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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

A Common Fixed Point of an Infinite Family of Pseudocontractive Maps

M. O. Nnakwe[1,*] **and B. C. Ifebude**[1,2]

[1] *African University of Science and Technology, Abuja, Nigeria*
e-mail : mondaynnakwe@gmail.com (M. O. Nnakwe)

[2] *Alex Ekwueme Federal University Ndufu-Alike Ikwo, Ebonyi State, Nigeria*
e-mail : bifebude@gmail.com (B. C. Ifebude)

Abstract A new inertial algorithm for approximating a common fixed point of an infinite family of strict pseudocontractions without any *compactness condition on the maps or their domains* is constructed. The sequence of the algorithm is proved to converge strongly to a common fixed point of the maps in a uniformly smooth real Banach space. This result is achieved as an application of a new inertial algorithm whose sequence approximates a common zero of an infinite family of inverse strongly accretive maps. In addition, numerical examples are given to compare the efficiency of the sequences of our algorithms over the sequences of some recent algorithms without inertial term. Finally, our theorems improve and complement some recent related results in the literature.

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1. Introduction

Let $\mathcal{D}^*$ be the dual space of a real normed space, $\mathcal{D}$ and H, a real Hilbert space except stated otherwise. A map $J : \mathcal{D} \rightrightarrows \mathcal{D}^*$ defined by

$$Ju := \{u^* \in \mathcal{D}^* : \langle u, u^* \rangle = \|u\| \|u^*\|, \|u\| = \|u^*\|\},$$

is called the *normalized duality map*, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between elements of $\mathcal{D}$ and $\mathcal{D}^*$. It is well known that if $\mathcal{D}$ is strictly convex, smooth and reflexive, then, J is one-to-one, single-valued and onto. For more properties of the normalized duality map (see, e.g., [1, 2]).

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*Corresponding author.

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A map $A : \mathcal{D} \rightrightarrows \mathcal{D}$ is called *accretive*, if for each $u, v \in \mathcal{D}$, there exists $j(u-v) \in J(u-v)$ such that,

$$\langle \eta_u - \eta_v, j(u-v)\rangle \geq 0, \quad \eta_u \in Au, \ \eta_v \in Av. \tag{1.1}$$

Consider the evolution inclusion $0 \in \frac{du}{dt} + Au$, where A is an accretive map. For solving this inclusion problem at equilibrium state, i.e.,

$$0 \in Au, \tag{1.2}$$

Browder [3] in the year 1967, introduced a self map, T on H defined by $T := I - A$, where I is the identity map. He called such a map *pseudocontractive.* Clearly, solutions of (1.2) correspond to fixed points of T. Therefore, approximating zeros of accretive maps is equivalent to approximating fixed points of *pseudocontractive maps*, assuming existence of such zeros, which is also, the equilibrium state of some dynamical systems.

An important class of pseudocontractive maps is the class of nonexpansive maps. It is well known that for a nonexpansive map T with a nonempty fixed point set, the classical Picard iterative sequence $u_{n+1} = Tu_n, \ u_0 \in D(T), \ n \geq 0$, where $D(T)$ denotes the domain of T does not always converge to a fixed point of T, assuming existence. However, following the pioneering research efforts by Mann [4], Krasnoselkii [5], Schaefer [6], Ishikawa [7], Edelstein [8–10], Reinermann [11], Edelstein and OBrian [10], Chidume [12], Reich [13] and a host of other authors, the following recursion formula, called *Mann recursion formula,* was developed and found to be effective for approximating fixed points of nonexpansive maps, assuming existence of solutions.

Let C be a nonempty and convex subset of a normed space, $\mathcal{D}$ and $T : C \to C$ be a nonexpansive map. Let the sequence $\{u_n\}$ in C be defined by

$$u_0 \in \ C, \quad u_{n+1} = (1-a_n)u_n + a_n Tu_n, \ n \geq 0, \tag{1.3}$$

where $\{a_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(i) $\lim a_n = 0$ and (ii) $\sum a_n = \infty$.

Ishikawa [7] proved that if, the sequence $\{u_n\}$ is bounded, then, the sequence is an approximate fixed point sequence.

Remark 1.1. The recursion formula (1.3) can only yield *weak convergence* to a fixed point of T (see e.g., Reich [13]). To obtain strong convergence to a fixed point of T, some compactness condition must be imposed either on the domain of the operator or the operator itself (see e.g., Chidume [14]).

For the more general class of Lipschitz pseudocontractive maps, attempts to use the Mann formula, which has been successfully employed for nonexpansive maps, to approximate a fixed point of a Lipschitz pseudocontractive map even on a compact convex domain in a real Hilbert space proved abortive (see, e..g., Chidume and Mutangadura [15]).

However, in the year 2007, Chidume *et al.* [16] proved the following theorem for a class of strictly pseudocontractive maps in a real Banach space.

