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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Strong Convergence of the Inertial Proximal Algorithm for the Split Variational Inclusion Problem in Hilbert Spaces

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Abstract In this paper, we study the split variational inclusion problem (SVIP). We introduce algorithm for the SVIP and prove the strong convergence theorem in the framework of real Hilbert spaces. Finally, we give some numerical experiments to signal recovery.

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1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space and let $B : H \to 2^H$ be a set-valued mapping with $\mathcal{D}(B) = \{x \in H : B(x) \neq \emptyset\}$. In 2011, Moudafi [1] introduced the split variational inclusion inclusion problem (SVIP) which is to find $x^* \in H_1$ such that

$$0_{H_1} \in B_1(x^*) \text{ and } 0_{H_2} \in B_2(Ax^*), \tag{1.1}$$

where H_1 and H_2 are real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $A : H_1 \to H_2$ is a linear and bounded operator, and A^* is the adjoint of A. This problem has a variety of specific applications in real world such as image reconstruction and signal processing. The split variational inclusion inclusion problem (SVIP) can be apply to split minimization problem, split feasibility problem, relaxed split feasibility problem and linear inverse problem. Some related works can be found in [2–13].

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In 2011, Byrne et al. [14] proposed the iterative method for solving the split variational inclusion problem in Hilbert spaces as following:

$$x_{n+1} = J_\beta^{B_1}(x_n - \gamma A^*(I - J_\beta^{B_2})Ax_n), \ n \in \mathbb{N}, \tag{1.2}$$

where β, γ are real positive numbers and $J_\beta^{B_1}(x) = (I + \beta B)^{-1}(x)$ for each $x \in H$ which is a resolvent mapping of B order β.

Recently, in 2016, Chuang [15] studied algorithm for solving the SVIP in Hilbert spaces as following:

Algorithm 1.1. Let $x_0, x_1 \in H_1$ and set

$$\begin{aligned} u_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= J_{\beta_n}^{B_1}(u_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Au_n), \end{aligned} \tag{1.3}$$

where $\{\theta_n\} \subseteq [0, 1)$ and $\gamma_n > 0$ satisfies

$$\gamma_n \|A^*(I - J_{\beta_n}^{B_2})Au_n - A^*(I - J_{\beta_n}^{B_2})Ay_n\| \leq \delta\|u_n - y_n\|, 0 < \delta < 1. \tag{1.4}$$

Define

$$x_{n+1} = J_{\beta_n}^{B_1}(u_n - \alpha_n D(u_n, \gamma_n)), \tag{1.5}$$

where

$$D(u_n, \gamma_n) = u_n - y_n + \gamma_n(A^*(I - J_{\beta_n}^{B_2})Ay_n - A^*(I - J_{\beta_n}^{B_2})Au_n)$$

and

$$\alpha_n = \frac{\langle u_n - y_n, D(u_n, \gamma_n)\rangle}{\|D(u_n, \gamma_n)\|^2}.$$

Chuang [15] proved its weak convergence theorem under suitable conditions.

In this paper, inspired by the algorithm proposed by Chuang [15], we introduce algorithm for the SVIP and prove its strong convergence in real Hilbert spaces. We give some numerical experiments in signal recovery to support our main results.

2. Preliminaries

In this section, we provide some basic definitions and lemmas which will be used in the sequel. Let H be a real Hilbert space. In what follows, we use the following notations:

- the symbols $\rightharpoonup$ stands for the weak convergence.
- the symbols $\rightarrow$ stands for the strong convergence.

Recall that a mapping $T : H \to H$ is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H.$$

(2) firmly-nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \ \forall x, y \in H.$$

We note that if T is firmly-nonexpansive, then $I - T$ is also firmly-nonexpansive. In a real Hilbert space H, we know the following relations:

$$\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2, \ \forall x, y \in H \tag{2.1}$$

and

$$\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2,\ \forall x,y \in H \text{ and } \alpha \in \mathbb{R}.$$

A mapping $f : H \to H$ is said to be a contraction on H if there exists a constant $a \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq a\|x - y\|,\ \forall x, y \in H. \tag{2.2}$$

A set-valued mapping $B : H \to 2^H$ is called monotone if for all $x, y \in H$

$$\langle u - v, x - y\rangle \geq 0,\ u \in Bx \text{ and } v \in By.$$

For a set-valued mapping B, graph(B) is defined as graph$(B) := \{(x,u) \in H \times H : u \in B(x)\}$. A monotone mapping $B : H \to 2^H$ is said to be maximal if the graph(B) is not properly contained in the graph of any other monotone mapping. Let $B : H \to 2^H$ be a set-valued maximal monotone operator. The resolvent operator $J_\beta^B : H \to H$ associated with B is defined by

$$J_\beta^B(x) = (I + \beta B)^{-1}(x),\ \forall x \in H,$$

where $\beta > 0$. It is well known that the resolvent operator is single-valued and firmly non-expansive.

Lemma 2.1 (Demiclosedness principle, [16])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. If $x_n \rightharpoonup x \in C$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $x = Tx$.*

In order to study the SVIP, we recall some lemmas which are needed in our proof. We denote by $B^{-1}(0) = \{x \in H : 0 \in Bx\}$, $\mathcal{D}(T)$ the domain of T and $Fix(T)$ the fixed point set of T, that is, $Fix(T) = \{x \in H : x = Tx\}$.

