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Pairwise T_S - Spaces

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Abstract : The aim of this paper is to introduce the concept of pairwise T_S - space and study its basic properties in bitopological spaces.

Keywords : pairwise T_s - space; pairwise T_b - space; pairwise ${}_{\alpha}T_b$ - space; pairwise ${}^*T_p$ - space; pairwise T_p^* - space.
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1 Introduction

Dunham, P. Bhattacharya and B.K. Lahiri, J. Duntchev, Y. Gnanambal introduced $T_{1/2}$, semi - $T_{1/2}$,semi pre - $T_{1/2}$ and pre regular $T_{1/2}$ spaces respectively. R. Devi et.al introduced T_b , T_d and ${}_{\alpha}T_b$, ${}_{\alpha}T_d$ spaces respectively. M.K.R.S. Veera Kumar introduced $T^*_{1/2}$, ${}^*T_{1/2}$, T_p^* , ${}^*T_p$ spaces. K.Chandrasekhara Rao and K. Joseph introduced the concept of s^*g - closed sets in topological spaces. Using this,K. Chandrasekhara Rao and D. Narasimhan introduced the concept of T_S spaces in topological spaces.
K. Chandrasekhara Rao and K. Kannan introduced the concept of $\tau_1\tau_2-s^*g$ closed sets in bitopological spaces. In this paper we introduce the concept of pairwise T_S space and study its basic properties in bitopological spaces.

2 Preliminaries

Let (X, τ_1, τ_2) or simply X denote a bitopological space. For any subset $A \subseteq X$, the closure [resp. semi closure, pre closure, $\alpha-$ closure] of a subset A of a space (X, τ_1, τ_2) is the intersection of all closed [resp. semi closed, pre closed, $\alpha-$ closed] sets that contain A and is denoted by cl(A) [resp. scl (A), pcl (A), αcl (A)]. Similarly, for any subset $A \subseteq X$, the interior [resp. semi interior] of a subset A of a space (X, τ_1, τ_2) is the intersection of all open [resp. semi open] sets that contained in A and is denoted by int (A) [resp. sint (A)]. A^C denotes the complement of the set A in X unless explicitly stated. We shall require the following known definitions.
Definitions 2.1
A set A of a bitopological space (X, τ_1, τ_2) is called

(a) $\tau_1\tau_2-$semi open if there exists an τ_1-open set U such that $U \subseteq A \subseteq \tau_2-cl(U)$,
(b) $\tau_1\tau_2-$semi closed if $X - A$ is $\tau_1\tau_2-$semi open,
Equivalently, a set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2-$semi closed if there exists a τ_1- closed set F such that τ_2-int $(F) \subseteq A \subseteq F$,
(c) $\tau_1\tau_2-$generalized closed ($\tau_1\tau_2-$ g closed) if τ_2-cl (A) $\subseteq$ U whenever A $\subseteq$ U and U is τ_1- open in X,
(d) $\tau_1\tau_2$- generalized open ($\tau_1\tau_2 - g$ open) if $X - A$ is $\tau_1\tau_2 - g$ closed,
(e) $\tau_1\tau_2-$semi generalized closed ($\tau_1\tau_2-$ sg closed) if τ_2-scl (A) $\subseteq$ U whenever A $\subseteq$ U and U is τ_1- semi open in X,
(f) $\tau_1\tau_2$- semi generalized open ($\tau_1\tau_2 - sg$ open) if $X - A$ is $\tau_1\tau_2 - sg$ closed,
(g) $\tau_1\tau_2$- generalized semi open ($\tau_1\tau_2 - gs$ open) if $F \subseteq \tau_2 - sint(A)$ whenever $F \subseteq A$ and F is τ_1- closed in X,
(h) $\tau_1\tau_2$- generalized semi closed ($\tau_1\tau_2 - gs$ closed) if $X - A$ is $\tau_1\tau_2 - gs$ open,
(i) $\tau_1\tau_2$- semi star generalized closed ($\tau_1\tau_2 - s^*g$ closed) if $\tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1- semi open in X,
(j) $\tau_1\tau_2$- semi star generalized open ($\tau_1\tau_2 - s^*g$ open) if $X - A$ is $\tau_1\tau_2 - s^*g$ closed in X,
(k) $\tau_1\tau_2 - \alpha$ open if $A \subseteq \tau_2 - int\{\tau_1 - cl[\tau_2 - int(A)\ ]\ \}$,
(l) $\tau_1\tau_2 - \alpha$ closed if $\tau_2 - cl\{\tau_1 - int[\tau_2 - cl(A)]\} \subseteq A$,
(m) $\tau_1\tau_2 - \alpha g$ closed if $\tau_2 - \alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1- open,
(n) $\tau_1\tau_2 - g^*$ closed if $\tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1 - g$ open,
(o) $\tau_1\tau_2 - g^*p$ closed if $\tau_2 - pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1 - g$ open,
(p) $\tau_1\tau_2 - gp$ closed if $\tau_2 - pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1- open.

