}^{N} a_i A_i$ is a nonexpansive mapping. ■

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F and C satisfy the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$,

$$\lim_{t \to 0^+} F\left(tz + (1-t)x, y\right) \leq F(x, y);$$

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.8 ([3])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP\left(F_i\right) \neq \emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\left(F_i\right),$$

where $a_i \in (0,1)$ *for every* $i = 1, 2, \ldots, N$ *and* $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.9 ([1]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.10 ([2]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i). T_r is single-valued;

(ii). T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(iii). $Fix\,(T_r) = EP(F)$;

(iv). $EP(F)$ is closed and convex.

Remark 2.11 ([3]). Since $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4), by Lemma 2.8 and Lemma 2.10, we obtain

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\,(F_i)\,,$$

where $a_i \in (0,1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$.

3. Strong convergence theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For $j = 1, 2, \ldots, \bar{N}$, let $B_j : C \to H$ be δ_j-inverse strongly monotone. Let $f, g : C \to C$ be contractive mappings with coefficients ρ_1 and ρ_2, respectively, with $\rho = \max_{i \in \{1,2\}} \rho_i$. Suppose that A is a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Assume that $\Omega_1 := \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$ and $\Omega_2 := \bigcap_{j=1}^{\bar{N}} VI(C, B_j) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} a_i F_i\,(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ v_n = P_C\left(I - \lambda\left(\displaystyle\sum_{j=1}^{\bar{N}} b_n^j B_j\right)\right) y_n, \\ x_{n+1} = (1 - \beta_n)\,x_n + \beta_n P_C\,(\alpha_n \gamma f(y_n) + (I - \alpha_n A)u_n)\,, \\ y_{n+1} = (1 - \beta_n)\,y_n + \beta_n P_C\,(\alpha_n \gamma g(x_n) + (I - \alpha_n A)v_n)\,, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{r_n\}, \{b_n^j\} \subseteq (0,1), j = 1, 2, \ldots, \bar{N}$ and $0 \leq a_i \leq 1$, for every $i = 1, 2, \ldots, N$, satisfy the following conditions:

(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii). $0 < \tau \le \beta_n \le \upsilon < 1$, *for some* $\tau, \upsilon > 0$;
(iii). $0 < \epsilon \le r_n \le \eta < \infty$, *for some* $\epsilon, \eta > 0$;
(iv). $0 < \lambda < 2\delta_j$, *for all* $j = 1, 2, \dots, \bar{N}$;
(v). $\sum_{i=1}^{N} a_i = 1$ *and* $\sum_{j=1}^{\bar{N}} b_n^j = 1$;
(vi). $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,
$\sum_{n=1}^{\infty} \left| b_{n+1}^j - b_n^j \right| < \infty$, *for all* $j = 1, 2, \dots, \bar{N}$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly* $\tilde{x} = P_{\Omega_1}(\tilde{x} - A\tilde{y} + \gamma f(\tilde{y}))$ *and* $\tilde{y} = P_{\Omega_2}(\tilde{y} - A\tilde{x} + \gamma f(\tilde{x}))$, *respectively.*

Proof. Since $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4), by Lemma 2.10 and Remark 2.11, we have $u_n = T_{r_n} x_n$ and $Fix(T_{r_n}) = \bigcap_{i=1}^{N} EP(F_i)$.
Moreover, from Lemma 2.5 and Lemma 2.7, we also obtain

$$\bigcap_{j=1}^{\bar{N}} VI(C, B_j) = VI\left(C, \sum_{j=1}^{\bar{N}} b_n^j B_j\right) = Fix\left(P_C\left(I - \lambda\left(\sum_{j=1}^{\bar{N}} b_n^j B_j\right)\right)\right).$$

Step 1 We show that $\{x_n\}$ and $\{y_n\}$ are bounded.
Let $x^* \in \Omega_1$ and $y^* \in \Omega_2$. Then we derive

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\beta_n)\|x_n - x^*\| + \beta_n \|P_C(\alpha_n \gamma f(y_n) + (I - \alpha_n A)u_n) - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\| \\
&\quad + \beta_n \|\alpha_n(\gamma f(y_n) - Ax^*) + (I - \alpha_n A)(u_n - x^*)\| \\
&\le (1-\beta_n)\|x_n - x^*\| \\
&\quad + \beta_n \Big[\alpha_n \gamma \|f(y_n) - f(y^*)\| + \alpha_n \|f(y^*) - Ax^*\| \\
&\quad + (1 - \alpha_n \bar{\gamma})\|u_n - x^*\|\Big] \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n \Big[\alpha_n \gamma \rho \|y_n - y^*\| + \alpha_n \|f(y^*) - Ax^*\| \\
&\quad + (1 - \alpha_n \bar{\gamma})\|x_n - x^*\|\Big] \\
&= (1 - \alpha_n \beta_n \bar{\gamma})\|x_n - x^*\| + \beta_n \alpha_n \gamma \rho \|y_n - y^*\| + \beta_n \alpha_n \|f(y^*) - Ax^*\|.
\end{aligned} \tag{3.1}$$

Using the same argument as (3.1), we also obtain

$$\|y_{n+1} - y^*\| \le (1 - \alpha_n \beta_n \bar{\gamma})\|y_n - y^*\| + \beta_n \alpha_n \gamma \rho \|x_n - x^*\| + \beta_n \alpha_n \|g(x^*) - Ay^*\|. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\| \leq (1-\alpha_n\beta_n(\bar{\gamma}-\gamma\rho))\Big[\,\|x_n-x^*\|+\|y_n-y^*\|\,\Big] + \alpha_n\beta_n\Big[\,\|f(y^*)-Ax^*\|+\|g(x^*)-Ay^*\|\,\Big].$$