Lemma 2.2 ([17, 18])**.** *Let H be a real Hilbert space, $B : H \to 2^H$ be a set-valued maximal monotone mapping. Thus,*

(i) J_β^B is a single-valued and firmly nonexpansive mapping for each $\beta > 0$;

(ii) $\mathcal{D}(J_\beta^B) = H$ and $Fix(J_\beta^B) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;

(iii) $\|x - J_\beta^B x\| \leq \|x - J_\gamma^B x\|$ for all $0 < \beta \leq \gamma$ and for all $x \in H$;

(iv) Suppose that $B^{-1}(0) \neq \emptyset$. Then $\|x - J_\beta^B x\|^2 + \|J_\beta^B - x^*\|^2 \leq \|x - x^*\|^2$ for each $x \in H$, each $x^* \in B^{-1}(0)$, and each $\beta > 0$.

(v) Suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\beta^B x, J_\beta^B x - w\rangle \geq 0$ for each $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.

The next lemma gives a crucial characterization of the solution sets of the SVIP and the fixed point sets of the resolvent operator.

Lemma 2.3 ([17])**.** *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $\beta > 0$, $\gamma > 0$, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone mappings. Given any $x^* \in H_1$.*

(i) If x^* is a solution of (SVIP), then $J_\beta^{B_1}(x^* - \gamma A^*(I - J_\beta^{B_2})Ax^*) = x^*$.

(ii) Suppose that $J_\beta^{B_1}(x^* - \gamma A^*(I - J_\beta^{B_2})Ax^*) = x^*$ and the solution set of (SVIP) is nonempty. Then x^* is a solution of (SVIP).

Lemma 2.4 ([17])**.** *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $\beta > 0$. Let $B : H_2 \to 2^{H_2}$ be a set-valued maximal monotone mapping.*

Define a mapping $T : H_1 \to H_1$ *by* $Tx := A^*(I - J^B_\beta)Ax$ *for each* $x \in H_1$*. Then*
(i) $\|(I - J^B_\beta)Ax - (I - J^B_\beta)Ay\|^2 \leq \langle Tx - Ty, x - y\rangle$ *for all* $x, y \in H_1$*;*
(ii) $\|A^*(I - J^B_\beta)Ax - A^*(I - J^B_\beta)Ay\|^2 \leq \|A\|^2 \cdot \langle Tx - Ty, x - y\rangle$ *for all* $x, y \in H_1$*.*

We also need the following tools in convergence analysis.

Lemma 2.5 ([19, 20])**.** *Let* $\{a_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \; n \geq 1, \tag{2.3}$$

where $\{\delta_n\}$ *is a sequence in* $(0, 1)$ *and* $\{b_n\}$ *is a real sequence. Assume* $\sum_{n=1}^\infty c_n < \infty$*. Then the following results hold:*
(i) If $b_n \leq \delta_n M$ *for some* $M \geq 0$*, then* $\{a_n\}$ *is a bounded sequence.*
(ii) If $\sum_{n=1}^\infty \delta_n = \infty$ *and* $\limsup_{n\to\infty} b_n/\delta_n \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$*.*

Lemma 2.6 ([21])**.** *Assume* $a_n \in [0, \infty)$ *and* $\delta_n \in [0, \infty)$ *satisfy:*
(i) $a_{n+1} - a_n \leq \theta_n(a_n - a_{n-1}) + \varphi_n$*,*
(ii) $\displaystyle\sum_{n=1}^{+\infty} \varphi_n < \infty$*,*
(iii) $\{\theta_n\} \subset [0, \theta]$*, where* $\theta \in [0, 1)$*. Then the sequence* $\{a_n\}$ *is convergent with* $\displaystyle\sum_{n=1}^{+\infty} [a_{n+1} - a_n]_+ < \infty$*, where* $[t]_+ := \max\{t, 0\}$ *(for any* $t \in \mathbb{R}$*).*

Lemma 2.7 ([22])**.** *Assume* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - c_n)s_n + c_n\eta_n, \tag{2.4}$$
$$s_{n+1} \leq s_n - \lambda_n + \varphi_n, \tag{2.5}$$

for all $n \geq 1$ *where* $\{c_n\}$ *is a sequence in* $(0, 1)$*,* $\{\lambda_n\}$ *is a sequence of nonnegative real numbers and* $\{\delta_n\}$ *and* $\{\varphi_n\}$ *are two sequences in* $\mathbb{R}$ *such that*
(i) $\displaystyle\sum_{n=1}^{\infty} c_n = \infty$*;*
(ii) $\displaystyle\lim_{n\to\infty} \varphi_n = 0$*;*
(iii) $\lim_{k\to\infty} \lambda_{n_k} = 0$ *implies* $\displaystyle\limsup_{k\to\infty} \eta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$*.*
Then $\displaystyle\lim_{n\to\infty} s_n = 0$*.*

3. Main Results

In this section, we introduce algorithm involving the linesearch and inertial techniques and prove the strong convergence theorem. Throughout this paper, we denote by Ω the solution set of the SVIP and assume that Ω is nonempty. Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operator, and A^* be the adjoint operator of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators.