Definition 2.2

A bitopological space (X, τ_1, τ_2) is called a
(a) pairwise $T_{1/2}$- space if every $\tau_1 - g$closed set is τ_2- closed and every $\tau_2 - g$ closed set is τ_1- closed,
(b) pairwise $T^*_{1/2}$- space if every $\tau_1\tau_2 - g^*$closed set is τ_2- closed and every $\tau_2\tau_1 - g^*$ closed set is τ_1- closed,
(c) pairwise T_b- space if every $\tau_1\tau_2 - gs$ closed set is τ_2- closed and every $\tau_2\tau_1 - gs$ closed set is τ_1- closed,
(d) pairwise ${}_\alpha T_b$- space if every $\tau_1\tau_2 - \alpha g$closed set is τ_2- closed and every $\tau_2\tau_1 - \alpha g$ closed set is τ_1- closed,
(e) pairwise ${}^*T_p$- space if $\tau_1\tau_2 - gp$ closed set is $\tau_1\tau_2 - g^*p$ closed,
(f) pairwise T^*_p- space if every $\tau_1\tau_2 - g^*p$- closed set is τ_2- closed,
(g) pairwise complemented space if every τ_2- open set is τ_1 -closed and τ_1- open set is τ_2 -closed
(h) pairwise door space if every sub set of X is either τ_1- open or τ_2- closed and τ_2- open or τ_1- closed

Remark 2.1

In X, every τ_2-closed set is $\tau_1\tau_2 - s^*g$ closed

Proof

Suppose that A is τ_2-closed. Then τ_2- cl $(A) = A$. Let U be τ_1-semi open and $A \subseteq U$. But then τ_2- cl $(A) = A \subseteq U$. Hence the remark is true.

3 Pairwise T_S- Spaces

Definition 3.1
A bitopological space (X, τ_1, τ_2) is called a pairwise T_S- space if every $\tau_1\tau_2 - s^*g$ closed set is τ_2- closed in X and every $\tau_2\tau_1 - s^*g$ closed set is τ_1- closed in X.
Example 3.1
Let $X = \{a, b, c\}$, τ_1 =$\{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{a, c\}\}$. Then $\{X, \tau_1, \tau_2\}$ is a pairwise T_S- space.
Proposition 3.1
Let (X, τ_1, τ_2) be a $\tau_1\tau_2 - T_S$ space.
(a) If Y is a τ_2- closed subspace of X, then $(Y, \tau_{1/Y}, \tau_{2/Y})$ is a $\tau_1\tau_2 - T_S$ space and
(b)If Y is a τ_1- closed subspace of X, then $(Y, \tau_{1/Y}, \tau_{2/Y})$ is a $\tau_2\tau_1 - T_S$ space
Proof
Let X be a pairwise T_S- space and Y be a τ_2- closed subspace of X. Let A be $\tau_1\tau_2 - s^*g$ closed in Y. Let $A \subseteq U$ and U is τ_1- semi open in Y
Then, $\tau_2 - cl_Y(A) \subseteq U$. Since U is τ_1- semi open in Y, we have $U = G \cap Y$ where G is τ_1- semi open in X. Therefore $A \subseteq G$ and G is τ_1- semi open in X. Since A is $\tau_1\tau_2 - s^*g$ closed in Y, we have $A = H \cap Y$ where H is $\tau_1\tau_2 - s^*g$ closed in X. But X is a pairwise T_S- space.
$\Rightarrow H$ is τ_2- closed in X.
$\Rightarrow H \cap Y$ is τ_2- closed in X.
$\Rightarrow A$ is τ_2- closed in X.
$\Rightarrow A \cap Y$ is τ_2- closed in Y.
$\Rightarrow A$ is τ_2- closed in Y.
(b) As we proved in (a)
Theorem 3.1
Let I be a index set. Let $\{X_i, i \in I\}$ be pairwise T_S- spaces. Then their product $X = \prod X_i$ is a pairwise T_S - space.
Proof
Let $A = p_j(A) \times \prod X_i$, $i \neq j$ be $\tau_1\tau_2 - s^*g$ closed in $X = \prod X_i$ where $p_j : \prod X_i \longrightarrow X_j$ be the j^{th} projection map which is a surjection. Then $\tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1- semi open in X. Since U is τ_1- semi open in $X = \prod X_i$, $U = \prod X_i \times U_j$, $j \neq i$, where U_j is τ_1- semi open in X_j. Since $p_j : \prod X_i \longrightarrow X_j$, $i \neq j$, be the j^{th} projection map, we have $p_j(U) = U_j$. Also $A \subseteq U$. Hence $p_j(A) \subseteq p_j(U) = U_j$. Since A is $\tau_1\tau_2 - s^*g$ closed in X , $p_j(A)$ is $\tau_1\tau_2 - s^*g$ closed in X_j. Since X_j is a pairwise T_S- space, we have $A_j = p_j(A)$ is τ_2- closed in X_j . Hence $A_j = \tau_2 - cl_{Xj}(Aj)$.Therefore $A_j \times \prod X_i = \tau_2$- $\mathrm{cl}_{Xj}(Aj) \times \prod X_i = \tau_2$- $\mathrm{cl}(Aj) \times \prod X_i$= τ_2- $\mathrm{cl}[(Aj) \times \prod X_i]$. Hence A is τ_2- closed in X.
Therefore every $\tau_1\tau_2 - s^*g$ closed set is τ_2- closed. Similarly, we can prove every $\tau_2\tau_1 - s^*g$ closed set is τ_1- closed. Hence X is a pairwise T_S- space.
Lemma 3.1
The inverse image of a $\tau_1\tau_2 - s^*g$ closed set under a pairwise continuous bijection map $f : X \to Y$ is $\tau_1\tau_2 - s^*g$ closed, where Y is another bitopological space.
Proof
Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise continuous bijection. Let A be