Theorem 1.2 (Chidume *et al.*, [16])**.** *Let $\mathcal{D}$ be a real Banach space. Let C be a nonempty, closed and convex subset of E. Let $T\colon C \to C$, be a strictly pseudocontractive map in the sense of Browder and Petryshyn with $Fix(T) \neq \emptyset$. For $u_0 \in C$, define the sequence $\{u_n\}$ by*

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n Tu_n, \forall \ n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ *is a real sequence satisfying the following conditions:*
(i) $\sum \alpha_n = \infty$, *(ii)* $\sum \alpha_n^2 < \infty$. *Then, (a)* $\{u_n\}$ *is bounded*
(b) $\lim ||u^* - u_n||$ *exists, for any* $u^* \in Fix(T)$ *(c)* $\liminf ||u_n - Tu_n|| = 0$.
If in addition, T *is demicompact, then* $\{u_n\}$ *converges strongly to fixed point of* T *in* C.

Motivated by the result of Chidume *et al.*, Zegeye and Shahzad [17] in the year 2009, proved the following theorem:

Theorem 1.3 (Zegeye and Shahzad, [17]). *Let* $\mathcal{D}$ *be a real reflexive Banach space which has a uniformly Gáteaux differentiable norm. Assume that every nonempty closed convex and bounded subset of* $\mathcal{D}$ *has the fixed point property for nonexpansive mappings. Let* $A_i : \mathcal{D} \to \mathcal{D}$, $i = 1, \dots$ *be a countably infinite family of* α_i*-inverse strongly accretive maps such that* $\cap_{i=1}^{\infty} A^{-1}(0) \neq \emptyset$. *Suppose that* $\alpha := \inf\{\alpha_i\} > 0$. *For any* $u \in \mathcal{D}$, *let* $\{x_n\}$ *be a sequence generated from arbitrary* $x_1 \in \mathcal{D}$

$$x_{n+1} = x_n - \lambda_n A x_n - \lambda_n \theta_n (x_n - u), \forall\ n \geq 1, \tag{1.5}$$

where $A := \sum_{i=1}^{\infty} a_i A_i$, *for* $a_i \in (0,1)$, $i = 1, 2, \cdots \sum_{i=1}^{\infty} a_i = 1$ *Then,* $\{x_n\}$ *convergences strongly to a common zero point* Qu *of* $\{A_1, A_2, \dots\}$, *where* Q *is the unique sunny nonexpansive retraction from* $\mathcal{D}$ *onto* $\cap_{i=1}^{\infty} A_i^{-1}(0)$.

In addition, they applied their theorem to prove a strong convergence theorem whose sequence approximates a fixed point of an infinite family of strictly pseudocontractive maps. In fact, they studied the following algorithm:

$$x_1 \in C, \ \ x_{n+1} = x_n - \lambda_n (x_n - T x_n) - \lambda_n \theta_n (x_n - u), \forall\ n \geq 1, \tag{1.6}$$

where T is a self map on C given by $T := \sum_{i=1}^{\infty} a_i (I - T_i)$, for $a_i \in (0,1)$, $i = 1, 2, \dots$ $\sum_{i=1}^{\infty} a_i = 1$ and C is a nonempty, closed and convex subset of E. Then, $\{x_n\}$ convergences strongly to a common fixed point Qu of $\{T_1, T_2, \dots\}$, where Q is the unique sunny nonexpansive retraction from $\mathcal{D}$ onto $\cap_{i=1}^{\infty} F(T_i)$.

Remark 1.4. The compactness condition imposed on the map T in the theorem of Chidume *et al.* has been dispensed with in the thoerem of Zegeye and Shahzad, although, the algorithms are slightly different.

For more and earlier results on approximation of fixed points of pseudocontractions, see e.g., Browder and Petryshyn [18], Chidume and Chidume [19], Chidume *et al.* [16], Reich [13], Bruck [20], Takahashi and Ueda [21], Schu [22], Kirk [23], Berinde [24], Chidume [14], Ofoedu *et al.*, [25], Ofoedu [26], Chidume [27, 28], Romanus *et al.* [29] and Monday [30].

It is well known that algorithms whose sequences approximate zeros of accretive operators are in general, very slow. This is because accretive operators are generally not differentiable. Therefore, algorithms which improve the speed of iterative sequences such as the *Newton-Kantorovich* algorithm may not be used. A lot of efforts are now being put in constructing iterative algorithms that improve and speed up convergence of sequences there by given a better approximation.

One method that is now being studied extensively is to incorporate *inertial extrapolation term* in the algorithms.

In 1964, Polyak [31] introduced and studied the *inertial extrapolation algorithm* from the heavy ball experiment of two order time dynamical system, given by:

$$w^{''}(s) + \xi w^{'}(s) + \nabla\psi(w(s)) = 0, \tag{1.7}$$

where $\xi > 0$ and $\psi : H \to \mathbb{R}$ is a differentiable function. The dynamical system (1.7) is discretized using numerical method such that, given the previous *two* iterates, w_n and w_{n-1}, the next iterate w_{n+1}, can be determined by

$$\frac{w_{n+1} - 2w_n + v_{n-1}}{k^2} + \gamma\frac{w_n - w_{n-1}}{k} + \nabla\phi(w(t)) = 0, \tag{1.8}$$

where k is the step size. Equation (1.8) gives the following algorithm:

$$w_{n+1} = w_n + \beta(w_n - w_{n-1}) - \alpha\nabla\psi(w_n),\ n \geq 0, \tag{1.9}$$

where $\beta = 1 - \gamma k$, $\alpha = k^2$ and $\beta(w_n - w_{n-1})$ is called the *inertial extrapolation term*, which is intended to speed up the convergence of the sequence generated by equation (1.9).