Algorithm 3.1. Let $\sigma > 0$, $\rho \in (0, 1)$, $\delta \in (0, 1)$, $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be a sequence in $[0, \theta) \subseteq [0, 1)$. Take arbitrarily $x_0, x_1 \in H_1$ and compute

$$\begin{aligned} u_n &= x_n + \theta_n(x_n - x_{n-1}) \qquad (3.1)\\ y_n &= J^{B_1}_{\beta_n}(u_n - \gamma_n A^*(I - J^{B_2}_{\beta_n})Au_n) \end{aligned}$$

where $\gamma_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\gamma_n\|A^*(I - J^{B_2}_{\beta_n})Au_n - A^*(I - J^{B_2}_{\beta_n})Ay_n\| \leq \delta\|u_n - y_n\|. \tag{3.2}$$

Define

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J^{B_1}_{\beta_n}(u_n - \phi\xi_n D(u_n,\gamma_n)) \tag{3.3}$$

where $\phi \in (0,2)$,

$$D(u_n,\gamma_n) = u_n - y_n - \gamma_n(A^*(I - J^{B_2}_{\beta_n})Au_n - A^*(I - J^{B_2}_{\beta_n})Ay_n)$$

and

$$\xi_n = \frac{\langle u_n - y_n, D(u_n,\gamma_n)\rangle + \gamma_n\|(I - J^{B_2}_{\beta_n})Ay_n\|^2}{\|D(u_n,\gamma_n)\|^2}. \tag{3.4}$$

Following the proof line as in [23], we obtain the following lemma:

Lemma 3.2. *The linesearch (3.2) is well defined. Besides, $\gamma' \leq \gamma_n \leq \sigma$, where $\gamma' = \min\{\sigma, \frac{\delta\rho}{L}\}$ and $L = \|A\|^2$.*

Theorem 3.3. *Assume that $\{\alpha_n\}$ and $\{\theta_n\}$ satisfy the assumptions:*
(a1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(a2) $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.
Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strong to a point $P_\Omega f(z)$ in Ω.

Proof. Let $z = P_\Omega f(z)$. Then $z \in B_1^{-1}(0)$ and $Az \in B_2^{-1}(0)$. By the definitions of y_n and $D(u_n,\gamma_n)$, we get

$$y_n = J^{B_1}_{\beta_n}(y_n - (\gamma_n A^*(I - J^{B_2}_{\beta_n})Ay_n - D(u_n,\gamma_n))). \tag{3.5}$$

By Lemma 2.2 (v), it follows that

$$\langle y_n - z, D(u_n,\gamma_n) - \gamma_n A^*(I - J^{B_2}_{\beta_n})Ay_n\rangle \geq 0 \tag{3.6}$$

which implies that

$$\langle y_n - z, D(u_n,\gamma_n)\rangle \geq \gamma_n\langle y_n - z, A^*(I - J^{B_2}_{\beta_n})Ay_n\rangle. \tag{3.7}$$

Moreover, we have $J^{B_2}_{\beta_n}Az = Az$. It also follows that

$$\begin{aligned}
\gamma_n\langle y_n - z, A^*(I - J^{B_2}_{\beta_n})Ay_n\rangle &= \gamma_n\langle y_n - z, A^*(I - J^{B_2}_{\beta_n})Ay_n - A^*(I - J^{B_2}_{\beta_n})Az\rangle \\
&= \gamma_n\langle Ay_n - Az, (I - J^{B_2}_{\beta_n})Ay_n - (I - J^{B_2}_{\beta_n})Az\rangle \\
&\geq \gamma_n\|(I - J^{B_2}_{\beta_n})Ay_n\|^2.
\end{aligned} \tag{3.8}$$

Using Lemma 2.2 (v) and (3.8), we get

$$\begin{aligned}
\langle y_n - z, D(u_n,\gamma_n)\rangle &= \langle y_n - z, u_n - y_n - \gamma_n(A^*(I - J^{B_2}_{\beta_n})Au_n - A^*(I - J^{B_2}_{\beta_n})Ay_n)\rangle \\
&= \langle y_n - z, u_n - y_n - \gamma_n A^*(I - J^{B_2}_{\beta_n})Au_n\rangle \\
&\quad +\gamma_n\langle y_n - z, A^*(I - J^{B_2}_{\beta_n})Ay_n\rangle \\
&\geq \gamma_n\|(I - J^{B_2}_{\beta_n})Ay_n\|^2.
\end{aligned} \tag{3.9}$$

From (3.9), we obtain

$$\begin{aligned}\langle u_n - z, D(u_n,\gamma_n)\rangle &= \langle u_n - y_n + y_n - z, D(u_n,\gamma_n)\rangle \\ &= \langle u_n - y_n, D(u_n,\gamma_n)\rangle + \langle y_n - z, D(u_n,\gamma_n)\rangle \\ &\geq \langle u_n - y_n, D(u_n,\gamma_n)\rangle + \gamma_n\|(I-J_{\beta_n}^{B_2})Ay_n\|^2. \qquad (3.10)\end{aligned}$$

We observe that, by the linesearch (3.2),

$$\begin{aligned}&\langle u_n - y_n, D(u_n,\gamma_n)\rangle \\ =\;& \langle u_n - y_n, u_n - y_n - \gamma_n(A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n)\rangle \\ =\;& \langle u_n - y_n, u_n - y_n\rangle - \langle u_n - y_n, \gamma_n(A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n)\rangle \\ =\;& \|u_n - y_n\|^2 - \gamma_n\langle u_n - y_n, A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n\rangle \\ \geq\;& \|u_n - y_n\|^2 - \gamma_n\|u_n - y_n\|\|A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n\| \\ \geq\;& \|u_n - y_n\|^2 - \delta\|u_n - y_n\|^2 \\ =\;& (1-\delta)\|u_n - y_n\|^2. \qquad (3.11)\end{aligned}$$