By induction, we get

$$\|x_n-x^*\|+\|y_n-y^*\| \leq \max\left\{\|x_1-x^*\|+\|y_1-y^*\|, \frac{\|\gamma f(y^*)-Ax^*\|+\|\gamma g(x^*)-Ay^*\|}{\bar{\gamma}-\gamma\rho}\right\}.$$

This implies that $\{x_n\}$ and $\{y_n\}$ are bounded. So are $\{u_n\}$ and $\{v_n\}$.
Step 2. Derive that $\|x_{n+1}-x_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$ as $n\to\infty$.
Using the same method as in [3], we get

$$\|u_n-u_{n-1}\| \leq \|x_n-x_{n-1}\|+\frac{1}{\epsilon}|r_n-r_{n-1}|\,\|u_n-x_n\|. \tag{3.3}$$

Take $p_n=\alpha_n\gamma f(y_n)+(I-\alpha_nA)u_n$ and $q_n=\alpha_n\gamma g(x_n)+(I-\alpha_nA)v_n$. Then, by (3.3), we obtain

$$\begin{aligned}
&\|p_n-p_{n-1}\| \\
&\quad\leq \alpha_n\gamma\|f(y_n)-f(y_{n-1})\|+\gamma|\alpha_n-\alpha_{n-1}|\,\|f(y_{n-1})\|+\|I-\alpha_nA\|\,\|u_n-u_{n-1}\| \\
&\qquad+\|(I-\alpha_nA)u_{n-1}-(I-\alpha_{n-1}A)u_{n-1}\| \\
&\quad\leq \alpha_n\gamma\rho\|y_n-y_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\,\|f(y_{n-1})\|+(1-\alpha_n\bar{\gamma})\|u_n-u_{n-1}\| \\
&\qquad+|\alpha_n-\alpha_{n-1}|\,\|Au_{n-1}\| \\
&\quad\leq \alpha_n\gamma\rho\|y_n-y_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\,\|f(y_{n-1})\|+(1-\alpha_n\bar{\gamma})\Big[\,\|x_n-x_{n-1}\| \\
&\qquad+\frac{1}{\epsilon}|r_n-r_{n-1}|\,\|u_n-x_n\|\Big]+|\alpha_n-\alpha_{n-1}|\,\|Au_{n-1}\| \\
&\quad\leq \alpha_n\gamma\rho\|y_n-y_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\,\|f(y_{n-1})\|+(1-\alpha_n\bar{\gamma})\|x_n-x_{n-1}\| \\
&\qquad+\frac{1}{\epsilon}|r_n-r_{n-1}|\,\|u_n-x_n\|+|\alpha_n-\alpha_{n-1}|\,\|Au_{n-1}\|.
\end{aligned} \tag{3.4}$$

From (3.4), we get

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad\leq (1-\beta_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+\beta_n\|p_n-p_{n-1}\| \\
&\qquad+|\beta_n-\beta_{n-1}|\,\|P_Cp_{n-1}\| \\
&\quad\leq \beta_n\alpha_n\gamma\rho\|y_n-y_{n-1}\|+(1-\beta_n\alpha_n\bar{\gamma})\|x_n-x_{n-1}\|+\frac{1}{\epsilon}|r_n-r_{n-1}|\,\|u_n-x_n\| \\
&\qquad+|\alpha_n-\alpha_{n-1}|(\gamma\|f(y_{n-1})\|+\|Au_{n-1}\|)+|\beta_n-\beta_{n-1}|(\|P_Cp_{n-1}\|+\|x_{n-1}\|).
\end{aligned} \tag{3.5}$$

Put $w_n = \sum_{j=1}^{\bar{N}} b_n^j B_j$. Next, we derive

$$\begin{aligned}
\|v_n - v_{n-1}\| &\leq \|P_C\left(I - \lambda w_n\right) y_n - P_C\left(I - \lambda w_n\right) y_{n-1}\| \\
&\quad + \|P_C\left(I - \lambda w_n\right) y_{n-1} - P_C\left(I - \lambda w_{n-1}\right) y_{n-1}\| \\
&\leq \|y_n - y_{n-1}\| + \|\left(I - \lambda w_n\right) y_{n-1} - \left(I - \lambda w_{n-1}\right) y_{n-1}\| \\
&= \|y_n - y_{n-1}\| + \lambda \|w_n y_{n-1} - w_{n-1} y_{n-1}\| \\
&= \|y_n - y_{n-1}\| + \lambda \left\|\sum_{j=1}^{\bar{N}} b_n^j B_j y_{n-1} - \sum_{j=1}^{\bar{N}} b_{n-1}^j B_j y_{n-1}\right\| \\
&= \|y_n - y_{n-1}\| + \lambda \left\|\sum_{j=1}^{\bar{N}} \left(b_n^j - b_{n-1}^j\right) B_j y_{n-1}\right\| \\
&\leq \|y_n - y_{n-1}\| + \lambda \sum_{j=1}^{\bar{N}} \left|b_n^j - b_{n-1}^j\right| \|B_j y_{n-1}\|.
\end{aligned} \tag{3.6}$$