An *inertial-type algorithm* is a two-step iterative process in which the next iterate is defined by making use of the previous *two* iterates.

The study of inertial algorithms, especially in connection with the PPA, forward and backward algorithm, double projection algorithm, extragradient and subgradient-extragradient algorithms, has become a flourishing area of research for numerous mathematicians. For more on the study of algorithms which speeds up convergence, see, for example, Alvarez [32], Thong and Van [33], Cholamjiak and Suantai [34], Dong *et al.* [35], Cholamjiak *et al.* [36], Shehu and Cholamjiak [37], Suantai *et al.* [38], Kesornprom and Cholamjiak [39], Chidume and Monday [40] Cholamjiak and Shehu [41].

Motivated by the results of Chidume et al. [16], Zegeye and Shahzad [17] and Polyak [31], we study in this paper, a *new inertial algorithm* for approximating a common fixed point of an infinite family of strict speudocontractions in a uniformly smooth real Banach space. The sequence of the algorithm is proved to converge strongly to a common fixed points of the maps. This result is achieved as an application of a new inertial algorithm whose sequence approximates a common zero of an infinite family of inverse strongly accretive maps. As far as we know, this is the first *inertial algorithm for approximating a common fixed point of an infinite family of strict speudocontractions* in this direction. Furthermore, numerical examples are given to compare the performance of the sequences of our algorithms over the sequences of some recent algorithms without inertial term. Finally, the theorems proved complement, improve and extend some related results in the literature.

2. Preliminaries

Let $\mathcal{D}$ be a nonempty and closed subset of a uniformly smooth real Banach space dual space M.

The following definitions and lemmas will be needed in the sequel.

Definition 2.1. A map $U : \mathcal{D} \to \mathcal{D}$ called *L-Lipschitz* if, there exists $L > 0$ such that

$$\|Uu - Uv\| \leq L\|u - v\|,\ \forall\ u, v \in \mathcal{D}.$$

If $L = 1$, then, the map U is nonexpansive.

A map $U : \mathcal{D} \to M$ is called *accretive* if, for each $u, v \in \mathcal{D}$ there exists $j(u-v) \in J(u-v)$ such that

$$\big\langle Uu - Uv, j(u-v)\big\rangle \geq 0.$$

A map U is called *η-inverse strongly accretive* if, there exists $\eta > 0$ such that for all $u, v \in \mathcal{D}$, there exists $j(u-v) \in J(u-v)$ such that

$$\big\langle Uu - Uv, j(u-v)\big\rangle \geq \eta \|Uu - Uv\|^2.$$

A map $T : \mathcal{D} \to \mathcal{D}$ is called *strictly-pseudocontractive* if, there exists a constant $\eta > 0$ such that

$$\langle u - v, Tu - Tv\rangle \leq ||u - v||^2 \; - \; \eta||(I-T)u - (I-T)v||^2, \; \forall \; u, v \in \mathcal{D}. \tag{2.1}$$

Remark 2.2. A map U is inverse strongly accretive, if and only if $U := (I - T) : \mathcal{D} \to \mathcal{D}$ is *strictly pseudocontractive.*

Lemma 2.3 (Xu, [42])**.** *Let $\{\beta_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$\beta_{n+1} \leq (1 - \sigma_n)\beta_n + \sigma_n b_n + c_n, \;\; n \geq 1,$$

where $\{\sigma_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the conditions:

(i) $\{\sigma_n\} \subset [0,1], \; \sum_{n=1}^{\infty} \sigma_n = \infty$; *(ii)* $\limsup_{n\to\infty} b_n \leq 0$; *(iii)* $c_n \geq 0, \; \sum_{n=1}^{\infty} c_n < \infty$.

Then, $\lim_{n\to\infty} \beta_n = 0$.

Lemma 2.4 (Zegeye and Shahzad, [17])**.** *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a uniformly smooth real Banach space, M. Let $U_i : \mathcal{D} \to M, \; i = 1, \; 2, \; \ldots$ be a family of η_i-inversely strongly accretive maps with $\eta := \inf_{i\geq 1} \eta_i > 0$ and $\cap_{i=1}^{\infty} U_i^{-1}(0) \neq \emptyset$. Define $U := \sum_{i=1}^{\infty} b_i U_i : \mathcal{D} \to M$, where $\{b_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} b_i = 1$. Setting $T_i := (I - U_i) : \mathcal{D} \to \mathcal{D}$. Then, the following conclusions hold:*

(a.) U is a η-inversely strongly accretive operators,

(b.) T_i is a strictly pseudocontractive for each $i = 1, 2, \ldots$.

(c.) $U^{-1}(0) = \cap_{i=1}^{\infty} U_i^{-1}(0) = \cap_{i=1}^{\infty} F(T_i)$.

Lemma 2.5 (Xu and Roach, [43])**.** *Let $\mathcal{D}$ be a uniformly smooth real Banach space. Then, there exist constants D and C such that for all $x, \; y \in \mathcal{D}, \; j(x) \in J(x)$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + D \max\Big\{\|x\| + \|y\|, \frac{1}{2}C\Big\}\rho_E(\|y\|),$$

where ρ_E denotes the modulus of smoothness of $\mathcal{D}$.