On the other hand, we obtain the following estimation

$$\begin{aligned}\|D(u_n,\gamma_n)\|^2 &= \|u_n - y_n - \gamma_n(A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n)\|^2 \\ &= \|u_n - y_n\|^2 + \gamma_n^2\|A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n\|^2 \\ &\quad -2\gamma_n\langle u_n - y_n, A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n)\rangle \\ &\leq \|u_n - y_n\|^2 + \delta^2\|u_n - y_n\|^2 \\ &\quad +2\gamma_n|\langle u_n - y_n, A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n)\rangle| \\ &\leq \|u_n - y_n\|^2 + \delta^2\|u_n - y_n\|^2 \\ &\quad +2\gamma_n\|u_n - y_n\|\|A^*(I-J_{\beta_n}^{B_2})Au_n - A^*(I-J_{\beta_n}^{B_2})Ay_n)\| \\ &\leq \|u_n - y_n\|^2 + \delta^2\|u_n - y_n\|^2 + 2\delta\|u_n - y_n\|^2 \\ &= (1+\delta)^2\|u_n - y_n\|^2. \qquad (3.12)\end{aligned}$$

Combining (3.11) and (3.12), we obtain

$$\gamma_n \geq \frac{\langle u_n - y_n, D(u_n,\gamma_n)\rangle}{\|D(u_n,\gamma_n)\|^2} \geq \frac{1-\delta}{(1+\delta)^2}. \qquad (3.13)$$

From (3.3), (3.4) and (3.10), we have

$$\begin{aligned}&\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - z\|^2 \\ =\;& \|u_n - \phi\xi_n D(u_n,\gamma_n) - z\|^2 - \|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2 \\ =\;& \|u_n - z\|^2 - 2\phi\xi_n\langle u_n - z, D(u_n,\gamma_n)\rangle + \phi^2\xi_n^2\|D(u_n,\gamma_n)\|^2 \\ & -\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2 \\ \leq\;& \|u_n - z\|^2 - 2\phi\xi_n(\langle u_n - y_n, D(u_n,\gamma_n)\rangle + \gamma_n\|(I-J_{\beta_n}^{B_2})Ay_n\|^2) \\ & +\phi^2\xi_n^2\|D(u_n,\gamma_n)\|^2 - \|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2 \\ =\;& \|u_n - z\|^2 - 2\phi\xi_n\left(\frac{\langle u_n - y_n, D(u_n,\gamma_n)\rangle + \gamma_n\|(I-J_{\beta_n}^{B_2})Ay_n\|^2}{\|D(u_n,\gamma_n)\|^2}\cdot\|D(u_n,\gamma_n)\|^2\right) \\ & +\phi^2\xi_n^2\|D(u_n,\gamma_n)\|^2 - \|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2\end{aligned}$$

$$\begin{aligned} &= \|u_n - z\|^2 - 2\phi\xi_n^2\|D(u_n,\gamma_n)\|^2 + \phi^2\xi_n^2\|D(u_n,\gamma_n)\|^2 \\ &\quad -\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2 \\ &= \|u_n - z\|^2 - \phi(2-\phi)\xi_n^2\|D(u_n,\gamma_n)\|^2 \\ &\quad -\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2. \end{aligned} \tag{3.14}$$

From (3.14) and $\phi \in (0,2)$, we have

$$\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - z\| \le \|u_n - z\| \tag{3.15}$$

On the other hand, we see that

$$\begin{aligned} \|u_n - z\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - z\|^2 \\ &= \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.16}$$

Using (2.1), we also obtain

$$\langle x_n - z, x_n - x_{n-1}\rangle = \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_{n-1} - z\|^2. \tag{3.17}$$

Combining (3.16) and (3.17), it follows that

$$\begin{aligned} \|u_n - z\|^2 &= \|x_n - z\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + \theta_n\|x_n - z\|^2 \\ &\quad +\theta_n\|x_n - x_{n-1}\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ &\le \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.18}$$

From (3.14) and (3.18), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(\alpha_n f(x_n) + (1-\alpha_n)J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n))) - z\|^2 \\ &= \langle \alpha_n f(x_n) + (1-\alpha_n)J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - z, x_{n+1} - z\rangle \\ &= \alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad +(1-\alpha_n)\langle J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - z, x_{n+1} - z\rangle \\ &\le \frac{1}{2}\alpha_n a\|x_n - z\|^2 + \frac{1}{2}\alpha_n\|x_{n+1} - z\|^2 + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad +\frac{1}{2}(1-\alpha_n)\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - z\|^2 + \frac{1}{2}(1-\alpha_n)\|x_{n+1} - z\|^2 \\ &\le \frac{1}{2}\alpha_n a\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \frac{1}{2}(1-\alpha_n)\|x_n - z\|^2 \\ &\quad +\frac{1}{2}(1-\alpha_n)\theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + (1-\alpha_n)\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad -\frac{1}{2}(1-\alpha_n)\phi(2-\phi)\xi_n^2\|D(u_n,\gamma_n)\|^2 \\ &\quad -\frac{1}{2}(1-\alpha_n)\|J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_n + \phi\xi_n D(u_n,\gamma_n)\|^2 \\ &\quad +\alpha_n\langle f(z) - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.19}$$

It follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-((1-a)\alpha_n))\|x_n-z\|^2\\ &+(1-\alpha_n)\theta_n\|x_{n-1}-x_n\|(\|x_n-z\|+\|x_{n-1}-z\|)\\ &+(1-\alpha_n)2\theta_n\|x_n-x_{n-1}\|^2-(1-\alpha_n)\phi(2-\phi)\xi_n^2\|D(u_n,\gamma_n)\|^2\\ &-(1-\alpha_n)\|J_{\beta_n}^{B_1}(u_n-\phi\xi_nD(u_n,\gamma_n))-u_n+\phi\xi_nD(u_n,\gamma_n)\|^2\\ &+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle. \qquad (3.20)\end{aligned}$$

Next, we will show that $\{x_n\}$ is bounded. We see that

$$\begin{aligned}\|u_n-z\| &= \|x_n-\theta_n(x_n-x_{n-1})-z\|\\ &\le \|x_n-z\|+\theta_n\|x_n-x_{n-1}\| \qquad (3.21)\end{aligned}$$

From (3.15), we have

$$\begin{aligned}\|x_{n+1}-z\| &= \|\alpha_nf(x_n)+(1-\alpha_n)J_{\beta_n}^{B_1}(u_n-\phi\xi_nD(u_n,\gamma_n))-z\|\\ &\le \alpha_n\|f(x_n)-z\|+(1-\alpha_n)\|J_{\beta_n}^{B_1}(u_n-\phi\xi_nD(u_n,\gamma_n))-z\|\\ &\le \alpha_n\|f(x_n)-f(z)\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|u_n-z\|\\ &\le \alpha_na\|x_n-z\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\|\\ &\quad+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|\\ &\le (1-\alpha_n(1-a))\|x_n-z\|+\alpha_n\|f(z)-z\|\\ &\quad+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|. \qquad (3.22)\end{aligned}$$

We see that $\psi_n=\frac{(1-\alpha_n)\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\to 0$. Hence it is bounded. Putting $M=\max\{\|f(z)-z\|,\sup_{n\ge1}\psi_n\}$ and using Lemma 2.5 (i), we conclude that the sequence $\{\|x_n-z\|\}$ is bounded.

Hence, we can show that $\{x_n\}$ is bounded. Employing Lemma 2.7 and (3.20), we set

$$\begin{aligned}s_n &= \|x_n-z\|^2\\ \varphi_n &= (1-\alpha_n)\theta_n\|x_{n-1}-x_n\|(\|x_n-z\|+\|x_{n-1}-z\|)+(1-\alpha_n)2\theta_n\|x_n-x_{n-1}\|^2\\ &\quad+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ \eta_n &= \frac{2}{1-a}\langle f(z)-z,x_{n+1}-z\rangle\\ &\quad+(1-\alpha_n)\frac{\theta_n}{(1-a)\alpha_n}\|x_{n-1}-x_n\|(\|x_n-z\|+\|x_{n-1}-z\|)\\ &\quad+(1-\alpha_n)\frac{2\theta_n}{(1-a)\alpha_n}\|x_n-x_{n-1}\|^2\\ \lambda_n &= (1-\alpha_n)\phi(2-\phi)\xi_n^2\|D(u_n,\gamma_n)\|^2\\ &\quad+(1-\alpha_n)\|J_{\beta_n}^{B_1}(u_n-\phi\xi_nD(u_n,\gamma_n))-u_n+\phi\xi_nD(u_n,\gamma_n)\|^2\\ c_n &= (1-a)\alpha_n. \qquad (3.23)\end{aligned}$$

So, (3.20) reduces to the inequalities

$$\begin{aligned}s_{n+1} &\le (1-c_n)s_n+c_n\eta_n \qquad (3.24)\\ s_{n+1} &\le s_n-\lambda_n+\varphi_n. \qquad (3.25)\end{aligned}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\lim_{k\to\infty} \lambda_{n_k} = 0. \tag{3.26}$$

It follows that

$$\lim_{k\to\infty} (1-\alpha_{n_k})\phi(2-\phi)\xi_{n_k}^2 \|D(u_{n_k},\gamma_{n_k})\|^2 = 0. \tag{3.27}$$

Using assumption of ϕ, we obtain

$$\lim_{k\to\infty} \|D(u_{n_k},\gamma_{n_k})\| = 0. \tag{3.28}$$

By definition of $D(u_n,\gamma_n)$, we have

$$\begin{aligned}
\|u_{n_k}-y_{n_k}\| &= \|D(u_{n_k},\gamma_{n_k}) + \gamma_{n_k}(A^*(I-J_{\beta_{n_k}}^{B_2})Au_{n_k} - A^*(I-J_{\beta_{n_k}}^{B_2})Ay_{n_k})\| \\
&\le \|D(u_{n_k},\gamma_{n_k})\| + \gamma_{n_k}\|A^*(I-J_{\beta_{n_k}}^{B_2})Au_{n_k} - A^*(I-J_{\beta_{n_k}}^{B_2})Ay_{n_k}\| \\
&\le \|D(u_{n_k},\gamma_{n_k})\| + \delta\|u_{n_k}-y_{n_k}\|.
\end{aligned}$$

This shows that

$$(1-\delta)\|u_{n_k}-y_{n_k}\| \le \|D(u_{n_k},\gamma_{n_k})\| \tag{3.29}$$

which yields

$$\lim_{k\to\infty} \|u_{n_k}-y_{n_k}\| = 0. \tag{3.30}$$

By (3.1), we see that

$$\lim_{k\to\infty} \|u_{n_k}-x_{n_k}\| = 0. \tag{3.31}$$

From (3.30) and (3.31), we obtain

$$\begin{aligned}
\|x_{n_k}-y_{n_k}\| &\le \|x_{n_k}-u_{n_k}\| + \|u_{n_k}-y_{n_k}\| \\
&\to 0 \text{ as } k\to\infty.
\end{aligned} \tag{3.32}$$