Using the same method as (3.4), from (3.6), we have

$$\begin{aligned}
\|q_n - q_{n-1}\| \leq& \alpha_n \gamma \rho \|x_n - x_{n-1}\| + \gamma \left|\alpha_n - \alpha_{n-1}\right| \|g\left(x_{n-1}\right)\| \\
& + \left(1 - \alpha_n \bar{\gamma}\right) \|v_n - v_{n-1}\| + \left|\alpha_n - \alpha_{n-1}\right| \|A v_{n-1}\| \\
\leq& \alpha_n \gamma \rho \|x_n - x_{n-1}\| + \gamma \left|\alpha_n - \alpha_{n-1}\right| \|g\left(x_{n-1}\right)\| + \left(1 - \alpha_n \bar{\gamma}\right) \Big[\|y_n - y_{n-1}\| \\
& + \lambda \sum_{j=1}^{\bar{N}} \left|b_n^j - b_{n-1}^j\right| \|B_j y_{n-1}\| \Big] + \left|\alpha_n - \alpha_{n-1}\right| \|A v_{n-1}\| \\
\leq& \alpha_n \gamma \rho \|x_n - x_{n-1}\| + \gamma \left|\alpha_n - \alpha_{n-1}\right| \|g\left(x_{n-1}\right)\| + \left(1 - \alpha_n \bar{\gamma}\right) \|y_n - y_{n-1}\| \\
& + \lambda \sum_{j=1}^{\bar{N}} \left|b_n^j - b_{n-1}^j\right| \|B_j y_{n-1}\| + \left|\alpha_n - \alpha_{n-1}\right| \|A v_{n-1}\|.
\end{aligned} \tag{3.7}$$

From (3.7), it yields that

$$\begin{aligned}
\|y_{n+1} - y_n\| \leq& \left(1 - \beta_n\right) \|y_n - y_{n-1}\| + \left|\beta_n - \beta_{n-1}\right| \|y_{n-1}\| + \beta_n \|q_n - q_{n-1}\| \\
& + \left|\beta_n - \beta_{n-1}\right| \|P_C q_{n-1}\| \\
\leq& \beta_n \alpha_n \gamma \rho \|x_n - x_{n-1}\| + \left(1 - \beta_n \alpha_n \bar{\gamma}\right) \|y_n - y_{n-1}\| \\
& + \lambda \sum_{j=1}^{\bar{N}} \left|b_n^j - b_{n-1}^j\right| \|B_j y_{n-1}\| + \left|\alpha_n - \alpha_{n-1}\right| \left(\gamma \|g(x_{n-1})\| + \|A v_{n-1}\|\right) \\
& + \left|\beta_n - \beta_{n-1}\right| \left(\|y_{n-1}\| + \|P_C q_{n-1}\|\right).
\end{aligned} \tag{3.8}$$

By (3.5) and (3.8), we obtain

$$\begin{aligned}
&\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\| \\
&\quad\leq (1-\alpha_n\beta_n(\bar{\gamma}-\gamma\rho))\Big[\,\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|\,\Big]+\frac{1}{\epsilon}\,|r_n-r_{n-1}|\,\|u_n-x_n\| \\
&\qquad+\lambda\sum_{j=1}^{\bar{N}}\left|b_n^j-b_{n-1}^j\right|\|B_jy_{n-1}\|+|\alpha_n-\alpha_{n-1}|\Big[\gamma\,\|f(y_{n-1})\|+\gamma\,\|g(x_{n-1})\| \\
&\qquad+\|Au_{n-1}\|+\|Av_{n-1}\|\Big]+|\beta_n-\beta_{n-1}|\Big[\,\|x_{n-1}\|+\|y_{n-1}\|+\|P_Cp_{n-1}\| \\
&\qquad+\|P_Cq_{n-1}\|\Big].
\end{aligned}$$

From Lemma 2.4 and the condition ((i).), ((ii).) and ((vi).), we get

$$\|x_{n+1}-x_n\|\to 0 \text{ as } n\to\infty \tag{3.9}$$

and

$$\|y_{n+1}-y_n\|\to 0 \text{ as } n\to\infty. \tag{3.10}$$

Step 3. Prove that $\lim_{n\to\infty}\|u_n-x_n\|=0$ and $\lim_{n\to\infty}\|v_n-y_n\|=0$.
Since $u_n=T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive mapping, we obtain

$$\begin{aligned}
\|x^*-T_{r_n}x_n\|^2 &= \|T_{r_n}x^*-T_{r_n}x_n\|^2 \\
&\leq \langle T_{r_n}x^*-T_{r_n}x_n, x^*-x_n\rangle \\
&= \frac{1}{2}\left(\|T_{r_n}x_n-x^*\|^2+\|x_n-x^*\|^2-\|T_{r_n}x_n-x_n\|^2\right),
\end{aligned}$$

which follows that

$$\|u_n-x^*\|^2\leq\|x_n-x^*\|^2-\|u_n-x_n\|^2. \tag{3.11}$$

By (3.11), thus we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad\leq (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|\alpha_n(\gamma f(y_n)-Au_n)+(u_n-x^*)\|^2 \\
&\quad\leq (1-\beta_n)\|x_n-x^*\|^2+\beta_n\Big[\,\|u_n-x^*\|^2+2\alpha_n\langle\gamma f(y_n)-Au_n, p_n-x^*\rangle\Big] \\
&\quad\leq (1-\beta_n)\|x_n-x^*\|^2+\beta_n\Big[\left(\|x_n-x^*\|^2-\|u_n-x_n\|^2\right) \\
&\qquad+2\alpha_n\langle\gamma f(y_n)-Au_n, p_n-x^*\rangle\Big] \\
&\quad\leq \|x_n-x^*\|^2-\beta_n\|u_n-x_n\|^2+2\alpha_n\beta_n\|\gamma f(y_n)-Au_n\|\,\|p_n-x^*\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\beta_n\|u_n-x_n\|^2 \\
&\quad\leq (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\,\|x_{n+1}-x_n\|+2\alpha_n\beta_n\|\gamma f(y_n)-Au_n\|\,\|p_n-x^*\|.
\end{aligned}$$