Lemma 2.6 (see e.g., Chidume, [14])**.** *Let $\mathcal{D}$ be a normed real linear space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \; \forall \; x, \; y \in \mathcal{D}, \; \forall \; j(x+y) \in J(x+y).$$

Lemma 2.7 (Reich, [44]). *Let $\mathcal{D}$ be a uniformly smooth real Banach space, and let $A : \mathcal{D} \rightrightarrows \mathcal{D}$ be m-accretive. Let $J_t x := (I + tA)^{-1}x$, $t > 0$ be the resolvent of A, and assume that $A^{-1}(0)$ is not empty. Then, for each $x \in \mathcal{D}$, $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}(0)$.*

Lemma 2.8 (Kamimura and Takahashi, [45]). *Let $\{u_n\}$ and $\{v_n\}$ be sequences in a uniformly convex and uniformly smooth real Banach space such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \phi(u_n, v_n) = 0$, then, $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Remark 2.9. It is easy to see that the converse of Lemma 2.8 is also true whenever $\{u_n\}$ *and* $\{v_n\}$ are bounded.

3. Main Result

The following conditions are required in Lemmas 3.1, 3.2 and 3.3 and Theorem 3.4:

(i) $\lim_{r\to\infty} \delta_r = 0$, $\{\delta_r\}$ is decreasing; (ii) $\sum \lambda_r \delta_r = \infty$; (iii) $\xi_r \le \lambda_r^4 \delta_r \gamma_0$;

(iv) $\lim_{r\to\infty} \frac{\frac{\delta_{r-1}-\delta_r}{\delta_r}}{\lambda_r\delta_r} = 0$, (v) $\frac{\rho_M(\lambda_r K)}{\lambda_r K} \le \delta_r^2 \gamma_0$,

for some constants $\gamma_0 > 0$, $K > 0$ and $\{\delta_r\}$, $\{\lambda_r\}$ and $\{\xi_r\}$ are sequences in $(0, 1)$.

First, we prove the following important lemmas of this section.

Lemma 3.1. *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a uniformly smooth real Banach space, M. Let $U : \mathcal{D} \to M$ be a η-inverse strongly accretive map with $U^{-1}(0) \neq \emptyset$. Let $\{v_r\}$ be a sequence in $\mathcal{D}$ generated by*

$$\begin{cases} v_0,\ v_1 \in \mathcal{D}, \\ y_r = v_r + \xi_r(v_{r-1} - v_r), \\ v_{r+1} = (1 - \lambda_r\delta_r)y_r - \lambda_r(Uy_r - \delta_r v_1),\ \forall\ r \ge 1. \end{cases} \tag{3.1}$$

Then, the sequence $\{v_r\}$ is bounded.

Proof. Let $v^* \in U^{-1}(0)$ and $v_1 \in \mathcal{D}$. Then, there exists $\alpha > 0$ (sufficiently large) such that $v_1 \in \overline{B(v^*, \frac{\alpha}{2})} := \{v \in \mathcal{D} : \|v - v^*\| \le \frac{\alpha}{2}\}$.

Define $B := \overline{B(v^*, \alpha)} = \{v \in \mathcal{D} : ||v^* - v|| \le \alpha\}$. It is sufficient to show that $\{v_r\} \in B$, for each r. We proceed by induction. By construction, we have that $v_1 \in B$. Assume that $v_r \in B$, up to some $r \ge 1$. Then, we have that $||v_r|| \le ||v^*|| + \alpha$. Now, we show that $v_{r+1} \in B$, i.e., $||v_{r+1} - v^*|| \le \alpha$. First, we observe that U is a bounded map. Now, define the following positive constants:

$$\begin{aligned} K_0 : &= \sup\{\|Uy + \delta(y - v_1)\| : y \in B,\ \delta \in (0,1)\} + 1, \\ K_1 : &= \sup\Big\{D \max\Big\{\|y - v^*\| + \lambda K_0, \frac{C}{2}\Big\} : y \in B,\ \lambda \in (0,1)\Big\}, \\ K : &= \max\{K_0, K_1\}, \quad \gamma_0 := \min\Big\{1, \frac{\alpha^2}{4(K^2+1)}\Big\}, \end{aligned}$$

where C and D are the constants appearing in Lemma 2.5. Using Lemmas 2.5, 2.6 and the recursion formula (3.1), we compute as follows:

$$\begin{aligned} ||v_{r+1} - v^*||^2 &= ||(1-\lambda_r\delta_r)y_r - \lambda_r(Uy_r - \delta_r v_1) - v^*||^2 \\ &\leq ||y_r - v^*||^2 - 2\lambda_r\langle\, Uy_r + \delta_r(y_r - v_1), j(y_r - v^*)\rangle \\ &\quad + D\max\left\{||y_r - v^*|| + \lambda_r||Uy_r + \delta_r(y_r - v_1)||, \frac{C}{2}\right\} \times \\ &\quad \rho_M\left(\lambda_r||Uy_r + \delta_r(y_r - v_1)||\right). \end{aligned} \tag{3.2}$$