Consider

$$\begin{aligned}
\|x_{n_k+1}-u_{n_k}\| &= \|\alpha_{n_k} f(x_{n_k}) + (1-\alpha_{n_k})J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_{n_k}\| \\
&= \|\alpha_{n_k} f(x_{n_k}) + (1-\alpha_{n_k})J_{\beta_n}^{B_1}(u_n - \phi\xi_n D(u_n,\gamma_n)) - u_{n_k} \\
&\quad + \phi\xi_n D(u_n,\gamma_n) - \phi\xi_n D(u_n,\gamma_n)\| \\
&\le \alpha_{n_k}\|f(x_{n_k}) - u_{n_k}\| + (1-\alpha_{n_k})\|J_{\beta_{n_k}}^{B_1}(u_{n_k} - \phi\xi_{n_k} D(u_{n_k},\gamma_{n_k})) \\
&\quad - u_{n_k} + \phi\xi_{n_k} D(u_{n_k},\gamma_{n_k})\| + (1-\alpha_{n_k})\phi\xi_{n_k}\|D(u_{n_k},\gamma_{n_k})\| \\
&\to 0, \text{ as } k\to\infty.
\end{aligned} \tag{3.33}$$

By (3.31) and (3.33), we get

$$\begin{aligned}
\|x_{n_k+1}-x_{n_k}\| &\le \|x_{n_k+1}-u_{n_k}\| + \|u_{n_k}-x_{n_k}\| \\
&\to 0 \text{ as } k\to\infty.
\end{aligned} \tag{3.34}$$

From Lemma 3.2, (3.9) and (3.28), we get

$$\begin{aligned}
\|Ay_{n_k} - J_{\beta_n}^{B_1}Ay_{n_k}\|^2 &\le \frac{1}{\gamma'}\langle y_{n_k}-z, D(u_{n_k},\gamma_{n_k})\rangle \\
&\le \frac{1}{\gamma'}\|y_{n_k}-z\|\|D(u_{n_k},\gamma_{n_k})\| \\
&\to 0 \text{ as } k\to\infty.
\end{aligned} \tag{3.35}$$

By Lemma 2.2(i), we have

$$\begin{aligned} & \|Au_{n_k} - J^{B_2}_{\beta_{n_k}} Au_{n_k}\| \\ \leq\ & \|Au_{n_k} - J^{B_2}_{\beta_{n_k}} Au_{n_k} - Ay_{n_k} + J^{B_2}_{\beta_{n_k}} Ay_{n_k}\| + \|Ay_{n_k} - J^{B_2}_{\beta_{n_k}} Ay_{n_k}\| \\ \leq\ & 2\|A\|\|u_{n_k} - y_{n_k}\| + \|Ay_{n_k} - J^{B_2}_{\beta_{n_k}} Ay_{n_k}\|. \end{aligned}$$

By (3.30) and (3.35), we have

$$\lim_{k\to\infty} \|Au_{n_k} - J^{B_2}_{\beta_{n_k}} Au_{n_k}\| = 0. \tag{3.36}$$

By Lemma 2.2(iii), we have

$$\lim_{k\to\infty} \|Au_{n_k} - J^{B_2}_{\beta} Au_{n_k}\| \leq \lim_{k\to\infty} \|Au_{n_k} - J^{B_2}_{\beta_{n_k}} Au_{n_k}\| = 0. \tag{3.37}$$

By (3.1), (3.36) and Lemma 2.2(i), we have

$$\begin{aligned} \|y_{n_k} - J^{B_1}_{\beta_{n_k}} u_{n_k}\| &= \|J^{B_1}_{\beta_{n_k}}(u_{n_k} - \gamma_{n_k} A^*(I - J^{B_2}_{\beta_{n_k}})Au_{n_k}) - J^{B_1}_{\beta_{n_k}} u_{n_k}\| \\ &\leq \gamma_{n_k}\|A^*\|\|(I - J^{B_2}_{\beta_{n_k}})Au_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.38}$$

From (3.30) and (3.38), we get

$$\begin{aligned} \|u_{n_k} - J^{B_1}_{\beta_{n_k}} u_{n_k}\| &= \|u_{n_k} - y_{n_k} + y_{n_k} - J^{B_1}_{\beta_{n_k}} u_{n_k}\| \\ &\leq \|u_{n_k} - y_{n_k}\| + \|y_{n_k} - J^{B_1}_{\beta_{n_k}} u_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.39}$$

By Lemma 2.2(iii), we obtain

$$\lim_{k\to\infty} \|u_{n_k} - J^{B_1}_{\beta} u_{n_k}\| \leq \lim_{k\to\infty} \|u_{n_k} - J^{B_1}_{\beta_{n_k}} u_{n_k}\| = 0. \tag{3.40}$$