From the condition ((i).), ((ii).) and (3.9), we have

$$\|u_n-x_n\|\to 0 \text{ as } n\to\infty. \tag{3.12}$$

From the definition of w_n, we have

$$\begin{aligned}\langle y_n - y^*, w_n y_n - w_n y^* \rangle &= \left\langle y_n - y^*, \sum_{j=1}^{\bar{N}} b_n^j B_j y_n - \sum_{j=1}^{\bar{N}} b_n^j B_j y^* \right\rangle \\ &= \left\langle y_n - y^*, \sum_{j=1}^{\bar{N}} b_n^j \left(B_j y_n - B_j y^*\right) \right\rangle \\ &= \sum_{j=1}^{\bar{N}} b_n^j \langle y_n - y^*, B_j y_n - B_j y^* \rangle \\ &\geq \sum_{j=1}^{\bar{N}} b_n^j \delta_j \left\| B_j y_n - B_j y^* \right\|^2 . \end{aligned} \tag{3.13}$$

Next, we derive

$$\begin{aligned}\left\| w_n y_n - w_n y^* \right\|^2 &= \left\| \sum_{j=1}^{\bar{N}} b_n^j B_j y_n - \sum_{j=1}^{\bar{N}} b_n^j B_j y^* \right\|^2 \\ &= \left\| \sum_{j=1}^{\bar{N}} b_n^j \left(B_j y_n - B_j y^*\right) \right\|^2 \\ &\leq \sum_{j=1}^{\bar{N}} b_n^j \left\| B_j y_n - B_j y^* \right\|^2 . \end{aligned} \tag{3.14}$$

Next, it implies by (3.13) and (3.14) that

$$\begin{aligned}\left\| v_n - y^* \right\|^2 &= \left\| P_C \left(I - \lambda w_n\right) y_n - P_C \left(I - \lambda w_n\right) y^* \right\|^2 \\ &\leq \left\| \left(I - \lambda w_n\right) y_n - \left(I - \lambda w_n\right) y^* \right\|^2 \\ &= \left\| y_n - y^* - \lambda \left(w_n y_n - w_n y^*\right) \right\|^2 \\ &= \left\| y_n - y^* \right\|^2 - 2\lambda \langle y_n - y^*, w_n y_n - w_n y^* \rangle + \lambda^2 \left\| w_n y_n - w_n y^* \right\|^2 \\ &\leq \left\| y_n - y^* \right\|^2 - 2\lambda \sum_{j=1}^{\bar{N}} b_n^j \delta_j \left\| B_j y_n - B_j y^* \right\|^2 + \lambda^2 \sum_{j=1}^{\bar{N}} b_n^j \left\| B_j y_n - B_j y^* \right\|^2 \\ &\leq \left\| y_n - y^* \right\|^2 - \sum_{j=1}^{\bar{N}} b_n^j \lambda \left(2\delta_j - \lambda\right) \left\| B_j y_n - B_j y^* \right\|^2 . \end{aligned} \tag{3.15}$$

From (3.15), we get

$$
\begin{aligned}
&\|y_{n+1} - y^*\|^2 \\
&\quad \le (1-\beta_n)\|y_n - y^*\|^2 + \beta_n \|q_n - y^*\|^2 \\
&\quad = (1-\beta_n)\|y_n - y^*\|^2 + \beta_n \|\alpha_n(\gamma g(x_n) - Av_n) + (v_n - y^*)\|^2 \\
&\quad \le (1-\beta_n)\|y_n - y^*\|^2 + \beta_n\Big[\|v_n - y^*\|^2 + 2\alpha_n\langle \gamma g(x_n) - Av_n, q_n - y^*\rangle\Big] \\
&\quad \le (1-\beta_n)\|y_n - y^*\|^2 + \beta_n\Big[\|y_n - y^*\|^2 - \sum_{j=1}^{\bar{N}} b_n^j \lambda(2\delta_j - \lambda)\|B_j y_n - B_j y^*\|^2 \\
&\qquad + 2\alpha_n\|\gamma g(x_n) - Av_n\|\|q_n - y^*\|\Big] \\
&\quad = \|y_n - y^*\|^2 - \beta_n\sum_{j=1}^{\bar{N}} b_n^j\lambda(2\delta_j - \lambda)\|B_j y_n - B_j y^*\|^2 \\
&\qquad + 2\alpha_n\beta_n\|\gamma g(x_n) - Av_n\|\|q_n - y^*\|\Big].
\end{aligned}
$$

This follows that

$$
\begin{aligned}
&\beta_n\sum_{j=1}^{\bar{N}} b_n^j\lambda(2\delta_j - \lambda)\|B_j y_n - B_j y^*\|^2 \\
&\quad \le (\|y_n - y^*\| + \|y_{n+1} - y^*\|)\|y_{n+1} - y_n\| + 2\alpha_n\beta_n\|\gamma g(x_n) - Av_n\|\|q_n - y^*\|.
\end{aligned}
$$

From (3.10) and the condition ((i).),((ii).) and ((iv).), we obtain

$$
\|B_j y_n - B_j y^*\| \to 0 \text{ as } n \to \infty, \text{ for any } j = 1, 2, \ldots, \bar{N}. \tag{3.16}
$$