Applying the definitions of U, γ_0 and Lemma 2.6, it follows from inequality (3.2) that:

$$\begin{aligned} ||v_{r+1} - v^*||^2 &\leq ||y_r - v^*||^2 - 2\lambda_r\langle Uy_r - Uv^*, j(y_r - v^*)\rangle - 2\lambda_r\delta_r\langle y_r - v_1, j(y_r - v^*)\rangle \\ &\quad + K\rho_M(\lambda_r K) \\ &\leq ||y_r - v^*||^2 - \lambda_r\eta||Uy_r - Uv^*||^2 - 2\lambda_r\delta_r||y_r - v^*||^2 + \lambda_r\delta_r||v^* - v_1||^2 \\ &\quad + \lambda_r\delta_r||y_r - v^*||^2 + K\rho_M(\lambda_r K) \\ &\leq (1-\lambda_r\delta_r)||y_r - v^*||^2 + \lambda_r\delta_r||v^* - v_1||^2 + K\rho_M(\lambda_r K) \\ &\leq (1-\lambda_r\delta_r)||v_r - v^*||^2 + 2K\xi_r + \lambda_r\delta_r||v^* - v_1||^2 + \frac{K\rho_M(\lambda_r K)}{\lambda_r K}\lambda_r K \\ &\leq (1-\lambda_r\delta_r)\alpha^2 + 2K\lambda_r\delta_r\gamma_0 + \frac{\lambda_r\delta_r\alpha^2}{4} + K^2\lambda_r\delta_r\gamma_0 \\ &= \alpha^2. \end{aligned} \tag{3.3}$$

Therefore, $\{v_r\}$ is bounded. Consequently, $\{y_r\}$ is bounded. The proof of this lemma is complete. ∎

Lemma 3.2. *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a uniformly smooth real Banach space, M. Let $U\colon \mathcal{D} \to M$ be a η-inverse strongly accretive map with $U^{-1}(0) \neq \emptyset$. Let $\{v_r\}$ be a sequence in $\mathcal{D}$ generated by*

$$\begin{cases} v_0,\ v_1 \in \mathcal{D}, \\ y_r = v_r + \xi_r(v_{r-1} - v_r), \\ v_{r+1} = (1-\lambda_r\delta_r)y_r - \lambda_r(Uy_r - \delta_r v_1),\ \forall\ r \geq 1. \end{cases} \tag{3.4}$$

Then, the sequence $\{v_r\}$ converges strongly to $v^* \in U^{-1}(0)$.

Proof. Set $p_r := J_{t_r}v_1$, where v_1 is an arbitrary fixed vector in $\mathcal{D} \subset M$, $t_r = \delta_r^{-1}$, $\forall\ r \geq 1$ in Lemma 2.7 and observe that with t_r, the sequence $\{p_r\}$ satisfies the following conditions:

$$\delta_r(p_r - v_1) + Ap_r = 0,\ \ \forall\ r \geq 1,\ \text{ and }\ p_r \to p^* \in A^{-1}0, \tag{3.5}$$

where A in this case is single-valued. Using Lemma 2.5, we have that:

$$\begin{aligned} ||v_{r+1} - p_r||^2 &= ||(1-\lambda_r\delta_r)y_r - \lambda_r(Uy_r - \delta_r v_1) - p_r||^2 \\ &\leq ||y_r - p_r||^2 - 2\lambda_r\langle Uy_r + \delta_r(y_r - v_1), j(y_r - p_r)\rangle + K\rho_M(\lambda_r K). \end{aligned} \tag{3.6}$$

Using the definition of U, we have that:

$$\begin{aligned} \langle Uy_r + \delta_r(y_r - v_1), j(y_r - p_r)\rangle &= \langle Uy_r - Up_r, j(y_r - p_r)\rangle + \delta_r||y_r - p_r||^2 \\ &\quad + \langle Up_r + \delta_r(p_r - v_1), j(y_r - p_r)\rangle \\ &\geq \eta||Uy_r - Up_r||^2 + \frac{\delta_r}{2}||y_r - p_r||^2. \end{aligned} \tag{3.7}$$

Substituting inequality (3.7) in inequality (3.6), we have that:

$$||v_{r+1} - p_r||^2 \leq (1 - \lambda_r\delta_r)\,||y_r - p_r||^2 \; - 2\lambda_r\eta||Uy_r - Up_r||^2 \; + \; K\rho_M\,(\lambda_r K)\,. \quad (3.8)$$

Using the definition of U and equation (3.5), we have that:

$$\begin{aligned} ||p_{r-1} - p_r|| &\leq ||p_{r-1} - p_r + \frac{1}{\delta_r}(Up_{r-1} - Up_r)|| \\ &\leq \left|\frac{\delta_{r-1} - \delta_r}{\delta_r}\right| (||y_{r-1}|| + ||v_1||)\,, \end{aligned} \quad (3.9)$$

By Lemma 2.6, we have that:

$$\begin{aligned} ||y_r - p_r||^2 &\leq ||y_r - p_{r-1}||^2 + \langle p_{r-1} - p_r, j(y_r - p_r)\rangle \\ &\leq ||y_r - p_{r-1}||^2 + ||p_{r-1} - p_r||\,||y_r - p_r||. \end{aligned} \quad (3.10)$$