So, by (3.31) and (3.37), we obtain

$$\begin{aligned} \|Ax_{n_k} - J^{B_2}_{\beta_{n_k}} Ax_{n_k}\| &= \|Ax_{n_k} - J^{B_2}_{\beta_{n_k}} Ax_{n_k} - Au_{n_k} + J^{B_2}_{\beta_{n_k}} Au_{n_k}\| \\ &\quad + \|Au_{n_k} - J^{B_2}_{\beta_{n_k}} Au_{n_k}\| \\ &\leq 2\|A\|\|x_{n_k} - u_{n_k}\| + \|Au_{n_k} - J^{B_2}_{\beta_{n_k}} Au_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.41}$$

By(3.31), (3.40) and Lemma 2.2(i), we get

$$\begin{aligned} \|x_{n_k} - J^{B_1}_{\beta} x_{n_k}\| &= \|x_{n_k} - u_{n_k} + u_{n_k} - J^{B_1}_{\beta} u_{n_k} + J^{B_1}_{\beta} u_{n_k} - J^{B_1}_{\beta} x_{n_k}\| \\ &\leq \|x_{n_k} - u_{n_k}\| + \|u_{n_k} - J^{B_1}_{\beta} u_{n_k}\| + \|J^{B_1}_{\beta} u_{n_k} - J^{B_1}_{\beta} x_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.42}$$

Since $\{x_{n_k}\}$ is bounded then there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup x^*$. By (3.41), (3.42), Lemma 2.1 and Lemma 2.2(ii), we have $x^* \in \Omega$.

From Lemma 2.2 (v), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle &= \lim_{i\to\infty} \langle f(z) - z, x_{n_{k_i}} - z\rangle \\ &= \langle f(z) - z, x^* - z\rangle \\ &\leq 0. \end{aligned} \tag{3.43}$$

From (3.34) and (3.43), we obtain

$$\limsup_{k\to\infty}\langle f(z)-z, x_{n_k+1}-z\rangle \le 0. \tag{3.44}$$

Hence, we get

$$\limsup_{k\to\infty}\eta_{n_k} \le 0. \tag{3.45}$$

Using Lemma 2.7, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_\Omega f(z)$. ■

4. Numerical Experiments

In this section, we give some numerical examples to the signal recovery in compressed sensing. Compressed sensing can be modeled as the following under determinated linear equation system:

$$y = Ax + \epsilon, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ϵ, and $A : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator. It is known that to solve (1.1) can be seen as solving the LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1, \tag{4.2}$$

where $\lambda > 0$. So we can apply our method for solving (1.1) in case $f(x) = \frac{1}{2}\|y - Ax\|_2^2$ and $g(x) = \lambda\|x\|_1$.

Find $x^* \in H_1$ such that

$$x^* \in \arg\min_{x\in H_1} f(x) \text{ and } Ax^* \in \arg\min_{y\in H_2} g(y),$$

where $f : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ are proper, lower semicontinuous, and convex functions.

In a real Hilbert space H, the proximal operator of f is defined by

$$prox_{\beta,f}(x) := \arg\min_{v\in H}\left\{f(v) + \frac{1}{2\beta}\|v - x\|^2\right\} \text{ for each } x \in H.$$

It is well-known that

$$prox_{\beta,f}(x) = (I + \beta\partial f)^{-1}(x) = J_\beta^{\partial f}(x),$$

where ∂f is the subdifferential of f defined by

$$\partial f(x) := \{x^* \in H : f(x) + \langle y - x, x^*\rangle \le f(y) \text{ for each } y \in H\}.$$

From [24], ∂f is a maximal monotone operator and $prox_{\beta,f}$ is firmly nonexpansive.

In our experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval [-2,2] with m nonzero elements. The matrix $A \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and variance one. The observation y is generated by white Gaussian noise with signal-to-noise ratio SNR=40. The restoration accuracy is measured by the error as follows:

$$E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}. \tag{4.3}$$

Choose $f(x) = \frac{x}{2}$, $\alpha_n = \frac{1}{100n}$, $x_0 = [0, 0, ..., 0]$, $x_1 = [1, 1, ..., 1]$, $\delta = 0.2$, $\sigma = 2$ and $\rho = 0.2$. Then the numerical results are reported as follows:

TABLE 1. Numerical results for Algorithm 3.1 with difference dimensions.

	$N = 512$ $M = 256$ $m = 10$	$N = 1024$ $M = 512$ $m = 20$	$N = 2048$ $M = 1024$ $m = 40$	$N = 4096$ $M = 2048$ $m = 100$
Iter	380	789	1460	2801
cpu time	1.0818	14.5757	176.3547	1.4355×10^3

We provide the graphs of original signal and recovered signal by Algorithm 3.1 for each dimensions in Figures 1-4, respectively. The convergence behavior of the error E_n for each dimensions in Figure 5.

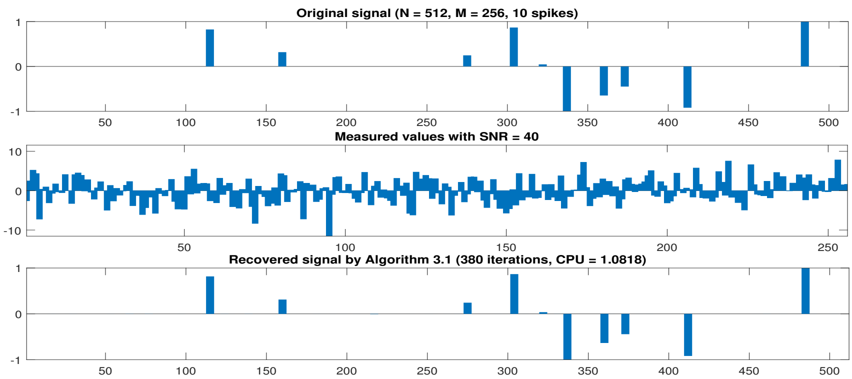

FIGURE 1. From top to bottom: original signal, observation data, recovered signal by Algorithm 3.1 with $N = 512$, $M = 256$ and $m = 10$, respectively.