$\sigma_1\sigma_2 - s^*g$ closed in Y. We shall show that$f^{-1}(A)$ is $\tau_1\tau_2 - s^*g$ closed in X. Let $f^{-1}(A) \subseteq U$, where U isτ_1- semi open in X. Then $A \subseteq f(U)$and $f(U)$ is σ_1-semi open in Y. Since A is $\sigma_1\sigma_2 - s^*g$ closed in Y, We have σ_2-cl$(A) \subseteq f(U)$. Therefore$\tau_2 - cl[f^{-1}(A)] \subseteq f^{-1}[\sigma_2 - cl(A)] \subseteq f^{-1}[f(U)] = U$ { Since f is pairwise continuous and bijection }.

$\Rightarrow \tau_2 - cl[f^{-1}(A)] \subseteq U$. Then $f^{-1}(A)$ is $\tau_1\tau_2 - s^*g$ closed in X.

Theorem 3.2

The image of a pairwise T_S - space under a pairwise continuous bijection map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$is a pairwise T_S - space, where Y is another bitopological space.

Proof

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise continuous bijection map. Since f is onto, we have $Y = f(X)$. Let A be $\sigma_1\sigma_2 - s^*g$ closed in Y. We shall show that A is σ_2- closed in Y. By Lemma 1, we have $f^{-1}(A)$ is $\tau_1\tau_2 - s^*g$ closed in X. But, X is a pairwise T_S - space. Hence $f^{-1}(A)$ is τ_2- closed in X. $\Rightarrow f^{-1}(A)$ $= \tau_2 - cl[f^{-1}(A)]$. This implies $A = f[\tau_2 - cl[f^{-1}(A)] \supseteq \sigma_2 - cl(A)$. Hence $\sigma_2 - cl(A) \subseteq A$. Obviously $A \subseteq \sigma_2 - cl(A)$.

Therefore, $\sigma_2 - cl(A) = A$. Now, $\sigma_2 - cl_Y(A) = \sigma_2 - cl(A) \cap Y = A \cap Y = A$. Therefore, A is σ_2- closed in Y. Similarliy we can prove every$\sigma_2\sigma_1 - s^*g$ closed set is σ_1- closed in Y. Hence Y is pairwiseT_S - space.

Theorem 3.3

In a pairwise T_S - space ,

(a) the intersection of two $\tau_1\tau_2 - s^*g$ closed sets is $\tau_1\tau_2 - s^*g$ closed,

(b) the union of two $\tau_1\tau_2 - s^*g$ open sets is $\tau_1\tau_2 - s^*g$ open.

Proof

(a) Let A and B be two $\tau_1\tau_2 - s^*g$ closed sets in (X, τ_1, τ_2). Since X is a pairwise T_S - space , A and B are τ_2- closed in X. Hence $A \cap B$ is τ_2- closed in X. Consequently$A \cap B$ is $\tau_1\tau_2 - s^*g$ closed in X.

(b) Let A and B be two $\tau_1\tau_2 - s^*g$ open sets in (X, τ_1, τ_2). Then A^C and B^C are $\tau_1\tau_2 - s^*g$ closed in X. By (a) , $A^C \cap B^C = (A \cup B)^C$ is $\tau_1\tau_2 - s^*g$ closed in X. Therefore $A \cup B$ is $\tau_1\tau_2 - s^*g$ open in X.