From inequalities (3.8), (3.9) and (3.10), and also, by Lemma 2.6 and for some constant $C^* > 0$, we have that:

$$\begin{aligned} ||v_{r+1} - p_r||^2 &\leq (1 - \lambda_r\delta_r)||y_r - p_{r-1}||^2 + C^*\left|\frac{\delta_{r-1} - \delta_r}{\delta_r}\right| + \; K\rho_M\,(\lambda_r K) \\ &= (1 - \lambda_r\delta_r)||v_r - p_{r-1}||^2 + 2K\xi_r + (\lambda_r\delta_r)\theta_r + \; K\frac{\rho_M(\lambda_r K)}{\lambda_r K}\lambda_r K, \\ &\leq (1 - \lambda_r\delta_r)||v_r - p_{r-1}||^2 + \lambda_r\delta_r K\,(\theta_r + \; \delta_r\gamma_0) + K\lambda_r^4\delta_r\gamma_0, \end{aligned}$$

where $\sigma_r := \lambda_r\delta_r$, $\theta_r := \frac{\left|\frac{\delta_{r-1}-\delta_r}{\delta_r}\right|}{\lambda_r\delta_r}$, $\beta_r := ||v_r - p_{r-1}||^2$, $b_r := K\,(\theta_r + \; \delta_r\gamma_0)\,$, $c_r := \lambda_r^4\delta_r$. By Lemma 2.8, $\lim_{r\to\infty}\|v_r - p_{r-1}\| = 0$. Since $\lim_{r\to\infty} p_r = p^* \in A^{-1}(0)$, then, $\{v_r\}$ converges to $p^* \in U^{-1}(0)$. This completes the proof. ■

Lemma 3.3. *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a uniformly smooth real Banach space, M. Let $\{U_i\}_{i=1}^{\infty}\colon \mathcal{D} \to M$ be a family of η_i-inverse strongly accretive maps $\eta := \inf_{i\geq 1}\eta_i > 0$ with $\cap_{i=1}^{\infty}U^{-1}(0) \neq \emptyset$. Let $\{b_i\}$ a positive sequence such that $\sum_{i=1}^{\infty} b_i = 1$ and $\{v_r\}$ be a sequence in $\mathcal{D}$ generated by*

$$\begin{cases} v_0,\, v_1 \in \mathcal{D}, \\ y_r = v_r + \xi_r(v_{r-1} - v_r), \\ v_{r+1} = (1 - \lambda_r\delta_r)y_r - \lambda_r\left(\sum_{i=1}^{\infty} b_i U_i y_r - \delta_r v_1\right), \; \forall\; r \geq 1. \end{cases} \quad (3.11)$$

Then, the sequence $\{v_r\}$ converges strongly to $v^* \in \cap_{i=1}^{\infty}U^{-1}(0)$.

Proof. Applying Lemma 2.4, we have that $U = \sum_{i=1}^{\infty} b_i U_i$ is an η_i-inverse strongly accretive, for each $i = 1,\, 2, \ldots$ and $U^{-1}(0) = \cap_{i=1}^{\infty}U^{-1}(0)$. Hence, by Lemma 3.2, the conclusion of Theorem 3.3 is immediate. ■

Now, we prove our main theorem.

Theorem 3.4. *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a uniformly smooth real Banach space, M. Let $\{T_i\}_{i=1}^{\infty} \colon \mathcal{D} \to \mathcal{D}$ be a family of η_i-strictly pseudocontractive maps such that $\eta := \inf_{i\geq 1} \eta_i > 0$ with $\cap_{i=1}^{\infty} U^{-1}(0) \neq \emptyset$. Let $\{b_i\}$ be a positive sequence such that $\sum\limits_{i=1}^{\infty} b_i = 1$ and $\{v_r\}$ be a sequence in $\mathcal{D}$ generated by*

$$\begin{cases} v_0,\, v_1 \in \mathcal{D}, \\ y_r = v_r + \xi_r(v_{r-1} - v_r), \\ v_{r+1} = (1 - \lambda_r\delta_r)y_r - \lambda_r \left(\sum\limits_{i=1}^{\infty} b_i(I - T_i)y_r - \delta_r v_1 \right), \ \forall\ r \geq 1. \end{cases} \tag{3.12}$$

Then, the sequence $\{v_r\}$ converges strongly to $v^* \in \cap_{i=1}^{\infty} F(T_i)$.

Proof. Setting $U_i := I - T_i$. By Lemma 2.4, U_i is an η_i-inverse strongly accretive map, for each $i = 1, \ldots,$ and $\cap_{i=1}^{\infty} U_i^{-1}(0) = \cap_{i=1}^{\infty} F(T_i)$. Thus, by Lemma 3.3, the proof of Theorem 3.4 is immediate. ∎

Remark 3.5. Theorem 3.4 is applicable in L_p, l_p *or* $W_p^m(\Omega)$ spaces, $1 < p < \infty$, where $W_p^m(\Omega)$ denote the usual Sobolev space, since these spaces are uniformly smooth. The analytical representations of the duality map in the spaces indicated where $p^{-1} + q^{-1} = 1$, are known precisely (see e.g., Theorem 3.1 of [1]; page 36).