By (3.14) and (3.16), it yields that

$$
\|w_n y_n - w_n y^*\| \to 0 \text{ as } n \to \infty. \tag{3.17}
$$

By the definition of v_n, hence we have

$$
\begin{aligned}
\|v_n - y^*\|^2 &= \|P_C(I - \lambda w_n)y_n - P_C(I - \lambda w_n)y^*\|^2 \\
&\le \langle (I - \lambda w_n)y_n - (I - \lambda w_n)y^*, P_C(I - \lambda w_n)y_n - y^*\rangle \\
&= \frac{1}{2}\Big[\|(I - \lambda w_n)y_n - (I - \lambda w_n)y^*\|^2 + \|P_C(I - \lambda w_n)y_n - y^*\|^2 \\
&\quad - \|(I - \lambda w_n)y_n - (I - \lambda w_n)y^* - (P_C(I - \lambda w_n)y_n - y^*)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|y_n - y^*\|^2 + \|P_C(I - \lambda w_n)y_n - y^*\|^2 \\
&\quad - \|y_n - P_C(I - \lambda w_n)y_n - \lambda(w_n y_n - w_n y^*)\|^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\Big[\|y_n-y^*\|^2+\|P_C(I-\lambda w_n)y_n-y^*\|^2\\
&-\|y_n-P_C(I-\lambda w_n)y_n\|^2-\lambda^2\|w_ny_n-w_ny^*\|^2\\
&+2\lambda\langle y_n-P_C(I-\lambda w_n)y_n,w_ny_n-w_ny^*\rangle\Big]\\
\le&\frac{1}{2}\Big[\|y_n-y^*\|^2+\|P_C(I-\lambda w_n)y_n-y^*\|^2\\
&-\|y_n-P_C(I-\lambda w_n)y_n\|^2-\lambda^2\|w_ny_n-w_ny^*\|^2\\
&+2\lambda\|y_n-P_C(I-\lambda w_n)y_n\|\|w_ny_n-w_ny^*\|\Big].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|v_n-y^*\|^2\le&\|y_n-y^*\|^2-\|y_n-P_C(I-\lambda w_n)y_n\|^2-\lambda^2\|w_ny_n-w_ny^*\|^2\\
&+2\lambda\|y_n-P_C(I-\lambda w_n)y_n\|\|w_ny_n-w_ny^*\|\\
\le&\|y_n-y^*\|^2-\|y_n-P_C(I-\lambda w_n)y_n\|^2\\
&+2\lambda\|y_n-P_C(I-\lambda w_n)y_n\|\|w_ny_n-w_ny^*\|. \qquad (3.18)
\end{aligned}
$$

It follows by (3.18) that

$$
\begin{aligned}
&\|y_{n+1}-y^*\|^2\\
&\le(1-\beta_n)\|y_n-y^*\|^2+\beta_n\|q_n-y^*\|^2\\
&=(1-\beta_n)\|y_n-y^*\|^2+\beta_n\|\alpha_n(\gamma g(x_n)-Av_n)+(v_n-y^*)\|^2\\
&\le(1-\beta_n)\|y_n-y^*\|^2+\beta_n\Big[\|v_n-y^*\|^2+2\alpha_n\langle\gamma g(x_n)-Av_n,q_n-y^*\rangle\Big]\\
&\le(1-\beta_n)\|y_n-y^*\|^2+\beta_n\Big[\|y_n-y^*\|^2-\|y_n-P_C(I-\lambda w_n)y_n\|^2\\
&\quad+2\lambda\|y_n-P_C(I-\lambda w_n)y_n\|\|w_ny_n-w_ny^*\|\\
&\quad+2\alpha_n\|\gamma g(x_n)-Av_n\|\|q_n-y^*\|\Big]\\
&=\|y_n-y^*\|^2-\beta_n\|y_n-P_C(I-\lambda w_n)y_n\|^2\\
&\quad+2\lambda\beta_n\|y_n-P_C(I-\lambda w_n)y_n\|\|w_ny_n-w_ny^*\|\\
&\quad+2\alpha_n\beta_n\|\gamma g(x_n)-Av_n\|\|q_n-y^*\|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\beta_n\|y_n-P_C(I-\lambda w_n)y_n\|^2\le&(\|y_n-y^*\|+\|y_{n+1}-y^*\|)\|y_{n+1}-y_n\|\\
&+2\lambda\beta_n\|y_n-P_C(I-\lambda w_n)y_n\|\|w_ny_n-w_ny^*\|\\
&+2\alpha_n\beta_n\|\gamma g(x_n)-Av_n\|\|q_n-y^*\|.
\end{aligned}
$$

Hence, by (3.10) and the conditions ((i).) and ((ii).), we get

$$
\|y_n-P_C(I-\lambda w_n)y_n\|\to 0 \text{ as } n\to\infty \text{ or } \|y_n-v_n\|\to 0 \text{ as } n\to\infty. \qquad (3.19)
$$

Step 4 Claim that $\limsup_{n\to\infty}\langle\gamma f(\tilde{y})-A\tilde{y},x_n-\tilde{x}\rangle\le 0$, where $\tilde{x}=P_{\Omega_1}(\tilde{x}-A\tilde{y}+\gamma f(\tilde{y}))$ and $\limsup_{n\to\infty}\langle\gamma g(\tilde{x})-\tilde{x},y_n-\tilde{y}\rangle\le 0$, where $\tilde{y}=P_{\Omega_2}(\tilde{y}-A\tilde{x}+\gamma g(\tilde{x}))$.

Without of generality, we can assume that $x_{n_k} \rightharpoonup \omega_1$ as $k \to \infty$. From (3.12), it follows that $u_{n_k} \rightharpoonup \omega_1$ as $k \to \infty$. Continuing the same method as in Step 4 of [3], we get

$$\omega_1 \in \Omega_1. \tag{3.20}$$

By (3.20) and $x_{n_k} \rightharpoonup \omega_1$ as $k \to \infty$, we derive that

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(\tilde{y}) - A\tilde{y}, x_n - \tilde{x}\rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{y}) - A\tilde{y}, x_{n_k} - \tilde{x}\rangle \\ &= \langle \gamma f(\tilde{y}) - A\tilde{y}, \omega_1 - \tilde{x}\rangle \\ &= \langle (\gamma f(\tilde{y}) - A\tilde{y} + \tilde{x}) - \tilde{x}, \omega_1 - \tilde{x}\rangle \\ &\leq 0. \end{aligned} \tag{3.21}$$