Theorem 3.4

(a) Every pairwise $T_{1/2}$ - space is a pairwise T_S- space,

(b) Every pairwise T_b- space is a pairwise T_S- space,

(c) Every pairwise ${}_\alpha T_b$- space is a pairwise T_S- space,

(d) Every pairwise door space is a pairwise T_S- space.

Proof

(a) Suppose that X is a pairwise $T_{1/2}$- space . Since every $\tau_1\tau_2 - s^*g$ closed set is τ_2- closed in a pairwise $T_{1/2}$- space , X is a pairwise T_S- space.

(b) Suppose that X is a pairwise T_b- space . Let A be $\tau_1\tau_2 - s^*g$ closed in X. Then A is $\tau_1\tau_2 - gs$ closed in X. Since X is a pairwise T_b- space, A is τ_2- closed in X. Hence X is a pairwise T_S- space.

(c) Suppose that X is a pairwise ${}_\alpha T_b$- space . Let A be $\tau_1\tau_2 - s^*g$ closed in X. Then A is $\tau_1\tau_2 - \alpha g$ closed in X. Since X is a pairwise ${}_\alpha T_b$- space, A is τ_2- closed in X. Therefore X is a pairwise T_S- space.

(d) Let X be a pairwie door space. Then X is pairwise $T_{1/2}$. From (a), we have X is a pairwise T_S- space.

Remark 3.1

The converses of the above theorem are not true as can be seen from the following example.

Example 3.2

In Example 3.1, (X, τ_1, τ_2) is a pairwise T_S- space but not a pairwise $T_{1/2}$- space , pairwise T_b- space,pairwise ${}_{\alpha}T_b$- space or a pairwise door space.

Theorem 3.5

(a) Every $\tau_1\tau_2 - gs$ closed set in a pairwise T_b - space is $\tau_1\tau_2 - s^*g$ closed,
(b) Every $\tau_1\tau_2 - sg$ closed set in a pairwise T_b - space is $\tau_1\tau_2 - s^*g$ closed,
(c) Every $\tau_1\tau_2 - \alpha g$ closed set in a pairwise ${}_{\alpha}T_b$- space is $\tau_1\tau_2 - s^*g$ closed.

Proof

(a) Let X be a pairwise T_b- space and A be $\tau_1\tau_2 - gs$ closed in X. Then A is τ_2- closed in X. Consequently,A is $\tau_1\tau_2 - s^*g$ closed in X.
(b) Let X be a pairwise T_b- space and A be $\tau_1\tau_2 - sg$ closed in X. Since A is $\tau_1\tau_2 - gs$ closed in X, A is $\tau_1\tau_2 - s^*g$ closed in X { by (a) }.
(c) Let X be a pairwise ${}_{\alpha}T_b$- space and A be $\tau_1\tau_2 - \alpha g$ closed in X. Then A is τ_2- closed in X. Consequently,A is $\tau_1\tau_2 - s^*g$ closed in X.

Corollary 3.1

(a) Every subset of a pairwise complemented T_b - space is $\tau_1\tau_2 - s^*g$ closed,
(b) Every subset of a pairwise complemented $T_{1/2}$- space is $\tau_1\tau_2 - s^*g$ closed,
(c) Every subset of a pairwise complemented ${}_{\alpha}T_b$- space is $\tau_1\tau_2 - s^*g$ closed.

Proof

(a) Since X is pairwise complemented, every subset of X is $\tau_1\tau_2 - gs$ closed in X. Since X is a pairwise T_b - space, every subset of X is $\tau_1\tau_2 - s^*g$ closed in X. { by Theorem 5 (a) }
(b) Since X is pairwise complemented, every subset of X is $\tau_1\tau_2 - g$ closed in X. Since X is a pairwise $T_{1/2}$- space, every subset of X is $\tau_1\tau_2 - s^*g$ closed in X.
(c) Since X is pairwise complemented , every subset of X is $\tau_1\tau_2 - \alpha g$ closed. Since X is a ${}_{\alpha}T_b$- space , every subset of X is $\tau_1\tau_2 - s^*g$ closed in X { by Theorem 5 (c) }.

Theorem 3.6

If (X, τ_1, τ_2) is both pairwise T_p^*- space and pairwise ${}^*T_p$ - space then X is a pairwise T_S- space.

Proof

Let A be $\tau_1\tau_2 - s^*g$ closed in X. Then A is $\tau_1\tau_2 - gp$ closed in X. Since X is a pairwise ${}^*T_p$- space, A is $\tau_1\tau_2 - g^*p$- closed in X. Therefore X is pairwise T_p^* space. Hence A is τ_2- closed in X. Consequently X is a pairwise T_S- space.

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