4. Numerical Illustration

In this section, we present some numerical examples to show the efficiency of our algorithms, Algorithms (3.11) and (3.12) over Algorithms (1.5) and (1.6). Numerical experiments are carried out on Matlab R2020a version. All programs are run on MacBook Pro with 2.6 GHz Dual-Core Intel Core i5 and 8GB 1600 MHz DDR3.

Example 4.1. Let $M = \mathbb{R}^n$ and $U, U_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, 2, \ldots$ be defined by $U_i v = 2v$ and $Uv = \sum_{i=1}^{\infty} \sigma_i U_i v = 2v$, for each $v \in \mathbb{R}^n$, where $b_i = \frac{1}{2^i}$, a sequence in $(0, 1)$. Clearly, U is $\frac{1}{2}$-inverse strongly monotone and $T := I - U$ is a strict pseudocontractive map. Furthermore, $v^* = 0$ is a solution of $Uv = 0$ which is also a fixed point of T. In Algorithms (3.11) and (3.12), we take $\lambda_r = (r+1)^{-\frac{1}{4}}$, $\delta_r = (r+1)^{-\frac{1}{8}}$ and $\xi_r = (r+1)^{-\frac{9}{8}}$, and in Algorithms (1.5) and (1.6), we take $\lambda_n = (n+1)^{-\frac{1}{4}}$ and $\theta_n = (n+1)^{-\frac{1}{8}}$, where $n = r \to \infty$. Then, we obtain the following iterates:

Table 1. **Computational results of Example 4.1**

N	Algorithm (1.5)	Algorithm (1.6)	Algorithm (3.11)	Algorithm (3.12)
	$\|\|v_N\|\|_2$	$\|\|v_N\|\|_2$	$\|\|v_N\|\|_2$	$\|\|v_N\|\|_2$
10	7.212899×10^{-1}	7.212899×10^{-1}	1.108282×10^{-3}	9.427619×10^{-9}
100	3.104400×10^{-1}	5.094462×10^{-1}	0.000000	0.000000
1000	2.462945×10^{-1}	4.195862×10^{-1}	0.000000	0.000000
10000	1.930853×10^{-1}	3.397905×10^{-1}	0.000000	0.000000

Table 2. **Computational time of Example 4.1**

N	Algorithm (1.5)	Algorithm (1.6)	Algorithm (3.11)	Algorithm (3.12)
	Comp. time	Comp. time	Comp. time	Comp. time
10	0.008321	0.005743	0.010976	0.018517
100	0.009927	0.019063	0.018546	0.037572
1000	0.016240	0.039444	0.019051	0.033098
10000	0.031547	0.042202	0.018616	0.030862

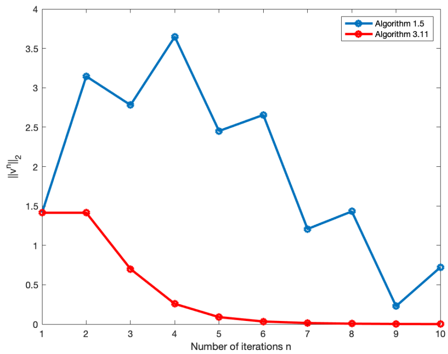

Figure 1.

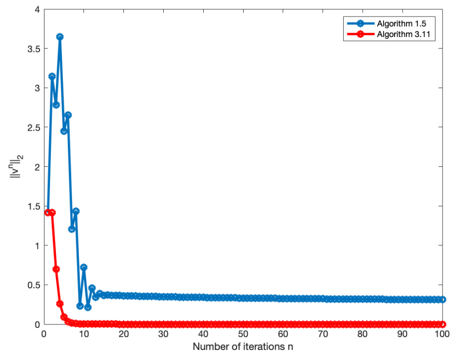

Figure 2.

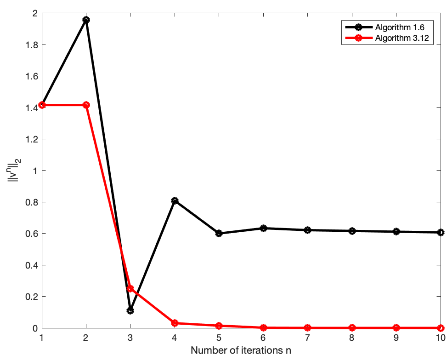

Figure 3.

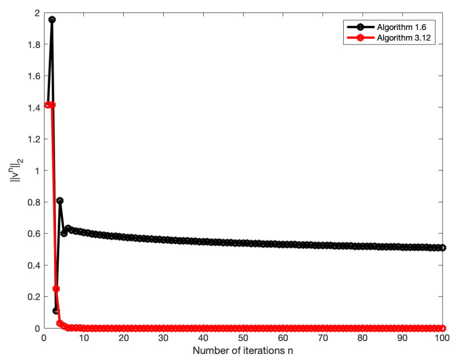

Figure 4.