Similarly, we can assume that $y_{n_k} \rightharpoonup \omega_2$ as $k \to \infty$ and, from (3.19), we have that $v_{n_k} \rightharpoonup \omega_2$ as $k \to \infty$. From Lemma 2.5 and Lemma 2.7, we also obtain

$$\begin{aligned}\bigcap_{j=1}^{\bar{N}} VI(C, B_j) &= VI\left(C, \sum_{j=1}^{\bar{N}} b_n^j B_j\right) \\ &= Fix\left(P_C\left(I - \lambda\left(\sum_{j=1}^{\bar{N}} b_n^j B_j\right)\right)\right) \\ &= Fix\left(P_C\left(I - \lambda w_n\right)\right).\end{aligned}$$

Assume that $\omega_2 \neq P_C(I - \lambda w_{n_k})\omega_2$. From Opial's condition, we get

$$\begin{aligned}\liminf_{k\to\infty} \|y_{n_k} - \omega_2\| &< \liminf_{k\to\infty} \|y_{n_k} - P_C(I - \lambda w_{n_k})\omega_2\| \\ &\leq \liminf_{k\to\infty} \|y_{n_k} - P_C(I - \lambda w_{n_k}) y_{n_k}\| \\ &\quad + \liminf_{k\to\infty} \|P_C(I - \lambda w_{n_k}) y_{n_k} - P_C(I - \lambda w_{n_k})\omega_2\| \\ &\leq \liminf_{k\to\infty} \|y_{n_k} - \omega_2\|.\end{aligned}$$

This is a contradiction. Then, we have

$$\omega_2 \in \bigcap_{j=1}^{\bar{N}} VI(C, B_j).$$

That is, $\omega_2 \in \Omega_2$. This follows that

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma g(\tilde{x}) - A\tilde{x}, y_n - \tilde{y}\rangle &= \lim_{k\to\infty} \langle \gamma g(\tilde{x}) - A\tilde{x}, y_{n_k} - \tilde{y}\rangle \\ &= \langle \gamma g(\tilde{x}) - A\tilde{x}, \omega_2 - \tilde{y}\rangle \\ &= \langle (\gamma g(\tilde{x}) - A\tilde{x} + \tilde{y}) - \tilde{y}, \omega_2 - \tilde{y}\rangle \\ &\leq 0. \end{aligned} \tag{3.22}$$

Step 5 Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1}(\tilde{x} - A\tilde{y} + \gamma f(\tilde{y}))$ and $\tilde{y} = P_{\Omega_2}(\tilde{y} - A\tilde{x} + \gamma f(\tilde{x}))$, respectively.

Hence, we obtain

$$
\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2\\
&=\Big\|(1-\beta_n)(x_n-\tilde{x})+\beta_n\Big[\alpha_n(\gamma f(y_n)-A\tilde{x})+(I-\alpha_nA)(u_n-\tilde{x})\Big]\Big\|^2\\
&=\|(1-\beta_n)(x_n-\tilde{x})+\alpha_n\beta_n(\gamma f(y_n)-A\tilde{x})+\beta_n(I-\alpha_nA)(u_n-\tilde{x})\|^2\\
&\le\|(1-\beta_n)(x_n-\tilde{x})+\beta_n(I-\alpha_nA)(u_n-\tilde{x})\|^2\\
&\quad+2\alpha_n\beta_n\langle\gamma f(y_n)-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
&\le\Big[(1-\beta_n)\|x_n-\tilde{x}\|+\beta_n(1-\alpha_n\bar{\gamma})\|x_n-\tilde{x}\|\Big]^2\\
&\quad+2\alpha_n\beta_n\gamma\|f(y_n)-f(\tilde{y})\|\|x_{n+1}-\tilde{x}\|+2\alpha_n\beta_n\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
&\le\Big[(1-\alpha_n\beta_n\bar{\gamma})\|x_n-\tilde{x}\|\Big]^2+2\alpha_n\beta_n\gamma\rho\|y_n-\tilde{y}\|\|x_{n+1}-\tilde{x}\|\\
&\quad+2\alpha_n\beta_n\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle\\
&\le(1-\alpha_n\beta_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2+\alpha_n\beta_n\gamma\rho\Big[\|y_n-\tilde{y}\|^2+\|x_{n+1}-\tilde{x}\|^2\Big]\\
&\quad+2\alpha_n\beta_n\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2\le&\frac{(1-\alpha_n\beta_n\bar{\gamma})^2}{1-\alpha_n\beta_n\gamma\rho}\|x_n-\tilde{x}\|^2+\frac{\alpha_n\beta_n\gamma\rho}{1-\alpha_n\beta_n\gamma\rho}\|y_n-\tilde{y}\|^2\\
&+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\gamma\rho}\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle.
\end{aligned}\tag{3.23}
$$

Similarly, we get

$$
\begin{aligned}
\|y_{n+1}-\tilde{y}\|^2\le&\frac{(1-\alpha_n\beta_n\bar{\gamma})^2}{1-\alpha_n\beta_n\gamma\rho}\|y_n-\tilde{y}\|^2+\frac{\alpha_n\beta_n\gamma\rho}{1-\alpha_n\beta_n\gamma\rho}\|x_n-\tilde{x}\|^2\\
&+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\gamma\rho}\langle\gamma g(\tilde{y})-A\tilde{y},y_{n+1}-\tilde{y}\rangle.
\end{aligned}\tag{3.24}
$$