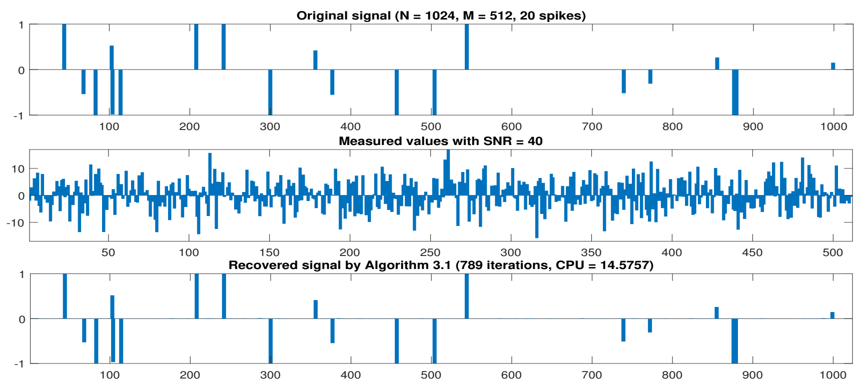

FIGURE 2. From top to bottom: original signal, observation data, recovered signal by Algorithm 3.1 with $N = 1024$, $M = 512$ and $m = 20$, respectively.

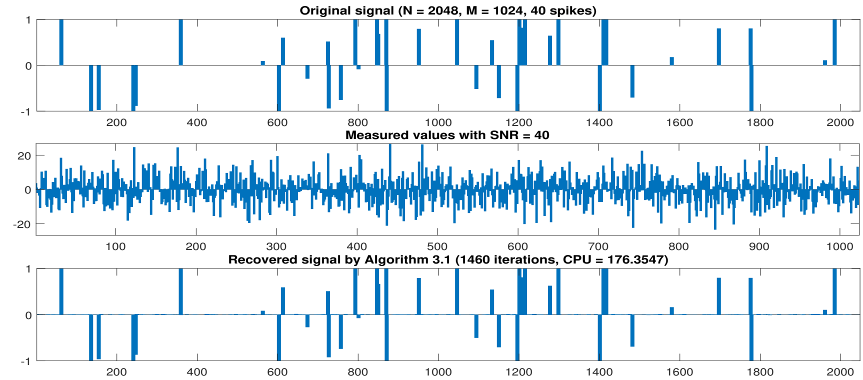

FIGURE 3. From top to bottom: original signal, observation data, recovered signal by Algorithm 3.1 with $N = 2048$, $M = 1024$ and $m = 40$, respectively.

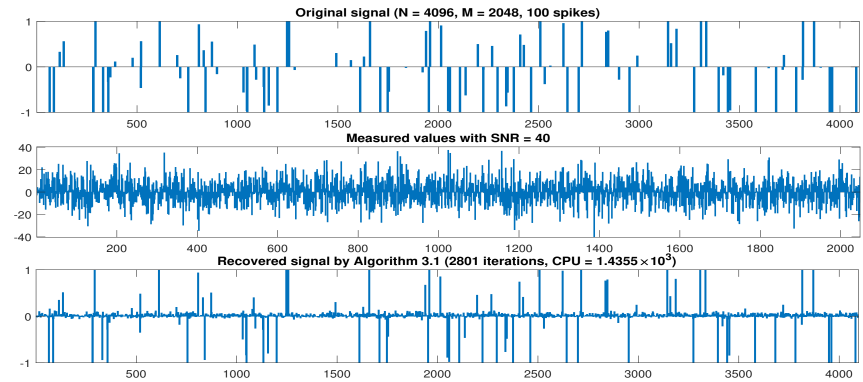

FIGURE 4. From top to bottom: original signal, observation data, recovered signal by Algorithm 3.1 with $N = 4096$, $M = 2048$ and $m = 100$, respectively.

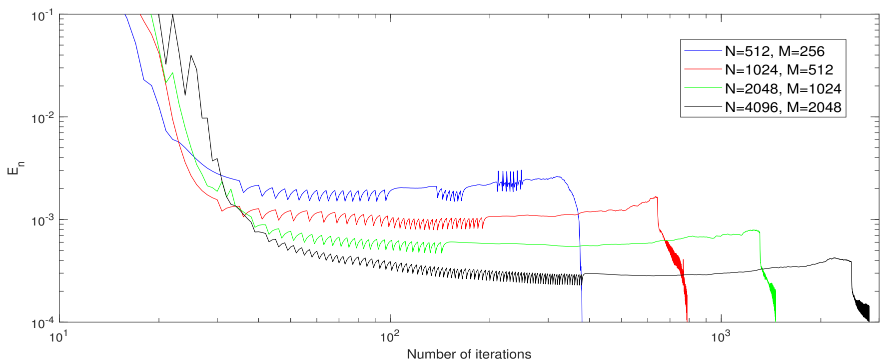

FIGURE 5. E_n versus number of iterations for each dimensions

5. Conclusions

In this work, we proposed algorithm for the split variational inclusion problem (SVIP) and proved strong convergence theorem. Numerical results in signal recovery show that our algorithm has efficiency.

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