Example 4.2. Let $M = \mathcal{L}_p^{\mathbb{R}}([0,1])$, $p := 3$. Let U, $\{U_i\}_{i=1}^{\infty} : \mathcal{L}_p^{\mathbb{R}}([0,1]) \to \mathcal{L}_p^{\mathbb{R}}([0,1])$ be defined by

$$(U_i v)(t) = \frac{1}{t+1}v(t) \quad \text{and} \quad (Uv)(t) = \sum_{i=1}^{\infty} a_i (U_i v)(t) = \frac{1}{t+1}v(t), \ \text{for each } t \in [0,1],$$

where $a_i = \frac{1}{2^i}$, a sequence in $(0,1)$. Clearly, U is inverse strongly monotone and $T := I - U$ is a strictly pseudocontraction map. Furthermore, $v^*(0) = 0$ is a solution of $(Uv)(t) = 0$ which is also a fixed point of T. In Algorithms (3.11), (3.12), Algorithms (1.5) and (1.6), we take our parameters as in Example 4.1. Then, we obtain the following iterates:

Table 3. **Computational results of Example 4.2**

N	Algorithm (1.5)	Algorithm (1.6)	Algorithm (3.11)	Algorithm (3.12)
	$\|\|v_N\|\|_2$	$\|\|v_N\|\|_2$	$\|\|v_N\|\|_2$	$\|\|v_N\|\|_2$
5	5.516058×10^{-1}	2.970616×10^{-1}	7.728360×10^{-3}	1.689166×10^{-2}
10	6.050354×10^{-1}	2.847487×10^{-1}	5.324458×10^{-4}	6.759885×10^{-4}

Table 4. **Computational time of Example 4.2**

N	Algorithm (1.5)	Algorithm (1.6)	Algorithm (3.11)	Algorithm (3.12)
	Comp. time	Comp. time	Comp. time	Comp. time
5	0.087513	0.094006	0.038490	0.034549
10	10.192401	68.222725	27.616478	30.107333

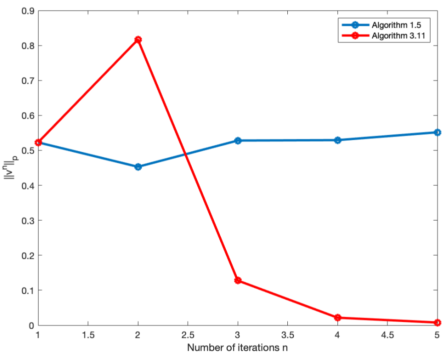

Figure 5.

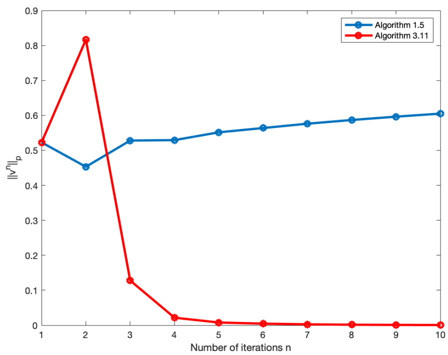

Figure 6.

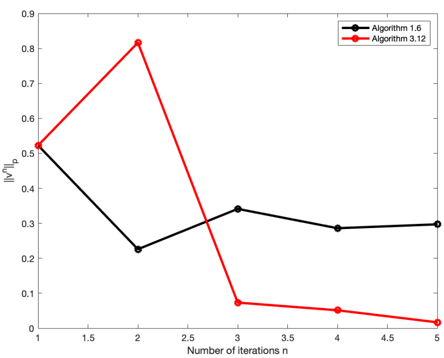

Figure 7.

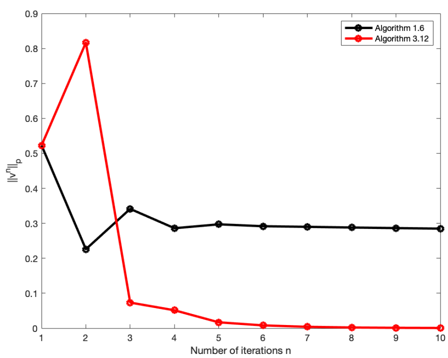

Figure 8.

Remark 4.3. Tables 1 - 4 represent computational results and CPU time of Algorithms (1.5), (1.6), (3.11) and (3.12). As observed in Figures 2 and 4, Algorithms (3.11) and (3.12) before 10 iterations, converged to a zero of U and a fixed point of T in 0.010976 seconds and 0.018517 seconds, respectively, whereas, Algorithms (1.5) and (1.6) are yet to converge to solution after 100. Similarly, in Figures 6 and 8, Algorithms (3.11) and

(3.12) before 6 iterations, converged to a zero of U and a fixed point of T in 0.038490 seconds and 0.034549 seconds, respectively, whereas, Algorithms (1.5) and (1.6) are yet to converge to solution after 10.

5. Conclusion

In this paper, we introduced a new inertial algorithm for approximating a common fixed point of an infinite family of strict speudocontractions without imposing any *compactness condition on the maps or their domains.* The sequence of the algorithm is proved to converge strongly to a common fixed point of the maps in a uniformly smooth real Banach space. This result is achieved as an application of a new inertial algorithm whose sequence approximates a common zero of an infinite family of inverse strongly accretive maps. Furthermore, given the test examples, the sequences of our inertial algorithms, Algorithms (3.11) and (3.12) performed much better when compared with the sequences of Algorithms (1.5) and (1.6) of Theorem 1.3 without inertial term. Finally, the theorems proved improve and complement most related results in the literature.

Competing Interest

The authors declare that they have no conflict of interest.

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