Combining (3.23) and (3.24), we obtain

$$
\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2+\|y_{n+1}-\tilde{y}\|^2\\
&\le\frac{(1-\alpha_n\beta_n\bar{\gamma})^2+\alpha_n\beta_n\gamma\rho}{1-\alpha_n\beta_n\gamma\rho}\Big[\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\Big]\\
&\quad+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\gamma\rho}\Big[\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle+\langle\gamma g(\tilde{y})-A\tilde{y},y_{n+1}-\tilde{y}\rangle\Big]\\
&=\left(1-\frac{2\alpha_n\beta_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\beta_n\gamma\rho}\right)\Big[\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\Big]\\
&\quad+\left(\frac{2\alpha_n\beta_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\beta_n\gamma\rho}\right)\Bigg[\frac{\alpha_n\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\alpha)}\left(\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\right)\\
&\quad+\frac{1}{\bar{\gamma}-\gamma\rho}\left(\langle\gamma f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle+\langle\gamma g(\tilde{y})-A\tilde{y},y_{n+1}-\tilde{y}\rangle\right)\Bigg].
\end{aligned}
$$

By Lemma 2.4 and the conditions ((i).),((ii).), we can conclude that $\{x_n\}$ and $\{y_n\}$ converge strongly $\tilde{x} = P_{\Omega_1}(\tilde{x} - A\tilde{y} + \gamma f(\tilde{y}))$ and $\tilde{y} = P_{\Omega_2}(\tilde{y} - A\tilde{x} + \gamma f(\tilde{x}))$, respectively. Moreover, from (3.12) and (3.19), thus we obtain $\{u_n\}$ and $\{v_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1}(\tilde{x} - A\tilde{y} + \gamma f(\tilde{y}))$ and $\tilde{y} = P_{\Omega_2}(\tilde{y} - A\tilde{x} + \gamma f(\tilde{x}))$, respectively. This completes the proof. ∎

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $B : C \to H$ be δ-inverse strongly monotone. Let $f, g : C \to C$ be contractive mappings with coefficients ρ_1 and ρ_2, respectively, with $\rho = \max_{i\in\{1,2\}} \rho_i$. Suppose that A is a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Assume that $\Omega_1 := \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$ and $\Omega_2 := \bigcap_{j=1}^{\bar{N}} VI(C, B_j) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = (1-\beta_n) x_n + \beta_n P_C (\alpha_n \gamma f(y_n) + (I - \alpha_n A) u_n), \\ y_{n+1} = (1-\beta_n) y_n + \beta_n P_C (\alpha_n \gamma g(x_n) + (I - \alpha_n A) P_C (I - \lambda B) y_n), \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{r_n\} \subseteq (0,1)$ satisfying the following conditions:

(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii). $0 < \tau \leq \beta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;
(iii). $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;
(iv). $0 < \lambda < 2\delta$;
(v). $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1}(\tilde{x} - A\tilde{y} + \gamma f(\tilde{y}))$ and $\tilde{y} = P_{\Omega_2}(\tilde{y} - A\tilde{x} + \gamma f(\tilde{x}))$, respectively.

Proof. Take $F \equiv F_i$ and $G \equiv G_j$, for all $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, \bar{N}$. Then, by Theorem 3.1, we can obtain the desired result. ∎

4. A Numerical Example

In this section, we give numerical examples to support our main theorem.

Example 4.1. Let $C = [-1, 1]$, and let $\mathbb{R}$ be the set of real numbers. For every $i = 1, 2, \ldots, N$, let $F_i : [-1,1] \times [-1,1] \to [-1,1]$ be defined by

$$F_i(x, y) = i(y - x)(y + 7x + 8), \text{ for all } x, y \in [-1, 1].$$

For all $j = 1, 2, \ldots, \bar{N}$, let $B_j : [-1, 1] \to \mathbb{R}$ be defined by

$$B_j(x) = \frac{x-1}{2j}, \text{ for all } x \in [-1, 1].$$

Suppose that $\lambda = \frac{1}{100}$ and $\gamma = \frac{1}{2}$. Let $A : \mathbb{R} \to \mathbb{R}$ be defined by

$$Ax = \frac{x}{2}, \text{ for all } x \in \mathbb{R}.$$

Moreover, let $f, g : [-1, 1] \to [-1, 1]$ be defined by

$$fx = \frac{x}{2}$$
$$gx = \frac{x}{6}, \text{ for all } x \in [-1, 1].$$

Put $a_i = \frac{7}{8^i} + \frac{1}{N8^N}$, for every $i = 1, 2, \ldots, N$. Let $\alpha_n = \frac{1}{100n}$, $\beta_n = \frac{3n}{5n+3}$, $r_n = \frac{3n+7}{4n+9}$ and $b_n^1 = \frac{2n+1}{9n+11}, b_n^2 = \frac{n+3}{9n+11}, b_n^3 = \frac{3n+2}{9n+11}, b_n^4 = \frac{2n+3}{9n+11}, b_n^5 = \frac{n+2}{9n+11}$, for every $n \in \mathbb{N}$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to -1 and 1, respectively.

Solution. Since $a_i = \frac{7}{8^i} + \frac{1}{N8^N}$, we obtain

$$\begin{aligned}\sum_{i=1}^{N} a_i F_i(x, y) &= \sum_{i=1}^{N} \left(\frac{7}{8^i} + \frac{1}{N8^N}\right) i(y - x)(y + 7x + 8)\\ &= \xi(y - x)(y + 7x + 8), \end{aligned} \tag{4.1}$$

where $\xi = \sum_{i=1}^{N} \left(\frac{7}{8^i} + \frac{1}{N8^N}\right) i$. It is clear to check that $\sum_{i=1}^{N} a_i F_i$ satisfies all conditions (A1)-(A4) and $-1 \in EP(\sum_{i=1}^{N} a_i F_i) = \bigcap_{i=1}^{N} EP(F_i)$.

It is easy to see that $\sum_{i=1}^{N} a_i F_i$ satisfies all conditions in Theorem 3.1 and $-1 \in EP(\sum_{i=1}^{N} a_i F_i) = \bigcap_{i=1}^{N} EP(F_i)$. By the definition of F, we have

$$\begin{aligned} 0 &\le \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\ &= \xi(y - x)(y + 7x + 8) + \frac{1}{r_n}(y - u_n)(u_n - x_n) \\ &\Leftrightarrow \\ 0 &\le \sigma r_n \xi(y - x)(y + 7x + 8) + (y - u_n)(u_n - x_n) \\ &= \xi r_n y^2 + (u_n + 8\xi r_n + 6\xi r_n u_n - x_n) y + u_n x_n - u_n^2 - 8\xi r_n u_n - 7\xi r_n u_n^2. \end{aligned}$$

Let $G(y) = \xi r_n y^2 + (u_n + 8\xi r_n + 6\xi r_n u_n - x_n) y + u_n x_n - u_n^2 - 8\xi r_n u_n - 7\xi r_n u_n^2$. Then $G(y)$ is a quadratic function of y with coefficient $a = \xi r_n$, $b = u_n + 8\xi r_n + 6\xi r_n u_n - x_n$, and $c = u_n x_n - u_n^2 - 8\xi r_n u_n - 7\xi r_n u_n^2$. Determine the discriminant Δ of G as follows

$$\begin{aligned} \Delta =& b^2 - 4ac \\ =& (u_n + 8\xi r_n + 6\xi r_n u_n - x_n)^2 - 4(\xi r_n)\left(u_n x_n - u_n^2 - 8\xi r_n u_n - 7\xi r_n u_n^2\right) \\ =& u_n^2 + 16\xi r_n u_n + 16\xi r_n u_n^2 + 64\xi^2 r_n^2 + 128\xi^2 r_n^2 u_n + 64\xi^2 r_n^2 u_n^2 - 2u_n x_n \\ & - 16\xi r_n x_n - 16\xi r_n u_n x_n + x_n^2 \\ =& (u_n + 8\xi r_n + 8\xi r_n u_n - x_n)^2. \end{aligned}$$

We know that $G(y) \ge 0, \forall y \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta \le 0$, so we obtain

$$u_n = \frac{x_n - 8\xi r_n}{1 + 8\xi r_n}, \text{where } \xi = \sum_{i=1}^{N} \left(\frac{7}{8^i} + \frac{1}{N8^N}\right) i. \tag{4.2}$$

Observe that $\{1\} = \bigcap_{j=1}^{\bar{N}} VI(C, B_j) = VI\left(C, \sum_{j=1}^{\bar{N}} b_n^j B_j\right)$. Clearly, all sequences and parameters are satisfied all conditions of Theorem 3.1. Hence, by Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to -1 and 1 respectively.

Table 1 and Figure 1 show the numerical results of sequences $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ with $x_1 = 1$, $y_1 = -1$, $N = 20$, $\bar{N} = 5$ and $n = 500$.

Remark 4.2. From the previous example, we can conclude that

(i). Table 1 and Figure 1 show that the sequences $\{u_n\}$, $\{x_n\}$ converge to $-1 \in \Omega_1$ and $\{v_n\}$,$\{y_n\}$ converge to $1 \in \Omega_2$, independently.

(ii). The convergence of $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ can be guaranteed by Theorem 3.1.

n	u_n	x_n	v_n	y_n
1	-0.751026	1.000000	-0.929028	-1.000000
2	-0.831844	0.343836	-0.899294	-0.971331
3	-0.899370	-0.198386	-0.865122	-0.936980
4	-0.943200	-0.548519	-0.829206	-0.900357
5	-0.969017	-0.754118	-0.792577	-0.862754
⋮	⋮	⋮	⋮	⋮
250	-0.999996	-0.999965	0.993388	0.993121
⋮	⋮	⋮	⋮	⋮
496	-0.999998	-0.999983	0.999660	0.999646
497	-0.999998	-0.999983	0.999661	0.999648
498	-0.999998	-0.999983	0.999662	0.999649
499	-0.999998	-0.999983	0.999664	0.999650
500	-0.999998	-0.999983	0.999665	0.999651

TABLE 1. The values of $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ with $x_1 = 1$, $y_1 = -1$, $N = 20$, $\bar{N} = 5$ and $n = 500$.

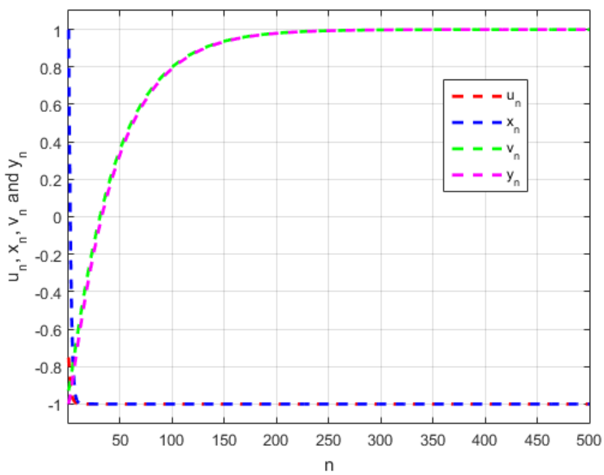

FIGURE 1. An independent convergence of $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ with $x_1 = 1$, $y_1 = -1$, $N = 20$, $\bar{N} = 5$ and $n = 500$.

5. Conclusion

The general intermixed iteration for two nonlinear mappings for solving the fixed point problem of two nonlinear mappings is introduced. This iterative method can be considered as an extension and modification of work by Yao [19] and Suwannaut [3]. Strong convergence theorem of the proposed algorithm is obtained under some control conditions.

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